Discrete entropy methods for nonlinear diffusive evolution equations

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- Continuous and discrete entropy methods
- Implicit Euler finite-volume scheme
- Higher-order time schemes
- Extensions

Entropy-dissipation method

Setting: u_∞ solves $A(u) = 0$, u solves

$$\partial_t u + A(u) = 0, \quad t > 0, \quad u(0) = u_0$$

- Lyapunov functional: $H[u]$ satisfies $\frac{dH}{dt}[u(t)] \leq 0$ for $t \geq 0$
- Entropy: convex Lyapunov functional $H[u]$ such that

$$D[u] := -\frac{dH}{dt}[u] = \langle A(u), H'[u] \rangle \geq 0$$

- Bakry-Emery approach: show that, for $\kappa > 0$,

$$\frac{d^2H}{dt^2}[u] \geq -\kappa \frac{dH}{dt}[u] \;\Rightarrow\; D[u] = -\frac{dH}{dt}[u] \geq \kappa H[u]$$

Consequences:

- $\frac{dH}{dt} \leq -\kappa H$ implies that $H[u(t)] \leq H[u(0)] e^{-\kappa t}\ \forall t > 0$
- $H[u] \leq \kappa^{-1} D[u]$ corresponds to convex Sobolev inequality

Introduction

Example: heat equation $\partial_t u = \Delta u$ on torus $\mathbb{T}^d$

Entropy: $H[u] = \int_\Omega u \log(u/u_\infty)dx$, u_∞: steady state

① Entropy-dissipation inequality:

$$D[u] = -\frac{dH}{dt}[u] = 4\int_{\mathbb{T}^d} |\nabla\sqrt{u}|^2 dx \geq 0$$

② Second-order time derivative:

$$\frac{d^2H}{dt^2}[u] = 4\int_{\mathbb{T}^d} \frac{\Delta\sqrt{u}}{\sqrt{u}}\Delta u dx \geq -\kappa\frac{dH}{dt} \quad \Rightarrow \quad \frac{dH}{dt} \leq -\kappa H$$

- Exponential decay to equilibrium:

$$H[u(t)] = \int_\Omega u \log\frac{u}{u_\infty}dx \leq H[u(0)]e^{-\kappa t}$$

- Logarithmic Sobolev inequality:

$$H[u] = \int_\Omega u \log\frac{u}{u_\infty}dx \leq \frac{1}{\kappa}D[u] = \frac{4}{\kappa}\int_{\mathbb{T}^d} |\nabla\sqrt{u}|^2 dx$$

Benefit: very robust, in particular for nonlinear problems

Introduction

Setting: $\partial_t u + A(u) = 0$, $t > 0$, $u(0) = u_0$

Task: Develop discrete entropy methods

Program:

- Implicit Euler scheme: $\frac{1}{\tau}(u^k - u^{k-1}) + A(u^k) = 0$
- Higher-order time scheme: $\partial_t^\tau u^k + A(u^k, u^{k-1}, \ldots) = 0$
- Finite-volume scheme: $\partial_t u_K + A(u_K) = 0$, u_K: const., K: control volume
- Fully discrete schemes, higher-order spatial discretizations
- Higher-order minimizing movement schemes

Questions: Is $H[u^k]$ dissipated? Rate of entropy decay?

Key idea: Translate entropy method to discrete settings

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✘ Fully discrete schemes, higher-order spatial discretizations

✘ Higher-order minimizing movement schemes (in progress)

Questions: Is $H[u^k]$ dissipated? Rate of entropy decay?

Key idea: Translate entropy method to discrete settings

Overview

- Introduction
- Implicit Euler finite-volume scheme
- Semi-discrete one-leg multistep scheme
- Semi-discrete Runge-Kutta scheme

Implicit Euler finite-volume scheme

Example: $\partial_t u = \Delta u^\beta$ with no-flux boundary conditions

Continuous case: entropy $H_\alpha[u] = \int_\Omega u^\alpha dx - (\int_\Omega u dx)^\alpha$

$$\frac{dH_\alpha}{dt} = \frac{d}{dt}\int_\Omega u^\alpha dx = \alpha \int_\Omega u^{\alpha-1}\Delta u^\beta dx$$

$$= -\frac{4\alpha\beta}{\alpha+\beta-1}\int_\Omega |\nabla u^{(\alpha+\beta-1)/2}|^2 dx \le -CH_\alpha[u]^{(\alpha+\beta-1)/\alpha}$$

"$\le$" follows from Beckner inequality: $(f = u^{(\alpha+\beta-1)/2})$

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \le C_B \|\nabla f\|^q_{L^2(\Omega)}, \quad q \le 2$$

