Introduction to Mechanism Design

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Mechanism Design

- A designer would like to make a collective decision according to agents' true preferences.
  - self-interested agents privately know their preferences.
  - when and how can the designer do it?
- Examples
  - monopolistic screening
  - design of auctions
  - optimal taxation
  - provision of public goods
  - design of voting procedures and constitution

Example: Single Object Allocation

- Designer wants to allocate one object among I buyers.
  - the designer's reservation value is normalized to be 0.
- Symmetric independent private values (SIPV)
  - buyers' "types" $\{\theta_i\}$ are independently drawn from $U[0, 1]$.
  - buyers' valuations for the object depend only on their own type.
- The designer wishes to "implement" the "efficient" allocation
  - efficient allocation: assign object to the bidder who values most.
  - how to do it?
- What if the designer wishes to maximize the revenue?

First-Price Sealed-Bid Auction

- "Mechanism"
  - each bidder i submits a bid m_i in a sealed envelope
  - bidder with the highest bid wins the object and pays his bid
- Observation
  - the mechanism specifies winner and payment given bid profile;
  - it "induces" a game where bidders' "strategies" are bids m_i;
  - payoff for bidder i: $\theta_i - m_i$ if winning, and 0 otherwise.
- Question: can it implement the efficient allocation?

Alternative Mechanism

- Second-price sealed bid auction
  - each bidder i submits a bid m_i in a sealed envelope
  - bidder with the highest bid wins the object but pays the second highest bid
- Questions:
  - can it implement the efficient allocation?
  - how does it compare to FPA: revenue, bidder payoff, etc.?
  - how should a revenue-maximizing designer adjust the auction mechanism?

Outline

- **Introduction to Bayesian games and mechanism design**
  - revelation principle
  - Gibbard-Satterthwaite impossibility theorem
- Quasilinear; uni-dimensional, independent, private types
- Quasilinear; multidimensional, independent, private types
- Nontransferrable utilities: single-peaked preferences

Bayesian Game

- Players: $i \in \mathcal{I} = \{1, ..., I\}$
- Types (players' private information): $\theta_i \in \Theta_i$
- Joint distribution of types (common prior and beliefs): $\Phi(\theta)$
- Strategies/messages $m_i : \Theta_i \to M_i$
- Preference over strategy profiles: $\widetilde{u}_i(m, \theta_i, \theta_{-i})$
- In mechanism design context (mechanism: (M,g))
  - outcome functions $g : M_1 \times \cdots \times M_2 \to Y$ (alternatives)
  - preference over Y: $u_i(y, \theta_i, \theta_{-i}) = u_i(g(m), \theta_i, \theta_{-i}) \equiv \widetilde{u}_i(m, \theta_i, \theta_{-i})$
- Bayesian game (with common prior): $[\mathcal{I}, \{M_i\}, \{\widetilde{u}_i\}, \{\Theta_i\}, \Phi(\cdot)]$

Equilibrium Concept

Definition

A strategy profile $(m_1^*(\cdot), ..., m_I^*(\cdot))$ is a dominant strategy equilibrium if, $\forall i, \forall \theta_i, \forall m_i \in M_i, \forall m_{-i} \in M_{-i}$,

$$\widetilde{u}_i(m_i^*(\theta_i), m_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \geq \widetilde{u}_i(m_i, m_{-i}(\theta_{-i}), \theta_i, \theta_{-i}).$$

Definition

A strategy profile $(m_1^*(\cdot), ..., m_I^*(\cdot))$ is a Bayesian Nash equilibrium if, $\forall i, \forall \theta_i, \forall m_i \in M_i$,

$$E_{\theta_{-i}}\left[\widetilde{u}_i\left(m_i^*(\theta_i), m_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}\right)\right] \geq E_{\theta_{-i}}\left[\widetilde{u}_i\left(m_i, m_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}\right)\right].$$

Mechanism Design Problem

- Consider a setting with I agents, $\mathcal{I} = \{1, ..., I\}$.
- The designer/principal must make a collective choice among a set of possible allocations Y.
- Each agent privately observes a signal (his type) $\theta_i \in \Theta_i$ that determines his preferences over Y, described by a utility function $u_i(y, \theta_i)$ for all $i \in \mathcal{I}$.
  - common prior: the prior distribution $\Phi(\theta)$ is common knowledge.
  - private values: utility depends only own type (and allocation).
  - type space: $\Theta = \Theta_1 \times ... \times \Theta_I$.
- A social choice function is a mapping $f : \Theta \rightarrow Y$.

Messages and Outcome Function

- Private information
  - information $\theta = (\theta_1, .., \theta_I)$ is dispersed among agents when the allocation y is to be decided.
  - notation: $\theta = (\theta_i, \theta_{-i})$, with $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_I)$.
- Messages
  - each agent can send a message $m_i : \Theta_i \to M_i$.
  - agents send their messages independently and simultaneously.
  - the message space M can be arbitrary: $M = M_1 \times ... \times M_I$.
- Outcome function is a mapping $g : M \to Y$.
  - after the agents transmit a message $m \in M$, a social allocation $y \in Y$ will be chosen according to g.

Mechanism and Implementation

Definition

A mechanism $\Gamma = (M_1, .., M_I, g(\cdot))$ is a collection of strategy sets $(M_1, .., M_I)$ and an outcome function $g : M \to Y$.

- A mechanism Γ, together with a type space Θ, a (joint) probability distribution $\Phi(\theta)$, and Bernoulli utility functions $(u_1(\cdot), ..., u_I(\cdot))$ induces a game with incomplete information where the strategy for agent i is a function $m_i : \Theta_i \to M_i$.

Definition

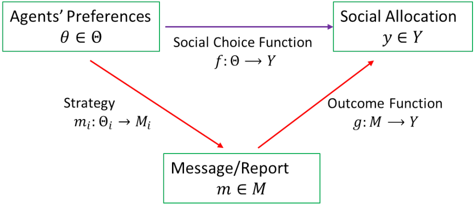
A mechanism $\Gamma = (M_1, .., M_I, g(\cdot))$ implements the social choice function $f(\cdot)$ if there is an equilibrium profile $(m_1^*(\theta_1), ..., m_I^*(\theta_I))$ of the game induced by Γ such that

$$g(m_1^*(\theta_1), ..., m_I^*(\theta_I)) = f(\theta_1, ..., \theta_I).$$

Partial vs. Full Implementation

- Partial/weak implementation (our focus)
  - a social choice function is partially implementable if it arises in an equilibrium where all agents report their information truthfully.
- Full/Maskin implemenation
  - a social choice function is fully implementable if it arises in every equilibria where all agents report their information truthfully.

