On Singularity Formation Under Mean Curvature Flow

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Mean Curvature Flow

The mean curvature flow is a family of hypersurfaces $M_t \subset \mathbb{R}^{d+1}$ whose smooth immersions $\psi(\cdot, t) : N \to M_t \subset \mathbb{R}^{d+1}$ satisfy the partial differential equation

$$(\partial_t \psi)^N = -H(\psi) \tag{1}$$

where $(\partial_t \psi)^N$ is the normal component of $\partial_t \psi$ and $H(x)$ is the mean curvature of M_t at a point $x \in M_t$.

Applications and Connections

- Material Science (interface motion between different materials or different phases).
- Image recognition.
- Connection to the Ricci flow.
- Topological classification of surfaces and submanifolds.

Some Key Works: Existence

- First mathematical treatment (using geometric measure theory): Brakke [1978];
- Short time existence: Brakke, Huisken, Evans and Spruck, Ilmanen, Ecker and Huisken [1991];
- Weak solutions: Evans and Spruck, Chen, Giga and Goto [1991];

Some Key Works: Singularities

The most interesting problem here is formation of singularities.

- Collapse of convex hypersurfaces: Huisken [1984], extensions: White [2000, 2003], Huisken and Sinestrari [2007-2009];
- Neckpinching for rotationally symmetric hypersurfaces: Grayson, Ecker, Huisken, M. Simon, Dziuk and Kawohl, Smoczyk, Altschuler, Angenent and Giga, Soner and Souganidis [1990-1995];
- MCF with surgery and topological classification of surfaces and submanifolds: Huisken and Sinestrari [2007-2009];
- Nature of the singular set: Huisken [1990], White [2000, 2003], Colding and Minicozzi [2012].

Huisken's Conjecture

Under MCF, the $\mathrm{vol}(M_t) \to 0$ as $t \to t_* \implies$ closed surfaces collapse. How this collapse takes place?
Besides planes, there are two explicit solutions of MCF:

- Collapsing Euclidean spheres with radii decreasing as $\sqrt{2d(t_* - t)}$;
- Collapsing Euclidean cylinders with radii decreasing as $\sqrt{2(d-1)(t_* - t)}$;

Conjecture [Huisken]: Generic sing. are spheres and cylinders.

Partial results: Huisken, White, Colding and Minicozzi

Results:

- The spherical collapse is asymptotically stable.
- The cylindrical collapse is unstable.

Neckpinching

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S) Let $d \geq 1$ and (informally for brevity)

M_0 be a surface close to a cylinder, $\mathcal{C}^{d+1}$,

M_0 has an arbitrary shallow waist and is even w.r.to the waist.

Then M_t is defined by an immersion

$$\psi(\omega, x, t) = (u(\omega, x, t)\omega, x) \qquad (2)$$

of C^{d+1}, where $(\omega, x) \in \mathcal{C}^{d+1}$ and $u(\omega, x, t)$ satisfies

(i) There exists a finite time t^* such that $\inf u(\cdot, t) > 0$ for $t < t^*$ and $\lim_{t \to t^*} \inf u(\cdot, t) \to 0$;

(ii) If $u_0 \partial_x^2 u_0 \geq -1$ then there exists a function $u_*(\omega, x) > 0$ such that $u(\omega, x, t) \geq u_*(\omega, x)$ for $\mathbb{R} \backslash \{0\}$ and $t \leq t^*$.

Dynamics of Scaling Parameter

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S)

(iii) There exist C^1 functions $\zeta(\omega, x, t)$, $\lambda(t)$ and $b(t)$ such that

$$u(\omega, x, t) = \lambda(t)[\sqrt{\frac{d + b(t)y^2}{a(t)}} + \zeta(\omega, y, t)]$$

with $y := x/\lambda(t)$, $a(t) = -\lambda(t)\partial_t\lambda(t)$ and

$$\|\langle y\rangle^{-m}\partial_y^n\zeta(\omega, y, t)\|_\infty \leq cb^2(t), \; m + n = 3.$$

(iv) The parameters $\lambda(t)$ and $b(t)$ satisfy (with $\tau := 2d(t^* - t)$)

$$\begin{aligned} \lambda(t) &= \tau^{\frac{1}{2}}(1 + o(1)) && \text{(scaling parameter)} \\ b(t) &= -\tfrac{d}{\ln\tau}(1 + O(\tfrac{1}{|\ln\tau|^{3/4}})) && \text{(shape parameter).} \end{aligned} \tag{3}$$

Comparison with Previous Results

A result similar to (ii) (axi-symmetric surfaces) but for a different set of initial conditions was proven by H.M.Soner and P.E.Souganidis.