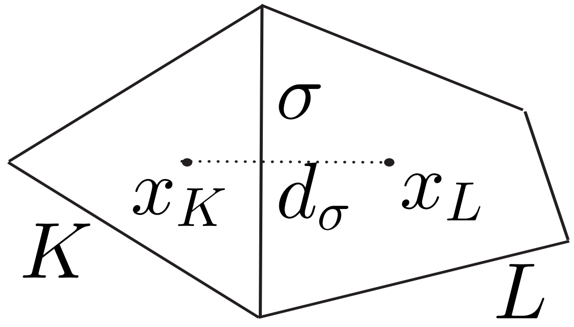
Standard Beckner inequality: $q = 2$

Proof: Differentiate L^p interpolation inequality (Dolbeault) and use generalized Poincaré-Wirtinger inequality

Task: Translate computations to discrete case

Implicit Euler finite-volume scheme

Finite-volume scheme: $\Omega = \cup K$



- Control volumes K, edges $\sigma = K|L$
- Transmissibility coeff.: $\tau_\sigma = |K|/d_\sigma$

$$|K|(u_K^k - u_K^{k-1}) + \tau \sum_{\sigma=K|L} \tau_\sigma((u_K^k)^\beta - (u_L^k)^\beta) = 0$$

Discrete case: $H_\alpha^d[u] = \sum_K |K| u_K^\alpha - (\sum_K |K| u_K)^\alpha$

$$H_\alpha^d[u^k] - H_\alpha^d[u^{k-1}] = \sum_K |K|((u_K^k)^\alpha - (u_K^{k-1})^\alpha)$$

$$\leq \sum_K |K|(u_K^k)^{\alpha-1}(u_K^k - u_K^{k-1})$$

$$\leq -C_1 |(u^k)^{(\alpha+\beta-1)/2}|_{H^1}^2 \leq -C_2 H_\alpha^d[u^k]^{(\alpha+\beta-1)/\alpha}$$

Follows from discrete Beckner inequality

Proof: Use discrete Poincaré-Wirtinger inequality

(Bessemoulin-Chatard, Chainais-Hillairet, Filbet 2012)

Implicit Euler finite-volume scheme

Theorem: (Chainais-Hillairet, A.J., Schuchnigg, 2013)

$$H_\alpha^d[u^k] \le (C_1 t_k + C_2)^{-\alpha/(\beta-1)}, \quad \alpha > 1, \ \beta > 1$$

$$H_\alpha^d[u^k] \le H_\alpha^d[u^0] e^{-\lambda t_k}, \quad 1 < \alpha \le 2, \ \beta > 0$$

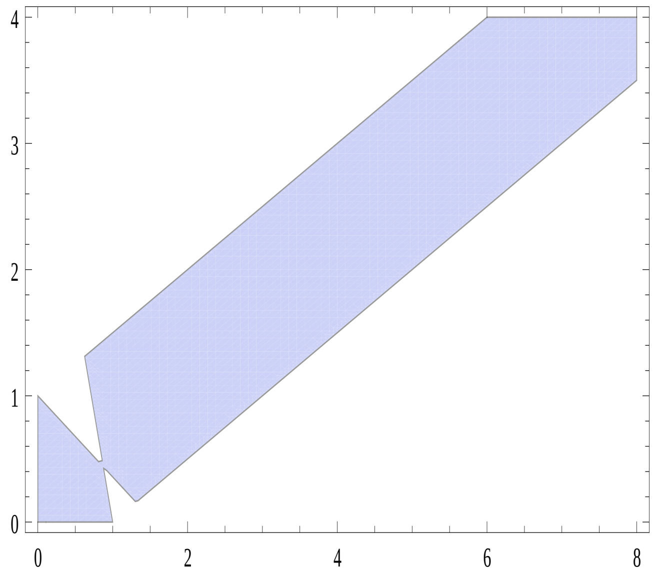
First-order entropies? $H_\alpha[u] = \int_\Omega |\nabla u^{\alpha/2}|^2 dx$

- Continuous case: Let $(\alpha, \beta) \in M_d$

$$\frac{dH_\alpha}{dt} = -\alpha \int_\Omega \operatorname{div}\,(u^{\alpha/2-1}\nabla u^{\alpha/2})\Delta u^\beta dx$$

$$\le -C \int_\Omega u^{\alpha+\beta-\gamma-1}(\Delta u^{\gamma/2})^2 dx$$

$$\le -C(\inf u_0) H_\alpha[u]$$



Proof: Systematic integration by parts (A.J.-Matthes 2006)