Mechanism Design as Reverse Engineering



- Social choice problem:
  - map agents' preference profiles into allocations.
- Implementation (or mechanism design) problem:
  - designer announces an outcome function mapping the agents' messages into allocations.
  - the outcome function induces a Bayesian game.
  - agents choose messages to reflect their preferences and to influence outcome.

Key Elements

- The objective of the designer
  - if it is welfare maximization: efficient mechanisms
  - if it is revenue maximization: optimal mechanisms
- Incentive constraints
  - the designer must give agents incentives to truthfully report their private information.
  - incentive provision is often costly, leading to inefficient allocation.
- Constrained maximization problem with two classes of constraints
  - the “participation” or “individual rationality” constraint
  - the “incentive compatibility” constraint

“Timing” of Mechanism Design Problem

Mechanism design as a three-step game of incomplete information

1. Principal announces and commits to a “mechanism” or “contract”.
2. Agents simultaneously decide whether to accept or reject.
3. Agents who accept play the game “induced” by the mechanism.
   - agents who reject get some exogenous “reservation utility”.

FPA vs. SPA

- Suppose there are two bidders, θ_1 and θ_2.
- Seller has cost 0, and $\theta_1, \theta_2 \sim U[0,1]$.
- The seller sets zero reserve price:

	First-price auction	Second-price auction
Eqm bidding	$\theta_i/2$	θ_i
Mechanism	indirect	direct
Solution concept	Bayesian	dominant strategy
Efficient?	yes	yes
Revenue	$1/3$	$1/3$

  - revenue-maximizing seller would set reserve $r = 1/2$.
  - both auction mechanisms would generate revenue $5/12$.

Dominant Strategy and Bayesian Implementation

Definition

The mechanism $\Gamma = (M, g(\cdot))$ implements the social choice function $f(\cdot)$ in dominant strategies if there exists a dominant strategy equilibrium of Γ, $m^*(\cdot) = (m_1^*(\cdot), ..., m_I^*(\cdot))$, such that $g(m^*(\theta)) = f(\theta)$ for all θ.

Definition

The mechanism $\Gamma = (M, g(\cdot))$ implements the social choice function $f(\cdot)$ in Bayesian strategies if there exists a Bayesian Nash equilibrium of Γ, $m^*(\cdot) = (m_1^*(\cdot), ..., m_I^*(\cdot))$, such that $g(m^*(\theta)) = f(\theta)$ for all θ.

Direct Revelation Mechanism

Definition

A direct revelation mechanism $\Gamma = (\Theta, f)$ is a mechanism in which $M_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all θ.

Definition

The social choice function $f(\cdot)$ is truthfully implementable (or incentive compatible*)* if the direct revelation mechanism $\Gamma = (\Theta, f(\cdot))$ has an equilibrium $(m_1^*(\theta_1), ..., m_I^*(\theta_I))$ in which $m_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$, for all i.

Revelation Principle

- Identification of implementable social choice function is complex
  - difficult to consider all possible mechanism $g(\cdot)$ on all possible domains of strategies M.
  - a celebrated result, the revelation principle, simplifies the task.

Theorem

Let $\Gamma = \{M, g(\cdot)\}$ be a mechanism that implements the social choice function $f(\cdot)$ in dominant strategies. Then $f(\cdot)$ is truthfully implementable in dominant strategies.

- Remark
  - valid also for implementation in Bayesian strategies.
  - sufficient to restrict attention to "direct revelation mechanisms."

Example of Direct Mechanism: Second-Price Auction

- One indivisible object, two agents with valuations θ_i, $i = 1, 2$.
- Quasi-linear preferences: $u_i\left(y_i, \theta_i\right) = \theta_i x_i + t_i$.
- An outcome (alternative) is a vector $y = \left(x_1, x_2, t_1, t_2\right)$
  - $x_i = 1$ if agent i gets the object, 0 otherwise;
  - t_i is the monetary transfer received by agent i;
  - hence, the set of alternatives is $Y = X \times T$.
- Direct mechanism $\Gamma = (M, g)$:
  - message space: $M_i = \Theta_i$,
  - outcome function $g : M \to Y$ with

$$g\left(m_1, m_2\right) = \begin{cases} x_1 = 1, x_2 = 0;\ t_1 = -m_2, t_2 = 0, & \text{if} \quad m_1 \geq m_2 \\ x_1 = 0, x_2 = 1;\ t_1 = 0, t_2 = -m_1, & \text{if} \quad m_1 < m_2 \end{cases}$$

  - it implements the efficient allocation in dominant strategies.

Dominant Strategy Implementation

- Dominant strategy implementation implements social choice function in a very robust way:
  - very weak informational requirement
  - independent of players' beliefs
  - the designer doesn't need to know $\Phi(\cdot)$ for implementation.
- But can we always implement in dominant strategies?
  - the answer is "no" in general.

Gibbard-Satterthwaite Impossibility Theorem

Definition

The social choice function $f(\cdot)$ is dictatorial if there is an agent i such that for all $\theta \in \Theta$,

$$f(\theta) \in \{z \in Y : u_i(z, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in Y\}.$$

Theorem

Suppose that Y contains at least three elements, preferences are rich (containing all possible rational preferences), and $f(\Theta) = Y$. Then f is truthfully implementable in dominant strategies if, and only if, it is dictatorial.