The previous result closest to ours is that by S. Angenent and D. Knopf on the axi-symmetric neckpinching for the Ricci flow.

Some ideas of the proof are close to those of Bricmont and Kupiainen on NLH.

All works mentioned above deal with *surfaces of revolution* of barbell shapes (*far from cylinders*) which are either compact (Dirichlet b.c.) or periodic (Neumann b.c.).

These works rely on parabolic maximum principle going back to Hamilton and monotonicity formulae for an entropy functional $\int_{M_t}$ backward heat kernel$(x,t)d\mu_t$, due to Huisken and Giga and Kohn.

Symmetries and Solitons

Collapsing spheres and cylinders are scaling solitons. The solitons correspond to symmetries of the MCF.

Given a generalized symmetry group, T_λ, of the MCF, i.e. one-parameter group satisfying

$$H(T_\lambda \psi) = b(\lambda) H(\psi) \qquad (\Rightarrow b(st) = b(s)b(t)),$$

we define the corresponding soliton as

$$\psi(t) = T_{\lambda(t)}\varphi.$$

Related to the translational, rotational and scaling symmetries of MCF are translational, rotational and scaling solitons.

We are interested in the solitons corresponding to the scaling sym.:

$$M(t) \equiv M^{\lambda(t)} := \lambda(t) M, \qquad \text{where } \lambda(t) > 0.$$

Rescaled MCF

To understand dynamics near a scaling soliton, we rescale the MCF:

$$\varphi(u,\tau) := \lambda^{-1}(t)\psi(u,t), \qquad \tau := \int_0^t \frac{dt'}{\lambda(t')^2} .$$

Important point: we do not fix $\lambda(t)$ but consider it as free parameter to be found from MCF. The rescaled MCF satisfies

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \qquad a = -\dot{\lambda}\lambda .$$

- The rescaled MCF is a gradient flow for the Huisken functional

$$V_a(\varphi) := \int_{M^\lambda} e^{-\frac{a}{2}|x|^2},$$

where $M^\lambda = \lambda^{-1}(t)M$ is the rescaled surface M.

(MCF is a gradient flow for the area functional $V(\psi) = V_{a=0}(\psi)$.)

Self-similar Surfaces

We traded the fast changing $\lambda(t)$ for slow changing $a(\tau) = -\dot{\lambda}\lambda$. We consider the rescaled MCF as an equation for φ and a:

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle. \tag{4}$$

- Its static solutions are self-similar surfaces,

$$H(\varphi) - a\langle \nu(\varphi), \varphi\rangle = 0, \qquad a \in \mathbb{R}.$$

Expect: as $\tau \to \infty$, solutions $\longrightarrow$ self-similar surfaces.

$\Rightarrow$ classify self-similar surfaces and determine their stability.

Theorem. (Huisken, Colding-Minicozzi) Under certain conditions, the only self-similar surfaces are planes, spheres and cylinders.

Cf. Bernstein conjecture for minimal surfaces ($a = 0$).

Linearized Stability

φ = a self-similar surface $\Longrightarrow$

$$\varphi_{\lambda,z,g} := T_g^{\mathrm{rot}} T_z^{\mathrm{transl}} T_\lambda^{\mathrm{scal}} \varphi$$

is also a self-similar surface. Consider the manifold

$$\mathcal{M}_{\mathrm{self-sim}} := \{\varphi_{\lambda,z,g} : (\lambda, z, g) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \times SO(d+1)\}. \quad (5)$$

Definition (Linearized stability of self-similar surfaces)

A self-similar surface φ, with $a > 0$, is *linearly stable* iff

$$\mathrm{Hess}^N V_a(\varphi) > 0 \quad \text{on} \quad (T_\varphi \mathcal{M}_{\mathrm{self-sim}})^\perp.$$

Note $T\mathcal{M}_{\mathrm{self-sim}} = \{$scaling, transl., rot. modes$\}$
(i.e. the only unstable motions allowed are scaling, transl., rot..)

Symmetries and Spectrum of Hessian

Theorem. The hessian $\mathrm{Hess}^N V_a(\varphi)$ of $V_a(\varphi)$ in the normal direction at a self-similar $d-$dimensional surface φ has

1. (Colding-Minicozzi) the simple eigenvalue $-2a$,
2. (Colding-Minicozzi) the eigenvalue $-a$ of multiplicity $d+1$,
3. the eigenv. 0 of multiplicity $\frac{1}{2}(d-1)d$ (unless φ is a sphere).