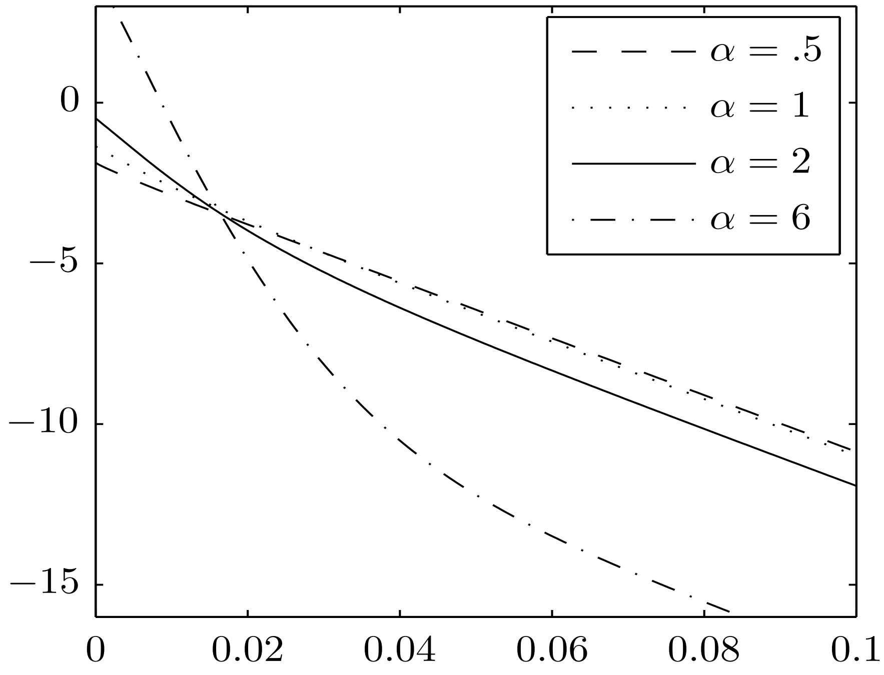
- Discrete case: If $\alpha = 2\beta$ then $H_\alpha^d[u^k]$ nonincreasing

  If 1-D and uniform grid, $H_\alpha^d[u^k] \le H_\alpha^d[u^0] e^{-\lambda t_k}$

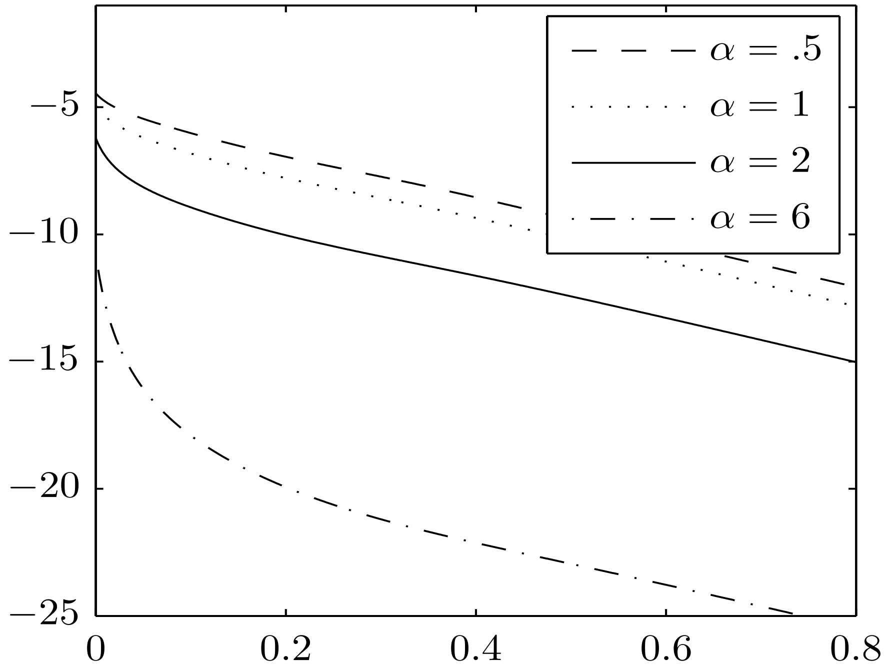
Implicit Euler finite-volume scheme

Numerical results: Zeroth-order entropies

$$\partial_t u = \Delta u^\beta, \quad H_\alpha[u] = \int_\Omega u^\alpha dx - \left(\int_\Omega u dx\right)^\alpha$$



$\beta = \frac{1}{2}$: $\log H_\alpha[u(t)]$ versus t



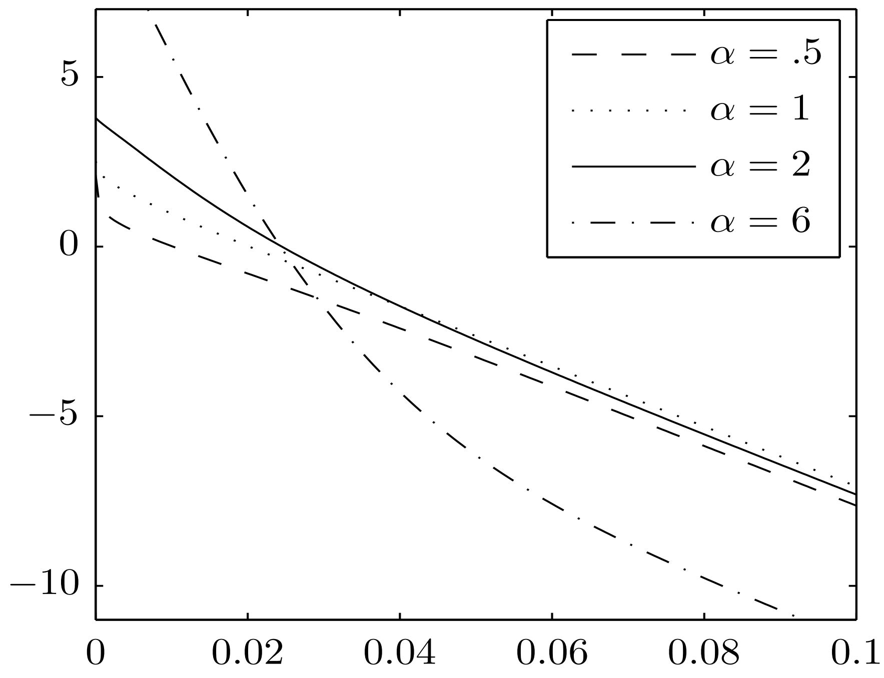
$\beta = 2$: $\log H_\alpha[u(t)]$ versus t