Outline

- Introduction to Bayesian games and mechanism design
- **Quasilinear; uni-dimensional, independent, private types**
  - efficient mechanisms: VCG mechanism, Roberts' theorem
  - optimal mechanisms: Myerson optimal auction
  - equivalence between Bayesian and dominant strategy implementation
- Quasilinear; multidimensional, independent, private types
- Nontransferrable utilities: single-peaked preferences

Quasilinear Environment

- How to get around this impossibility theorem?
  - relax the dominant strategy requirement
  - focus on restricted domain of preferences:
    1. quasilinear preferences
    2. single-peaked preferences
- Quasilinear preferences: $u_i(x, \theta_i) = v_i(x, \theta_i) + t_i$.
  - social choice function: $f(\cdot) = (x(\cdot), t_1(\cdot), ..., t_I(\cdot))$, with allocation $x(\theta) \in X$ and transfer $t_i \in T_i$.
  - set of social allocations $Y = X \times T$.
  - an allocation $x^*(\theta)$ is ex-post efficient if

$$\sum_{j=1}^{I} v_j(x^*(\theta), \theta_j) \geq \sum_{j=1}^{I} v_j(x, \theta_j) \text{ for all } x \in X.$$

VCG Mechanism

Theorem (Vickrey-Clarke-Groves)

The social choice function $f(\cdot) = (x^*(\cdot), t_1(\cdot), ..., t_I(\cdot))$ *is truthfully implementable in dominant strategies if, for all* $i = 1, ..., I$,

$$t_i(\theta_i, \theta_{-i}) = \left[\sum_{j \neq i} v_j(x^*(\theta_i, \theta_{-i}), \theta_j)\right] - \left[\sum_{j \neq i} v_j\left(x^*_{-i}(\theta_{-i}), \theta_j\right)\right].$$

- Remarks:
  - agent i is pivotal iff $x^*(\widehat{\theta}_i, \theta_{-i}) \neq x^*_{-i}(\theta_{-i})$.
  - agent i pays only when pivotal: pivotal mechanism.
  - agent i payoff in a pivotal mechanism equals his marginal contribution to social surplus:

$$\sum_j v_j(x^*(\theta_i, \theta_{-i}), \theta_j) - \sum_{j \neq i} v_j\left(x^*_{-i}(\theta_{-i}), \theta_j\right).$$

Proof

- Suppose truth-telling is not a dominant strategy for some agent i.
- Then there exist $\theta_i, \widehat{\theta}_i$, and θ_{-i} such that

$$v_i(x^*(\widehat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\widehat{\theta}_i, \theta_{-i}) > v_i\left(x^*\left(\theta_i, \theta_{-i}\right), \theta_i\right) + t_i\left(\theta_i, \theta_{-i}\right)$$

- Substituting $t_i(\widehat{\theta}_i, \theta_{-i})$ and $t_i\left(\theta_i, \theta_{-i}\right)$ yields

$$\sum_{j=1}^{I} v_j(x^*(\widehat{\theta}_i, \theta_{-i}), \theta_j) > \sum_{j=1}^{I} v_j\left(x^*\left(\theta\right), \theta_j\right),$$

  which contradicts $x^*\left(\cdot\right)$ being an optimal policy.
- Thus, $f\left(\cdot\right)$ must be truthfully implementable in dominant strategies.

Form of VCG Mechanisms

- Vickrey auctions (second-price sealed-bid auctions)
  - $t_i(\theta_i, \theta_{-i}) = 0$ if $x_i(\theta_i, \theta_{-i}) = 0$, and
  - $t_i(\theta_i, \theta_{-i}) = -\max_{j \neq i} v_j(x, \theta_j)$ if $x_i(\theta_i, \theta_{-i}) = 1$.
  - a special case of VCG mechanism
- More general form of VCG mechanism
  - set the transfer function $\widetilde{t}_i(\theta_i, \theta_{-i})$ as

$$\widetilde{t}_i(\theta_i, \theta_{-i}) = t_i(\theta_i, \theta_{-i}) + h_i(\theta_{-i})$$

where $h_i(\theta_{-i})$ some functions does not depend on θ_i.

Uniqueness of VCG Mechanism

Theorem (Green and Laffont, 1977)

Suppose that for each i, $\Theta_i = \left[\underline{\theta}_i, \overline{\theta}_i\right]$, *or that* Θ_i *is smoothly path connected. That is, for each two points* $\theta, \theta' \in \Theta$, *there is a differentiable function* $f : [0, 1] \to \Theta$ *such that* $f(0) = \theta$ *and* $f(1) = \theta'$. *In addition, for each decision outcome* x, $v_i(x, \theta_i)$ *is differentiable in its second argument. Then any efficient, dominant strategy incentive compatible direct mechanism is a VCG mechanism.*

Roberts' Theorem

Theorem (Roberts, 1979)

Let $v_i(x) \in V_i$ *denote agent* i*'s resulting value if alternative* x *is chosen, where* V_i *is the space of all possible types of agent* i*. Suppose the set of allocation* X *is finite,* $|X| \geq 3$*, and the domain of preferences is unrestricted with* $V = \mathbb{R}^{|X|}$*. Then, for every DIC allocation rule* $x : V \rightarrow X$*, there exist non-negative weights* $k_1, ..., k_I$*, not all of them equal to zero, and a deterministic real-valued function* $C : X \rightarrow \mathbb{R}$ *such that, for all* $v \in V$*,*

$$x(v) \in \arg\max_{x \in X} \left\{ \sum_{i=1}^{I} k_i v_i(x) + C(x) \right\}.$$

Remark

- If $x(v)$ is DIC, then

$$x(v) \in \arg\max_{x \in X} \left\{ \sum_{i=1}^{I} k_i v_i(x) + C(x) \right\}.$$

  - quasilinear preferences, but possibly multi-dimensional types.
- Every DIC allocation rule must be weighted VCG.
- Relation to Gibbard-Satterthwaite Theorem:
  - suppose transfers are not allowed.
  - with unrestricted domain, if $k_i > 0$, agent i can misreport some v_i such that $v_i(x) - v_i(y)$ for all $y \neq x$ is suitably large, so that agent i can ensure that any alternative x is chosen; thus, if $k_i > 0$, we must have $v_i(x(v)) \geq v_i(y)$ for all y.
  - similarly, if $k_j > 0$, $j \neq i$, it must be $v_j(x(v)) \geq v_j(y)$ for all y.
  - but by suitable choice of v, this is not always possible, so only one $k_i > 0$, i.e., dictatorship.

Bayesian (Efficient) Implementation

- Implementation in dominant strategies often too demanding.
  - VCG is ex post efficient, but
  - it generally does not satisfy budget balance.
- Under a weaker solution concept of Bayesian Nash equilibrium, we can implement ex post efficient outcome with budget balance
  - expected externality mechanism or AGV mechanisms
  - d'Aspremont and Gerard-Varet (1979), and Arrow (1979).
- Myerson-Satterthwaite impossibility theorem
  - no efficient mechanism satisfies interim IR, IC and BB.