These eigenvalues are due to breaking φ *scaling, translation and rotation* symmetry of MCF. The eigenvalue 0 distinguishes between a *sphere, a cylinder and a generic surface*.

Proof. To prove say the 1st statement, we observe that, if φ is self-similar, then it satisfies the equation

$$H_{\lambda^{-2}a}(\lambda\varphi) = \lambda^{-1} H_a(\varphi), \ \forall \lambda \in \mathbb{R}_+.$$

Differentiating this equation w.r.to λ at $\lambda = 1$, and reparametrizing the result, we arrive at the desired eigenvalue equation. $\square$

Spectrum and Stability

The spectral theorem above gives the tangent spaces to the unstable and central manifolds. They correspond to the eigenvalues $-2a$, $-a$ and 0.

Hence, if these are the only non-positive eigenvalues, then we expect the stability in the transverse direction to $\mathcal{M}_{\mathrm{self-sim}}$. Otherwise, we expect instability.

Spectrum and Mean convexity

The spectral information tells us about the geometry of φ. In particular, we have the following result

Theorem

Let φ be a self-similar surface. Then:

(a) (Colding-Minicozzi) For $a > 0$ (shrinker),
$\text{Hess}^N V_a(\varphi) \geq -2a$ *iff* $H(\varphi) > 0$.

(b) For $a < 0$ (expander), $H(\varphi)$ changes the sign.

Proof.

One shows that the normal hessian, $\text{Hess}^N V_a(\varphi)$, has a positivity improving property. Therefore the Perron-Frobenius theory applies and gives the result. □

Spectral Picture of Collapse: Sphere and Cylinder

For the $d-$sphere of the radius $\sqrt{\frac{a}{d}}$, the normal hessian > 0 on (scaling and translational modes)$^{\perp} \Rightarrow \mathbb{S}^d_{\sqrt{\frac{a}{d}}}$ is linearly stable.

For the $(d+1)-$cylinder of the radius $\sqrt{\frac{a}{d}}$, the normal hessian has, in addition to the eigenvalues above,

1. the eigenvalue $-a$ of multiplicity 1, due to translations along the axis of the cylinder,
2. the eigenvalue 0 of multiplicity $d+1$, which originates in a "shape instability".

Hence the $(d+1)-$cylinder is linearly unstable.

Modulated Cylinders

Consider cylinders. We have to expand the manifold of cyliners to incorporate the additional central manifold found above.

Using the eigenfunction corresponding to the shape instability eigenvalue, we find the approximate neck profile

$$\varphi_{ab}(y,\omega) := (y, \rho_{ab}^{\mathrm{neck}}(y)\omega), \qquad \rho_{ab}^{\mathrm{neck}} := \sqrt{\frac{d+by^2}{a}}, \ b>0. \quad (6)$$

We extend the manifold of self-similar solutions, $\mathcal{M}_{\mathrm{self-sim}}$, to the manifold of modulated cylinders or necks

$$M_{neck} := \{\lambda g \varphi_{ab} + z : (\lambda,\ z,\ g, a, b) \in \mathcal{P}\}, \qquad (7)$$

where $\varphi_{ab}(y,\omega) := (y, \rho_{ab}^{\mathrm{neck}}(y)\omega)$ and $\mathcal{P} := G_{sym} \times \mathbb{R}^+ \times \mathbb{R}^+$.

Hessian on the Neck

Consider the Hessian on the neck $\varphi_{ab} = \text{graph}_{\mathcal{C}^{d+1}}\, \rho_{ab}^{\text{neck}}$ in the direction transversal to the neck manifold M_{neck}:

$$Hess^N V_a(\varphi_{ab}) = \underbrace{-\partial_y^2 + ay\partial_y - 2a - \frac{a}{d}\Delta_{\mathbb{S}^d}}_{\text{normal hess on cyl}} + W_{ab}(y,\omega). \qquad (8)$$

($W_{ab}(y,\omega)$ is generated by ρ_{ab}^{neck}.) Now, one can show that

$$Hess^N V_a(\varphi_{ab}) > 0 \quad \text{on} \quad M_{neck}^{\perp}$$

$\Rightarrow$ The evolution is linearly stable in transverse directions.

Key Estimate

Linearize MCF on the neck manifold M_{neck} to obtain

$$\partial_\tau \phi = L_{ab}\phi,$$

where $L_{ab} := Hess^N V_a(\varphi_{ab})$.

Let $U(\tau,\sigma)$, $\tau \geq \sigma \geq 0$, be the propagator generated by $-L_{ab}$.