- 2-D scheme, uniform grid, initial data: Barenblatt profile
- Exponential time decay for all α

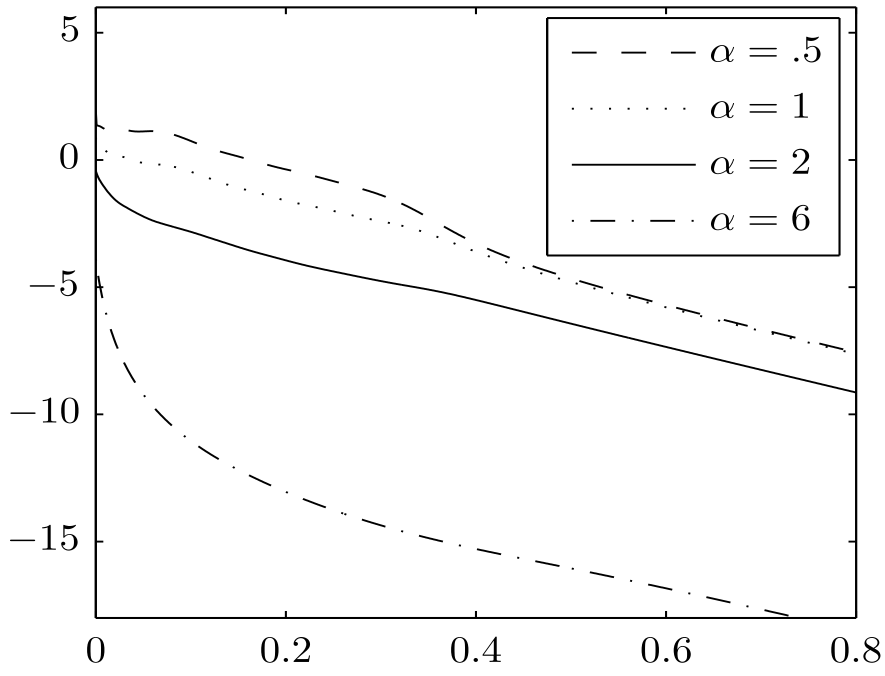
Implicit Euler finite-volume scheme

Numerical results: First-order entropies

$$\partial_t u = \Delta u^\beta, \quad H_\alpha[u] = \int_\Omega |\nabla u^{\alpha/2}|^2 dx$$



$\beta = \frac{1}{2}$: $\log H_\alpha[u(t)]$ versus t



$\beta = 2$: $\log H_\alpha[u(t)]$ versus t

- 2-D scheme, uniform grid, initial data: truncated polynom.
- Exponential time decay for all α

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Semi-discrete multistep scheme

Equation: $\partial_t u + A(u) = 0,\ t > 0,\ u(0) = u_0$

"Energy" method: Let A satisfy $\langle A(u), u\rangle \geq 0$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \langle \partial_t u, u\rangle = -\langle A(u), u\rangle \leq 0$$

"Entropy" method: Let $\langle A(u), H'(u)\rangle \geq 0$

$$\frac{1}{2}\frac{dH}{dt}[u] = \langle \partial_t u, H'(u)\rangle = -\langle A(u), H'(u)\rangle \leq 0$$

$\rightarrow$ entropy method generalizes from quadratic structure

One-leg multistep scheme:

$$\tau^{-1}\rho(E)u^k + A(\sigma(E)u^k) = 0, \quad u^k \approx u(t_k)$$

- Approximation of $\partial_t u(t_k)$: $\frac{1}{\tau}\rho(E)u^k = \frac{1}{\tau}\sum_{j=0}^p \alpha_j u^{k+j}$
- Approximation of $u(t_k)$: $\sigma(E)u^k = \sum_{j=0}^p \beta_j u^{k+j}$

Question: $H[u^k]$ generally not dissipated – what can we do?