Optimal Auction Design

- Auction design problem:
  - how to sell an object to I potential bidders to maximize revenue?
- We follow a two-step procedure to characterize optimal mechanisms:
  - first characterize the implementable mechanisms,
  - then find the one that maximizes the seller's revenue.
- As a by-product, we also prove the revenue equivalence theorem.

Setup

- A seller wants to sell an indivisible object to one of I buyers.
- Independent private values, one-dimensional types
  - the value of the object to individual i is θ_i,
  - θ_i is randomly drawn from commonly known distribution F_i with support $\left[\underline{\theta}_i, \overline{\theta}_i\right]$,
  - types are assumed to be statistically independent.
- The seller's reservation value for the object is normalized to 0.

Direct Revelation Mechanisms

- By the revelation principle, we can focus on direct mechanisms.
- A direct mechanism consists of a pair of functions:
  - allocation rule $x_i(\theta)$: the probability of agent i getting the object
    - $x_i = 0$ if agent i does not get the object,
    - $x_i = 1$ if agent i gets the object.
  - payment rule $t_i(\theta)$: the monetary transfer from agent i.

IC and IR Constraints

- Given the selling mechanism $(x(\cdot), t(\cdot))$, a type-θ_i bidder's expected payoff by reporting $\hat{\theta}_i$ is

$$\mathbb{E}_{\theta_{-i}}\left[u_i(\hat{\theta}_i, \theta_i; \theta_{-i})\right] = \theta_i \mathbb{E}_{\theta_{-i}}\left[x_i(\hat{\theta}_i, \theta_{-i})\right] - \mathbb{E}_{\theta_{-i}}\left[t_i(\hat{\theta}_i, \theta_{-i})\right].$$

- Feasible mechanisms
  - individually rational:

$$\mathbb{E}_{\theta_{-i}}[u_i(\theta_i, \theta_i; \theta_{-i})] \geq 0 \text{ for all } \theta_i \tag{IR}$$

  - incentive compatible:

$$\theta_i \in \arg\max_{\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]} \mathbb{E}_{\theta_{-i}}\left[u_i(\hat{\theta}_i, \theta_i; \theta_{-i})\right] \text{ for all } \theta_i \tag{IC}$$

Envelope Condition

- Define bidder i's expected utility with truth-telling as

$$\begin{aligned} U_i(\theta_i) &\equiv \mathbb{E}_{\theta_{-i}}\left[u_i(\theta_i, \theta_i; \theta_{-i})\right] \\ &= \max_{\hat{\theta}_i} \mathbb{E}_{\theta_{-i}}\left[u_i(\hat{\theta}_i, \theta_i; \theta_{-i})\right] \\ &= \max_{\hat{\theta}_i} \mathbb{E}_{\theta_{-i}}\left[\theta_i x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i})\right]. \end{aligned}$$

- The envelope theorem implies

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \mathbb{E}_{\theta_{-i}} \int_{\underline{\theta}_i}^{\theta_i} x_i(s, \theta_{-i})\, ds.$$

Characterization of IC Constraints

Theorem (Myerson 1981)

A selling mechanism $(x(\theta), t(\theta))$ *is Bayesian incentive compatible (BIC) iff, for all* i *and* θ_i*, (i)* $\mathbb{E}_{\theta_{-i}}\left[x_i\left(\theta_i, \theta_{-i}\right)\right]$ *is nondecreasing in* θ_i*, and (ii)* $U_i\left(\theta_i\right) = U_i\left(\underline{\theta}_i\right) + \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[x_i\left(s, \theta_{-i}\right)\right] ds.$

Theorem (Maskin and Laffont, 1979)

A selling mechanism $(x(\theta), t(\theta))$ *is dominant strategy incentive compatible (DIC) iff, for all* i*, and for all* θ*, (i)* $x_i\left(\theta_i, \theta_{-i}\right)$ *is nondecreasing in* θ_i*, and (ii)* $u_i\left(\theta_i, \theta_i; \theta_{-i}\right) = u_i\left(\underline{\theta}_i, \underline{\theta}_i; \theta_{-i}\right) + \int_{\underline{\theta}_i}^{\theta_i} x_i\left(s, \theta_{-i}\right) ds.$

- Remark: we also say allocation rule $x(\theta)$ is BIC (DIC) if there exists a transfer $t(\theta)$ such that (x, t) is BIC (DIC).
- Remark: allocation rule $x(\theta)$ is BIC (DIC) if it is "average" (component-wise) monotone.

Proof of Necessity (BIC)

- IC constraints imply that for $\theta_i > \hat{\theta}_i$,

$$\begin{aligned} \mathbb{E}_{\theta_{-i}}[\theta_i x_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})] &\geq \mathbb{E}_{\theta_{-i}}[\theta_i x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i})] \\ \mathbb{E}_{\theta_{-i}}[\hat{\theta}_i x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i})] &\geq \mathbb{E}_{\theta_{-i}}[\hat{\theta}_i x_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})] \end{aligned}$$

  Add two inequalities together and simplify

$$(\theta_i - \hat{\theta}_i)\mathbb{E}_{\theta_{-i}}\left[x_i(\theta_i, \theta_{-i}) - x_i(\hat{\theta}_i, \theta_{-i})\right] \geq 0.$$

  Thus, $\mathbb{E}_{\theta_{-i}}[x_i(\theta_i, \theta_{-i}) - x_i(\hat{\theta}_i, \theta_{-i})] \geq 0$.
- The FOC condition follows from the envelope theorem.

Proof of Sufficiency (BIC)

- Suppose θ_i wants to pretend $\hat{\theta}_i < \theta_i$.
- By FOC, we have

$$\begin{aligned} U_i(\theta_i) - U_i(\hat{\theta}_i) &= \int_{\hat{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[x_i(s, \theta_{-i})\right] ds \geq \int_{\hat{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[x_i(\hat{\theta}_i, \theta_{-i})\right] ds \\ &= (\theta_i - \hat{\theta}_i)\mathbb{E}_{\theta_{-i}}\left[x_i(\hat{\theta}_i, \theta_{-i})\right] \end{aligned}$$

Hence

$$\begin{aligned} U_i(\theta_i) &\geq U_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)\mathbb{E}_{\theta_{-i}}\left[x_i(\hat{\theta}_i, \theta_{-i})\right] \\ &= \mathbb{E}_{\theta_{-i}}\left[\hat{\theta}_i x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i})\right] \\ &\quad + (\theta_i - \hat{\theta}_i)\mathbb{E}_{\theta_{-i}}\left[x_i(\hat{\theta}_i, \theta_{-i})\right] \\ &= \mathbb{E}_{\theta_{-i}}[\theta_i x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i})] \end{aligned}$$

- The case with $\theta_i < \hat{\theta}_i$ can proved analogously.