The main step: showing the *key propagation estimate*:

$$\|\langle z\rangle^{-3} U(\tau,\sigma) g\|_\infty \lesssim e^{-c(\tau-\sigma)} \|\langle z\rangle^{-3} g\|_\infty, \tag{9}$$

$$\forall g \in (TM_{neck})^\perp \approx (\mathrm{Span}\{1,\ a(\tau)y^2 - 1\})^\perp.$$

Here $\perp$ is in the sense of $L^2(\mathbb{R}\times\mathbb{S}^d, e^{-\frac{a(\tau)}{2}y^2} dy d\omega)$.

Estimating the Linear Propagator. I

Write $L_{ab} = L_{a0} + W$, with $L_{a0} := -\partial_y^2 + ay\partial_y - 2a$ (the normal hessian at the cylinder), and use that W is slowly varying in y to do a multiplicativ perturbation (adiabatic) theory.
For the integral kernel $K(x,y)$ of $U(\tau,\sigma)$ (for simplicity, we do not display the variables of $\mathbb{S}^d$), we have the representation

$$K(x,y) = K_0(x,y)\langle e^W\rangle(x,y),$$

where $K_0(x,y)$ is the integral kernel of the operator $e^{-(\tau-\sigma)L_{a0}}$ and

$$\langle e^W\rangle(x,y) = \int e^{\int_\sigma^\tau W(\omega(s)+\omega_0(s),s)ds} d\mu(\omega).$$

Here $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma,\tau] \to \mathbb{R}$ with the boundary condition $\omega(\sigma) = \omega(\tau) = 0$ and

$$(-\partial_s^2 + a^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x.$$

Estimating the Linear Propagator. II

To estimate $U(x, y)$ for $e^{a(\tau-\sigma)} \leq b^{-1/32}(\tau)$ we use the explicit formula

$$K_0(x, y) = 4\pi(1 - e^{-2ar})^{-\frac{1}{2}}\sqrt{a}e^{2ar}e^{-a\frac{(x-e^{-ar}y)^2}{2(1-e^{-2ar})}},$$

where $r := \tau - \sigma$, and the bound

$$|\partial_y \langle e^W \rangle(x, y)| \leq b^{\frac{1}{2}} r,$$

which follows from the definition of $\langle e^W \rangle$ and the properties

$$W(y, \tau) \geq 0 \text{ and } |\partial_y W(y, \tau)| \lesssim b^{\frac{1}{2}}(\tau).$$

Then we iterate using the semi-group property $\Rightarrow$ control the rescaled MCF.

Thank-you for your attention

Extensions

We do not fix the cylinder and look for surfaces of the form

$$\psi(x, \omega, t) = \lambda(t) g(t) \varphi(y, \omega, \tau) + z(t),$$

where $(\lambda,\ z,\ g) : [0, T) \to \mathbb{R} \times \mathbb{R}^{d+2} \times SO(d+2)$,
to be determined later,
$y = \lambda^{-1}(t)x, \qquad \tau = \tau(t) := \int_0^t \lambda^{-2}(s)ds$, and
$\varphi(\cdot, \tau) : \mathcal{C}^{d+1} \to \mathbb{R}^{d+2}$ is a normal graph over the fixed cylinder.

The time dependent parameters $\lambda(t),\ z(t),\ g(t)$ are chosen so that $\varphi(\cdot, \tau)$ is orthogonal to the non-positive (scaling, translation and rotation) modes of the normal hessian on the cylinder.

Then we proceed as before.

*Comparisons

Compare the dynamics for the scaling parameter $\lambda(t)$ for (MCF) and the critical Yang-Mills equation

$$\lambda\ddot{\lambda} = \frac{3}{4}\dot{\lambda}^4,$$

the critical wave map equation

$$\dot{\lambda}^2 = \lambda\ddot{\lambda}\ln\frac{c}{\lambda\ddot{\lambda}}\ , \ c = 0.122.$$

and the Keller-Segel equations, for $a(\tau) = -\lambda(t)\dot{\lambda}(t)$,

$$a_\tau = -\frac{2a^2}{\ln(\frac{1}{a})}. \tag{10}$$

For the critical Keller-Segel equations:

$$\lambda(t) = (T-t)^{\frac{1}{2}}e^{-|\frac{1}{2}\ln(T-t)|^{\frac{1}{2}}}(c_1 + o(1)). \tag{11}$$

For the critical Yang-Mills equation this gives

$$\lambda \approx \sqrt{\frac{2}{3}\frac{t_*-t}{\sqrt{-\ln(t_*-t)}}}.$$