Semi-discrete multistep scheme

Discrete "energy" method: Assume Hilbert space structure

$$\tau^{-1}\rho(E)u^k + A(\sigma(E)u^k) = 0$$

$$\rho(E)u^k = \sum_{j=0}^{p} \alpha_j u^{k+j}, \quad \sigma(E)u^k = \sum_{j=0}^{p} \beta_j u^{k+j}$$

- Conditions on (ρ, σ) yield second-order scheme
- Dahlquist 1963: (ρ, σ) A-stable $\Rightarrow p \leq 2$
- Energy dissipation: If (ρ, σ) A-stable then G-stable, i.e., $\exists$ symmetric positive definite matrix (G_{ij}) such that

$$(\rho(E)u^k, \sigma(E)u^k) \geq \tfrac{1}{2}\|U^{k+1}\|_G^2 - \tfrac{1}{2}\|U^k\|_G^2$$

where $U^k = (u^k, \ldots, u^{k+p-1})$, $\|U^k\|_G^2 = \sum_{i,j} G_{ij}(u^{k+i}, u^{k+j})$

Energy dissipation: (Hill 1997)

$$\tfrac{1}{2}\|U^{k+1}\|_G^2 - \tfrac{1}{2}\|U^k\|_G^2 \leq -\tau(A(\sigma(E)u^k), \sigma(E)u^k) \leq 0$$

Semi-discrete multistep scheme

Discrete "entropy" method:

Aim: Develop entropy-dissipative one-leg multistep scheme

Difficulty: Energy dissipation based on quadratic $\frac{1}{2}\|u\|_G^2$

Key idea: Enforce quadratic structure by $v^2 = H(u)$

$$\partial_t u + A(u) = 0 \Rightarrow H(u)^{1/2} H'(u)^{-1} \partial_t v + \frac{1}{2} A(u) = 0$$

Semi-discrete scheme:

$$H(w^k)^{1/2} H'(w^k)^{-1} \rho(E) v^k + \frac{\tau}{2} A(w^k) = 0$$

$$w^k = H^{-1}((\sigma(E) v^k)^2)$$

Let $H(u) = u^\alpha$, $\alpha \geq 1$:

$$\rho(E) v^k + \tau B(\sigma(E) v^k) = 0, \;\; B(v) = \frac{\alpha}{2} v^{1-2/\alpha} A(v^{2/\alpha})$$

- Is the scheme well-posed? Yes, under conditions on A
- Entropy dissipativity & positivity preservation? Yes!
- Numerical convergence order? Maximal order two

Semi-discrete multistep scheme

$$\rho(E)v^k + \tau B(\sigma(E)v^k) = 0, \ B(v) = \tfrac{\alpha}{2} v^{1-2/\alpha} A(v^{2/\alpha})$$

Proposition (Entropy dissipation): Let (ρ, σ) be G-stable. Then $H[V^k] = \frac{1}{2}\|V^k\|_G^2$ with $V^k = (v^k, \ldots, v^{k+p-1})$ is nonincreasing in k. (Recall that $(\sigma(E)v^k)^{2/\alpha} \approx u(t_k)$.)

Proof: By G-stability and assumption on A,

$$H[V^{k+1}] - H[V^k] = \tfrac{1}{2}\|V^{k+1}\|_G^2 - \tfrac{1}{2}\|V^k\|_G^2 \le (\rho(E)v^k, \underbrace{\sigma(E)v^k}_{=(w^k)^{\alpha/2}})$$

$$= \tfrac{\tau}{2}\langle A(w^k), H'(w^k)\rangle \le 0$$

Theorem (Convergence rate): Let (ρ, σ) be G-stable and of second order. Let u be smooth, $B + \kappa\,\mathrm{Id}$ be positive, and $p = 2$. Then, for $\tau > 0$ small, $\|v^k - u(t_k)^{\alpha/2}\| \le C\tau^2$.

Proof: Use idea of Hundsdorfer/Steininger 1991

Population model (Shigesada-Kawasaki-Teramoto 1979)

- Motivation: Models segregation of population species
- Population densities: u_1, u_2, periodic boundary conditions

$$\partial_t u_1 - \mathrm{div}\,((d_1 + a_1 u_1 + u_2)\nabla u_1 + u_1 \nabla u_2) = 0$$

$$\partial_t u_2 - \mathrm{div}\,((d_2 + a_2 u_2 + u_1)\nabla u_2 + u_2 \nabla u_1) = 0$$

Theorem: (A.J.-Milišić, *NMPDE* 2014, to appear)
Let $d \leq 3$, $1 < \alpha < 2$, $4a_1 a_2 \geq \max\{a_1, a_2\} + 1$, (ρ, σ) G-stable. Then $\exists$ solution $(v_1^k, v_2^k, w_1^k, w_2^k) \in W^{1,3/2}(\mathbb{T}^d)$ such that w_j^k, $\sigma(E)v_j^k \geq 0$ and

$$H[V^{k+1}] + \frac{2\tau}{\alpha^2}(\alpha - 1)\int_{\mathbb{T}^d} \sum_{j=1}^{2} d_j |\nabla (w_j^k)^{\alpha/2}|^2 dx \leq H[V^k]$$