From Allocation-Transfers to Allocation-Utilities

- By definition of $U_i(\theta_i)$,

$$
\begin{aligned}
\mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i,\theta_{-i})\right] &= \mathbb{E}_{\theta_{-i}}[\theta_i x_i(\theta_i,\theta_{-i})] - U_i(\theta_i) \\
&= \mathbb{E}_{\theta_{-i}}[\theta_i x_i(\theta_i,\theta_{-i})] - U_i(\underline{\theta}_i) - \mathbb{E}_{\theta_{-i}}\int_{\underline{\theta}_i}^{\theta_i} x_i(s,\theta_{-i})\,ds.
\end{aligned}
$$

- Hence, we can write $\mathbb{E}_\theta\left[t_i(\theta)\right]$ as

$$
\begin{aligned}
& \mathbb{E}_\theta[\theta_i x_i(\theta)] - U_i(\underline{\theta}_i) - \mathbb{E}_{\theta_{-i}}\int_{\underline{\theta}_i}^{\bar{\theta}_i}\left[\int_{\underline{\theta}_i}^{\theta_i} x_i(s,\theta_{-i})\,ds\right] f_i(\theta_i)\,d\theta_i \\
= \; & \mathbb{E}_\theta[\theta_i x_i(\theta)] - U_i(\underline{\theta}_i) - \mathbb{E}_{\theta_{-i}}\int_{\underline{\theta}_i}^{\bar{\theta}_i}(1-F_i(\theta_i))\,x_i(\theta_i,\theta_{-i})\,d\theta_i \\
= \; & \mathbb{E}_\theta\left[\left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right)x_i(\theta)\right] - U_i(\underline{\theta}_i)
\end{aligned}
$$

Reformulating the Seller's Problem

- Thus, the seller's revenue can be written as

$$\Pi = \sum_{i=1}^{I} \mathbb{E}_\theta \left[t_i (\theta) \right] = -\sum_{i=1}^{I} U_i \left(\underline{\theta}_i \right) + \mathbb{E}_\theta \sum_{i=1}^{I} \left[\left(\theta_i - \frac{1 - F_i (\theta_i)}{f_i (\theta_i)} \right) x_i (\theta) \right]$$

- Therefore, the seller's maximization problem is to choose $\{x_i (\theta)\}$ to maximize Π subject to

$$\begin{aligned} \text{IR} &: U_i \left(\underline{\theta}_i \right) \geq 0 \text{ for all } i \\ \text{Monotonicity} &: \mathbb{E}_{\theta_{-i}} \left[x_i (\theta_i, \theta_{-i}) \right] \text{ is nondecreasing in } \theta_i. \end{aligned}$$

Revenue Equivalence Theorem

Theorem

Suppose a pair of BNEs of two different auction procedures are such that, for every buyer i,

1. *buyer i has the same probability of winning the object for each possible realization of $\theta = (\theta_1, ..., \theta_I)$;*
2. *buyer i with type $\underline{\theta}_i$ has the same expected utility.*
   Then these two auctions generate the same revenue.

Solving Optimal Mechanism

- First notice that the optimal selling mechanism should set

$$U_i\left(\underline{\theta}_i\right) = 0.$$

- Second, since there is only one object, the allocation function $x_i(\theta)$ has to satisfy

$$x_i(\theta) \in [0,1] \text{ and } \sum_{i=1}^{I} x_i(\theta) \leq 1.$$

Virtual Surplus Function

- Define the virtue surplus function $J_i(\theta_i)$ as

$$J_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)},$$

- The optimal allocation rule should maximize $\mathbb{E}_\theta \left[\sum_{i=1}^{I} J_i(\theta_i) x_i(\theta)\right]$, subject to

$$x_i(\theta) \in [0,1], \sum_{i=1}^{I} x_i(\theta) \leq 1,$$

$$\mathbb{E}_{\theta_{-i}}\left[x_i(\theta_i, \theta_{-i})\right] \text{ is nondecreasing in } \theta_i$$

Pointwise Maximization

- Since $x_i(\theta)$ is nonnegative and $\sum_{i=1}^{I} x_i(\theta) \leq 1$, we can write

$$\sum_{i=1}^{I} J_i(\theta_i)\, x_i(\theta) = \sum_{i=1}^{I} x_i(\theta)\, J_i(\theta_i) + \left(1 - \sum_{i=1}^{I} x_i(\theta)\right) \cdot 0$$

  which is just a weighted average of $I+1$ numbers:

$$J_1(\theta_1), J_2(\theta_2), \ldots., J_I(\theta_I), 0,$$

  with weights being

$$x_1(\theta), x_2(\theta), \ldots., x_I(\theta), \left(1 - \sum_{i=1}^{I} x_i(\theta)\right).$$

- Optimal allocation (weight):
  - $x_i(\theta) = 0$ if $J_i(\theta_i) < 0$,
  - $x_i(\theta) = 0$ if $J_i(\theta_i) < J_k(\theta_k)$ with $k \neq i$,
  - $x_i(\theta) = 1$ if $J_i(\theta_i) > \max\{0, \max_{k \neq i} J_k(\theta_k)\}$.

Optimal Auction

- The optimal probability for agent i to win the object is

$$x_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \text{if} \quad J_i(\theta_i) > \max\{0, \max_{k \neq i} J_k(\theta_k)\} \\ 0 & \text{otherwise} \end{cases}.$$

  – note that $J_i(\theta_i) = \max\{0, \max_{k \neq i} J_k(\theta_k)\}$ has probability zero.

- If we assume $J_i(\theta_i)$ is nondecreasing in θ_i, then $x_i(\theta_i, \theta_{-i})$ is nondecreasing in θ_i, which in turn implies

$$\mathbb{E}_{\theta_{-i}}[x_i(\theta_i, \theta_{-i})] \text{ is nondecreasing in } \theta_i.$$

  Therefore, above $x_i(\theta_i, \theta_{-i})$ actually solves the original problem.

Symmetric Environment

- Suppose buyers are ex-ante symmetric, i.e., $F_i = F$ for all i.
- Suppose further that F has monotone hazard rate, that is, $f(\theta_i)/[1 - F(\theta_i)]$ is nondecreasing in θ_i.
- As a result $J_i(\theta_i) = J(\theta_i)$ for all i and $J(\theta_i)$ is increasing in θ_i.

Optimal Auction: SPA with Reserve Price

- The optimal selling mechanism sets

$$x_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \text{if} \quad J(\theta_i) > \max\{0, \max_{k \neq i}[J(\theta_k)]\} \\ 0 & \text{otherwise} \end{cases},$$

or equivalently

$$x_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \text{if} \quad \theta_i > \max\{r, \max_{k \neq i} \theta_k\} \\ 0 & \text{otherwise} \end{cases}.$$

- Optimal selling mechanism: SPA with optimal reserve r solves

$$r - [1 - F(r)]/f(r) = 0.$$

- RET: all standard auctions with optimal r are optimal.

Equivalence between Bayesian and Dominant Strategy Implementation

- Revenue (more generally payoff) equivalence theorem
  - first price auction (BIC) $=$ second price auction (DIC)
  - equivalence in terms of allocation and transfers
- Equivalence in terms of interim utility holds more generally.
  - linear utilities, private, uni-dimensional, independent types
  - Gershkov et al. (2013), applying a theorem due to Gutmann et al. (1991)
  - for any BIC mechanism, there exists a DIC mechanism that delivers the same interim utilities for all agents and the same ex ante expected social surplus.

Gutmann et al. (1991)

Theorem

Let $x(\theta_1, \theta_2)$ be measurable on $[0,1]^2$ and such that $0 \leq x(\theta_1, \theta_2) \leq 1$,

$$\xi(\theta_1) = \int_0^1 x(\theta_1, \theta_2)\, d\theta_1 \text{ is nondecreasing in } \theta_1,$$

$$\eta(\theta_2) = \int_0^1 x(\theta_1, \theta_2)\, d\theta_2 \text{ is nondecreasing in } \theta_2.$$

Then there exists $\widehat{x}(\theta_1, \theta_2)$ measurable $[0,1]^2$ satisfying $0 \leq \widehat{x}(\theta_1, \theta_2) \leq 1$, having the same marginals as x, and such that $\widehat{x}(\theta_1, \theta_2)$ is nondecreasing in θ_1 and θ_2 separately.

Recall BIC and DIC Characterization in Auction Setting

Theorem (Myerson 1981)

A selling mechanism $(x(\theta), t(\theta))$ *is Bayesian incentive compatible (BIC) iff, for all* i *and* θ_i*, (i)* $\mathbb{E}_{\theta_{-i}}\left[x_i\left(\theta_i, \theta_{-i}\right)\right]$ *is nondecreasing in* θ_i*, and (ii)* $U_i\left(\theta_i\right) = U_i\left(\underline{\theta}_i\right) + \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[x_i\left(s, \theta_{-i}\right)\right] ds.$

Theorem (Maskin and Laffont, 1979)

A selling mechanism $(x(\theta), t(\theta))$ *is dominant strategy incentive compatible (DIC) iff, for all* i*, and for all* θ*, (i)* $x_i\left(\theta_i, \theta_{-i}\right)$ *is nondecreasing in* θ_i*, and (ii)* $u_i\left(\theta_i, \theta_i; \theta_{-i}\right) = u_i\left(\underline{\theta}_i, \underline{\theta}_i; \theta_{-i}\right) + \int_{\underline{\theta}_i}^{\theta_i} x_i\left(s, \theta_{-i}\right) ds.$

Discrete Version

Theorem

Let (x_{ij}) be $m \times n$ matrix with $0 \leq x_{ij} \leq 1$ having nondecreasing row sums and nondecreasing column sums. Then there exists another $m \times n$ matrix $(\widehat{x}_{ij})$ with $0 \leq \widehat{x}_{ij} \leq 1$, which has exactly the same row sums and column sums as (x_{ij}), such that $\widehat{x}_{ij}$ is nondecreasing in both i and j.

Proof.

- Consider the (unique) $m \times n$ matrix $(\widehat{x}_{ij})$ with $0 \leq \widehat{x}_{ij} \leq 1$, having the same row sum and column sum as (x_{ij}), and minimizing $\sum_{i,j}(\widehat{x}_{ij})^2$.
- Suppose $0 \leq \widehat{x}_{i+1,j} < \widehat{x}_{ij} \leq 1$ for some i, j. Since $\sum_k \widehat{x}_{ik} \leq \sum_k \widehat{x}_{i+1,k}$ (row-sum monotonicity), there exists $1 \leq k \leq n$ for which $0 \leq \widehat{x}_{ik} < \widehat{x}_{i+1,k} \leq 1$.
- Now increase $\widehat{x}_{i+1,j}$ and $\widehat{x}_{ik}$ by ε, and decrease $\widehat{x}_{ij}$ and $\widehat{x}_{i+1,k}$ by ε. We get a new matrix $(\widetilde{x}_{ij})$ with $0 \leq \widehat{x}_{ij} \leq 1$, with the same row sums and column sums, but $\sum_{i,j}(\widetilde{x}_{ij})^2 < \sum_{i,j}(\widehat{x}_{ij})^2$. A contradiction.

Example

- Symmetric single-unit auction, two bidders, two equally-likely types, $\underline{\theta}$ and $\overline{\theta}$.
  - allocation rule can be represented by a 2×2 matrix.
- Consider the BIC but not DIC allocation rule:

$$x(\theta_1, \theta_2) = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$$

  - rows = agent 1's type, columns = agent 2's type.
  - entries = probabilities that the object is assigned to either agent.
- Family of allocation rules with the same marginals $(0 \leq \varepsilon \leq 1)$:

$$x_\varepsilon(\theta_1, \theta_2) = \begin{pmatrix} 1/2 - \varepsilon & 1/4 + \varepsilon \\ 1/4 + \varepsilon & 1/2 - \varepsilon \end{pmatrix} \Longrightarrow \widehat{x}(\theta_1, \theta_2) = \begin{pmatrix} 3/8 & 3/8 \\ 3/8 & 3/8 \end{pmatrix}.$$

  - minimizing the sum of squared entries of $x_\varepsilon(\theta_1, \theta_2)$ yields $\varepsilon = 1/8$.
  - $\widehat{x}(\theta_1, \theta_2)$ is everywhere non-decreasing, so DIC.