$\rightarrow$ Scheme is nonnegativity-preserving and entropy-dissipative

Quantum diffusion model

$$\partial_t u + \nabla^2 : (u\nabla^2 \log u) = 0 \text{ in } \mathbb{T}^d, \quad u(0) = u_0$$

Theorem: (A.J.-Milišić, *NMPDE* 2014, to appear)
Let $d \leq 3$, $1 < \alpha < \frac{(\sqrt{d}+1)^2}{d+2}$, (ρ, σ) G-stable. Then $\exists$ solution (v^k, w^k) with $(w^k)^{\alpha/2} \in H^2(\mathbb{T}^d)$, $w^k, \rho(E)v^k \geq 0$ to

$$\frac{2}{\alpha\tau}(w^k)^{1-\alpha/2-1}\rho(E)v^k + \nabla^2 : (w^k\nabla^2 \log w^k) = 0 \quad \text{in } \mathbb{T}^d$$

satisfying discrete entropy inequality

$$H[V^{k+1}] + \frac{\alpha\tau}{2}\kappa_\alpha \int_{\mathbb{T}^d} (\Delta(w^k)^{\alpha/2})^2 dx \leq H[V^k]$$

$\rightarrow$ Scheme is positivity-preserving and entropy-dissipative

Conclusion: Method works well if u^α and $u \log u$ are entropies and entropy dissipation gives Sobolev estimates

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- Introduction
- Implicit Euler finite-volume scheme
- Semi-discrete one-leg multistep scheme
- Semi-discrete Runge-Kutta scheme

Semi-discrete Runge-Kutta scheme

Equation: $\partial_t u + A(u) = 0,\ t > 0,\ u(0) = u_0$

Entropy: $H[u] = \int_\Omega u^\alpha dx$

Discretization:

$$u^{k+1} = u^k + \tau \sum_{i=1}^{s} b_i K_i, \quad K_i = A\Big(u^i + \tau \sum_{j=1}^{s} a_{ij} K_j\Big)$$

Objective: Show that

$$H[u^{k+1}] - H[u^k] \le -\tau\alpha \int_\Omega (u^{k+1})^{\alpha-1} A(u^{k+1}) dx \le 0$$

Idea: Fix $u := u^{k+1}$, interpret $u^k = v(\tau)$

- Define $G(\tau) = H[u] - H[v(\tau)]$ and take $\tau > 0$ small:

$$G(\tau) = \underbrace{G(0)}_{=0} + \tau G'(0) + \frac{\tau^2}{2}\Big(\underbrace{G''(0)}_{<0} + \frac{\tau}{3}\underbrace{G'''(\xi)}_{<\infty}\Big) \le \tau G'(0)$$
$$= -\tau\alpha \int_\Omega u^{\alpha-1} A(u) dx$$

- To show: $G'(0) \le 0,\ G''(0) < 0,\ G'''(\xi) < \infty$

Semi-discrete Runge-Kutta scheme

$$G'(0) = -\tau\alpha \int_\Omega u^{\alpha-1} A(u)dx, \quad A \text{ diff. operator order } p$$

$$G''(0) = -\alpha \int_\Omega u^{\alpha-2}\big[uDA(u)(A(u)) - (\alpha-1)(A(u))^2\big]dx$$

- $G'(0)$ involves derivatives of order p
- $G''(0)$ involves derivatives of order $2p$

Key idea: Systematic integration by parts (A.J.-Matthes 2006)

Example: $\partial_t u = \Delta(u^\beta)$ in $\mathbb{T}^d$

$$G'(0) = \alpha \int_{\mathbb{T}^d} u^{\alpha-1}\Delta(u^\beta)dx$$

$$= -\alpha(\alpha-1)\beta \int_{\mathbb{T}^d} u^{\alpha+\beta-3}|\nabla u|^2 dx \le 0$$

$$G''(0) = -\alpha\int_{\mathbb{T}^d} \big[\beta u^{\beta-1}\Delta(u^{\alpha-1})\Delta(u^\beta)+(\alpha-1)u^{\alpha-1}(\Delta(u^\beta))^2\big]dx$$

Semi-discrete Runge-Kutta scheme

$$G''(0) = -\alpha \int_{\mathbb{T}^d} \left[\beta u^{\beta-1}\Delta(u^{\alpha-1})\Delta(u^\beta) + (\alpha-1)u^{\alpha-1}(\Delta(u^\beta))^2\right] dx$$

Systematic integration by parts:

- Formulate $G''(0)$ in terms of $\xi_G = \frac{|\nabla u|}{u}$, $\xi_L = \frac{\Delta u}{u}$:

$$G''(0) = \int_{\mathbb{T}^d} u^{\alpha+2\beta-2} P(\xi_G, \xi_L) dx, \;\; P(\xi_G, \xi_L) \text{ polynomial}$$