Gershkov et al. (2013)

- Consider the following general social choice environment
  - linear utilities, private, uni-dimensional, independent types
  - K alternatives: $u_i^k(\theta_i, t_i) = a_i^k \theta_i + c_i^k + t_i$
  - direct mechanisms: $\left\{x^k(\theta)\right\}_{k=1}^K$ and $\{t_i(\theta)\}_{i=1}^I$
  - relevant function: $v_i(\theta) \equiv \sum_{k=1}^K a_i^k x^k(\theta)$
- Allocation rule $\left\{x^k(\theta)\right\}$ is BIC (DIC) iff $v_i(\theta_i, \theta_{-i})$ is average (component-wise) monotone.

Gershkov et al. (2013)

Theorem

Let Θ_i be connected for all $i \in \mathcal{I}$ and let (x,t) denote a BIC mechanism. An interim-utility equivalent DIC mechanism is given by $(\widehat{x},\widehat{t})$, where the allocation rule $\widehat{x}$ solves

$$\min_{\{\widehat{x}^k(\theta)\}} \mathbb{E}_\theta \sum_{i\in\mathcal{I}} \left[\widehat{v}_i(\theta)\right]^2 ,$$

subject to $\widehat{x}^k(\theta) \geq 0, \forall\theta, \forall k, \sum_{k=1}^K \widehat{x}^k(\theta) = 1,\ \forall\theta,$ and

$$\begin{aligned} \mathbb{E}_{\theta_{-i}}\left[\widehat{v}_i(\theta)\right] &= \mathbb{E}_{\theta_{-i}}\left[v_i(\theta)\right], \forall\theta_i, \forall i, \\ \mathbb{E}_\theta[\widehat{x}^k(\theta)] &= \mathbb{E}_\theta\left[x^k(\theta)\right], \forall k. \end{aligned}$$

- Limits of BIC-DIC equivalence
  - stronger equivalence concept; interdependent values; multi-dimensional types; nonlinear utilities

Outline

- Introduction to Bayesian games and mechanism design
- Quasilinear; uni-dimensional, independent, private types
- **Quasilinear; multidimensional, independent, private types**
  - Rochet theorem: cyclical monotonicity
- Nontransferrable utilities: single-peaked preferences

Rochet (1987): Setup

- Quasilinear preferences

$$u(\theta, x, t) = v(x, \theta) - t$$

  - allocation rule x, transfer t, and type $\theta \in \Theta$
  - DIC and private values: without loss to consider single agent problem

- An allocation rule x is DIC if there exists $t : \Theta \to \mathbb{R}$ such that

$$v(x(\theta), \theta) - t(\theta) \geq v\left(x\left(\theta'\right), \theta\right) - t\left(\theta'\right) \ \forall \theta, \theta' \in \Theta$$

Rochet's Theorem

Theorem (Rochet, 1987)

A necessary and sufficient condition for $x(\cdot)$ to be DIC is that, for all finite cycles $\theta_0, \theta_1, ..., \theta_{N+1} = \theta_0$ in Θ,

$$\sum_{k=0}^{N} \left[v\left(x\left(\theta_k\right), \theta_{k+1}\right) - v\left(x\left(\theta_k\right), \theta_k\right)\right] \leq 0.$$

If types are one dimensional, the above theorem is equvalent to

Theorem (Spence 1974, Mirrless 1976)

Suppose $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, and v is twice differentiable satisfying

$$\frac{\partial^2 v\left(x, \theta\right)}{\partial\theta\partial x} > 0 \text{ for all } \theta \text{ and } x$$

Then cyclical monotonicity is equivalent to the monotonicity of $x\left(\theta\right)$.

Proof of Rochet's Theorem: Necessity

- Let $x(\cdot)$ be DIC with transfer $t(\cdot)$, and $\theta_0, \theta_1, ..., \theta_{N+1} = \theta_0$ be a finite cycle.
- DIC implies that, for all $k \in \{0, ..., N\}$, type θ_{k+1} will not mimic type θ_k:

$$v(x(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1}) \geq v(x(\theta_k), \theta_{k+1}) - t(\theta_k)$$

  which is equivalent to

$$t(\theta_k) - t(\theta_{k+1}) \geq v(x(\theta_k), \theta_{k+1}) - v(x(\theta_{k+1}), \theta_{k+1})$$

- Adding up yields

$$\sum_{k=0}^{N} \left[v(x(\theta_k), \theta_{k+1}) - v(x(\theta_{k+1}), \theta_{k+1})\right] \leq 0,$$

  which is equivalent to

$$\sum_{k=0}^{N} \left[v(x(\theta_k), \theta_{k+1}) - v(x(\theta_k), \theta_k)\right] \leq 0.$$

Proof: Sufficiency

- Suppose cyclic mononicity holds.
- Take an arbitrary $\theta_0 \in \Theta$, and set for any θ in Θ

$$U(\theta) \equiv \sup_{\{\text{all chains from } \theta_0 \text{ to } \theta_{N+1}=\theta\}} \sum_{k=0}^{N} \left[v\left(x\left(\theta_k\right), \theta_{k+1}\right) - v\left(x\left(\theta_k\right), \theta_k\right)\right].$$

- By definition, $U(\theta_0) = 0$ and $U(\theta)$ is finite because

$$U(\theta_0) \geq U(\theta) + v(x(\theta), \theta_0) - v(x(\theta), \theta).$$

- By definition again,

$$U(\theta) \geq U(\theta') + v(x(\theta'), \theta) - v(x(\theta'), \theta').$$

- By setting $t(\theta) = v(x(\theta), \theta) - U(\theta)$, we have

$$v(x(\theta), \theta) - t(\theta) \geq v(x(\theta'), \theta) - t(\theta') \ \forall \theta, \theta' \in \Theta.$$

Linear Utilities

Theorem

Let Θ be a convex subset of $\mathbb{R}^k$, v be linear in θ and twice continuously differentiable in x. Then a continuously differentiable allocation rule $x(\cdot)$ is DIC iff there exists a function $U : \Theta \to \mathbb{R}$ such that, $\forall \theta \in \Theta$,

$$\frac{\partial v(x(\theta), \theta)}{\partial \theta} = \nabla U(\theta)$$

and $\forall \theta_0, \theta_1 \in \Theta$,

$$v(x(\theta_0), \theta_1) - v(x(\theta_0), \theta_0) + v(x(\theta_1), \theta_0) - v(x(\theta_1), \theta_1) \leq 0.$$

- Remark
  - multidimensional analoge of Myerson (1981), Maskin and Laffont (1979). non-differentiable-x
  - the first condition is often called integrability condition.
  - the second condition is called weak (2-cycle) monotonicity.