- Interpret integrations by parts as polynomial manipulation:

$$0 = \int_{\mathbb{T}^d} \operatorname{div}\left(u^{\alpha+2\beta-2}|\nabla u|^2 \nabla u\right) dx = \int_{\mathbb{T}^d} u^{\alpha+2\beta-2} T_1(\xi_G, \xi_L) dx$$

$$0 = \int_{\mathbb{T}^d} \operatorname{div}\left(u^{\alpha+2\beta-2}(D^2 u - \Delta u \mathbb{I})\nabla u\right) dx = \int_{\mathbb{T}^d} u^{\alpha+2\beta-2} T_2 dx$$

- Find c_1, $c_2 \in \mathbb{R}$ such that for all ξ_G, $\xi_L \in \mathbb{R}$:

$$P(\xi_G, \xi_L) = P(\xi_G, \xi_L) + c_1 T_1(\xi_G, \xi_L) + c_2 T_2(\xi_G, \xi_L) > 0$$

Semi-discrete Runge-Kutta scheme

Polynomial decision problem: Solve by quantifier elimination

$$\exists c_1, c_2 : \forall \xi_G, \xi_L : P(\xi_G, \xi_L) = (P + c_1 T_1 + c_2 T_2)(\xi_G, \xi_L) > 0$$

Tarski 1930: Such quantified statements can be reduced to a quantifier-free statement in an algorithmic way.

+ Implementations in `Mathematica`, `QEPCAD` available
+ Gives complete, exact answer and proof
− Algorithms are doubly exponential in no. of ξ_i, c_i

Consequence: $G''(0) = \int_{\mathbb{T}^d} u^{2\alpha+\beta-2} P dx < 0$

$$\Rightarrow G(\tau) = G(0) + \tau G(0) + \frac{1}{2}\tau^2 G''(0) + \frac{1}{6}\tau^3 G'''(\xi) \leq \tau G'(0)$$

$$G(\tau) = H[u^{k+1}] - H[u^k] \leq -\tau G'(0)$$

$$= -\alpha(\alpha - 1)\beta\tau \int_{\mathbb{T}^d} (u^{k+1})^{\alpha+\beta-3} |\nabla u^{k+1}|^2 dx \leq 0$$

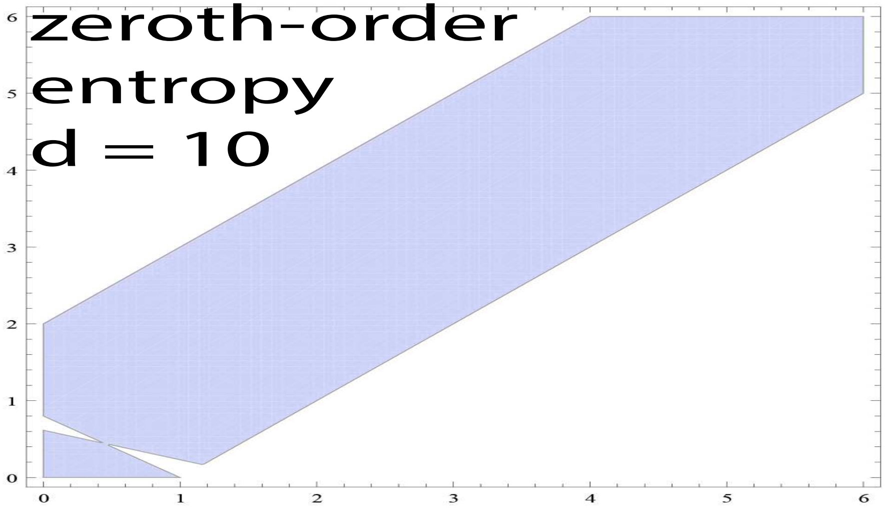
Semi-discrete Runge-Kutta scheme

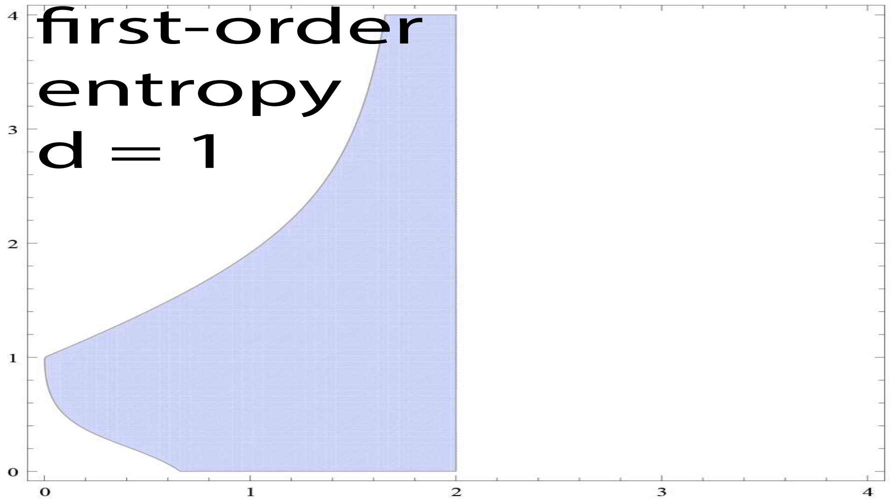
$$\partial_t u = \Delta(u^\beta) \quad \text{in } \mathbb{T}^d,\ t > 0, \quad u(0) = u_0$$

Theorem: (A.J.-Schuchnigg, 2014)
There exists a parameter range for (α, β) such that any implicit Runge-Kutta schema dissipates the entropy:

$$H[u^{k+1}] = \int_{\mathbb{T}^d} (u^{k+1})^\alpha dx \le H[u^k], \quad \tau > 0 \text{ small}$$

$$F[u^{k+1}] = \frac{1}{2}\int_{\mathbb{T}} |(u^{\alpha/2})_x|^2 dx \le F[u^k], \quad \text{only 1-D}$$





Intermediate summary

Implicit Euler finite-volume scheme:

- Discrete exponential/algebraic equilibration rates
- Tools: Compute discrete $\frac{dH}{dt}$, use discrete Beckner ineq.

Higher-order time schemes:

- One-leg multistep: G-norm is dissipative, up to order two
- Tools: Enforce quadratic structure, G-stability theory
- Implicit Runge-Kutta: Discrete entropy is dissipative
- Tools: Taylor expansion, systematic integration by parts

Questions:

- What about spatially discrete Bakry-Emery approach?
- What about higher-order minimizing movement scheme?

Extensions

Question: What about discrete Bakry-Emery?

Mielke 2013: Geodesic λ-convexity for discrete Fokker-Planck

$$\partial_t u = \mathrm{div}\,(\nabla u + u\nabla\phi) = \mathrm{div}\,\left(u_\infty \frac{u}{u_\infty}\nabla \log\frac{u}{u_\infty}\right),\ \ u_\infty = ce^{-\phi}$$

Discrete entropy: $H[u] = \sum_i u_i \log\frac{u_i}{u_{\infty,i}}$

Finite-volume discretization: uniform 1-D mesh with size $\triangle x$

$$\partial_t u = S^\top L S \log\frac{u}{u_\infty},\ \ L = \mathrm{diag}(L_i)$$

- $S^\top$, S: discrete divergence, gradient respectively
- $L_i = \sqrt{u_{\infty,i}u_{\infty,i+1}}\Lambda(\frac{u_i}{u_{\infty,i}}, \frac{u_{i+1}}{u_{\infty,i+1}})$ and $\Lambda(a,b) = \frac{a-b}{\log a - \log b}$

Theorem: If $\frac{1}{(\triangle x)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \geq 2\kappa > 0$ then

$$\frac{d^2H}{dt^2} \geq -\frac{4}{(\triangle x)^2}(1 - e^{-\kappa(\triangle x)^2})\frac{dH}{dt} \ \Rightarrow\ \text{exp. decay}$$

$\rightarrow$ Asymptotically sharp rate. Extension to nonlinear eqs.?

Extensions

Question: Higher-order minimizing movement scheme?

Restrict first to Hilbert space setting: $\partial_t u = \nabla\phi(u)$

First-order minimizing movement scheme:

$$u^k - u^{k-1} = \tau\nabla\phi(u^k),\ u^k = \operatorname*{arg\,min}_{v\in H}\left(\tfrac{1}{2\tau}\|u^{k-1} - v\|^2 + \phi(v)\right)$$

$\to$ Gradient flow in the L^2-Wasserstein distance
(Ambrosio, Otto, Savaré,...)

Higher-order minimizing movement scheme: one-leg scheme

$$\rho(E)u^k = \tau\nabla\phi(\sigma(E)u^k),\ w = \operatorname*{arg\,min}_{v\in H}\left(\tfrac{1}{2\tau}\|\eta + v\|^2 + \phi(v)\right)$$

$$\eta = \sum_{j=0}^{p-1}\left(\frac{\alpha_j\beta_p}{\alpha_p} - \beta_j\right)u^{k+j},\ u^{k+p} = \beta_k^{-1}\left(w - \sum_{j=0}^{p-1}\beta_j u^{k+j}\right)$$

$\to$ Discrete entropy in G-norm is dissipated (A.J.-Fuchs 2014)
Extensions: differ. inclusions, metric/Wasserstein spaces?

Summary

✔ Implicit Euler finite-volume scheme:

Exponential/algebraic decay of discrete entropy $H[u^k]$

(Tool: Discrete generalized Beckner inequality)

✔ Higher-order one-leg multistep scheme:

Discrete entropy in G-norm dissipated

(Tool: G-stability theory of Dahlquist)

✔ Higher-order Runge-Kutta scheme:

Discrete entropy $H[u^k]$ dissipated

(Tool: Systematic integration by parts)

✘ Discrete Bakry-Emery method:

Exponential entropy decay & discrete convex Sobolev ineq.

✘ Higher-order minimizing movement schemes:

Discrete entropy in G-norm is dissipated