DIC Implementation with Multi-dimensional Types

- Private, independent types, and quasilinear preferences
- Any domain:
  - cyclical monotonicity (Rochet 1987, Rockafellar 1970)
- Restricted domain
  - finite # of alternatives and convex domain: weak (2-cycle) monotonicity sufficient
  - Bikhchandani et al. (2006), Saks and Yu (2005), Ashlagi et al. (2010)
- Unrestricted domain
  - all DIC rules are weighted VCGs (Roberts 1979).

Outline

- Introduction to Bayesian games and mechanism design
- Quasilinear; uni-dimensional, independent, private types
- Quasilinear; multidimensional, independent, private types
- **Nontransferrable utilities: single-peaked preferences**
  - Moulin (1980)'s theorem: generalized median voter schemes

Moulin (1980)

- I agents and a linearly ordered set A of alternatives (say, $A = \mathbb{R}$).
- Full domain of single-peaked preferences on A.
- Each agent i is assumed to report only the peak x_i of their preferences.

Theorem

A voting scheme $\pi : \mathbb{R}^I \to \mathbb{R}$ is strategy-proof, efficient, and anonymous if, and only if there exist $(I-1)$ real numbers $\alpha_1, ..., \alpha_{n-1} \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ such that, $\forall (x_1, ..., x_I)$,

$$\pi(x_1, ..., x_I) = median\,(x_1, ..., x_I, \alpha_1, ..., \alpha_{n-1}).$$

- Remark: later literature shows that "top-only" restriction can be removed.

Implementation without Transfers

- Strategy proof rules with single-peaked preferences

Preferences	Quasilinear	Single-peaked
simple rule	VCG	median voter scheme
full domain	weighted VCG	generalized median (Moulin, 1980)
any domain	cyclical monotonicity	????
restricted	many papers	many papers

- Gershkov, Moldovanu and Shi (2014): single-crossing preferences
  - a modified successive voting procedure can replicate the outcome of any anonymous, unanimous and strategy-proof rule.
  - alternatives are voted in a pre-specified order, and at each step an alternative is either adopted (and voting stops), or eliminated from further consideration (and the next alternative is considered).
  - characterize utitilarian optimal voting rule.

Other Topics

- Correlated types, full surplus extraction, robust mechanism design
  - Myerson (1981) example
  - Cremer/McLean (1985, 1988), Bergemann/Morris (2005)
- Interdependent values and information externality example
  - impossibility theorem (Maskin, 1992, Jehiel and Moldovanu, 2001)
- Dynamic mechanism design
  - Courty and Li (2000), Eso and Szentes (2007), Gershkov and Moldovanu (2009), Pavan, Segal and Toikka (2013)
  - Bergemann and Valimaki (2010), Athey and Segal (2014)
- Endogenous information structure
  - Bergemann and Valimaki (2002), Shi (2012)
  - Bergemann and Pesendorfer (2007), Eso and Szentes (2007), Li and Shi (2013)

Selected References

- Books
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  - Borgers (2014), *An Introduction to the Theory of Mechanism Design*.
  - Vohra (2011), *Mechanism Design: A Linear Programming Approach*.
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  - Myerson (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 58-71.
  - Rochet (1987), "A Necessary and Sufficient Condition for Rationalizability in a Quasilinear Context," *Journal of Mathematical Economics*, 191-200.
  - Roberts (1979), "The Characterization of Implementable Choice Rules," in *Aggregation and Revelation of Preferences*, J.J. Laffont eds, 321-349.
  - Moulin (1980), "On Strategy-Proofness and Single Peakedness," *Public Choice*, 437-455.

It is one of the first duties of a professor, for example, in any subject, to exaggerate a little both the importance of his subject and his own importance in it.

— G. H. Hardy (1940), A Mathematician's Apology

Linear Utilities with General Allocation Rule

Theorem

Let Θ *be a convex subset of* $\mathbb{R}^k$, v *be linear in* θ *and continuously differentiable in* x. *Then an allocation rule* $x(\cdot)$ *is DIC iff there exists a convex function* $U : \Theta \to \mathbb{R}$ *such that*

$$\forall \theta \in \Theta, \ \frac{\partial v(x(\theta), \theta)}{\partial \theta} \in \partial U(\theta)$$

where $\partial U(\theta)$ *is the subdifferential of* U *at* θ.

Proof. ($\Rightarrow$) Define $U(\theta) \equiv \sup_{\theta' \in \Theta} \left\{ v\left(\theta, x\left(\theta'\right)\right) - t\left(\theta'\right) \right\}$. This implies $U(\theta) \geq U\left(\theta'\right) + v\left(\theta, x\left(\theta'\right)\right) - v\left(\theta', x\left(\theta'\right)\right)$. It follows from linearity that $U(\theta) \geq U\left(\theta'\right) + \frac{\partial v(x(\theta'), \theta')}{\partial \theta}\left(\theta - \theta'\right)$. ($\Leftarrow$) Set $t(\theta) = v(\theta, x(\theta)) - U(\theta)$ and apply the definition of $\partial U(\theta)$ and linearity of v. goback

Correlated Types/Signals

- Two bidders, each may have a valuation $\theta_i = 10$ or $\theta_i = 100$.
- Joint probability distribution for (θ_1, θ_2) is

	$\theta_2 = 10$	$\theta_2 = 100$
$\theta_1 = 10$	$1/3$	$1/6$
$\theta_1 = 100$	$1/6$	$1/3$

  so these two values are not independent.
- The seller's valuation is 0.

Full Surplus Extraction Mechanism

- Consider the following auction mechanism
  - $(100, 100)$: sell it to either bidder for $\$100$ with equal probability.
  - $(100, 10)$ or $(10, 100)$: sell it to high bidder for $\$100$ and charge low bidder $\$30$.
  - $(10, 10)$: give $\$15$ to one of them, and give the object and $\$5$ to the other, with equal probability.
- Seller extracts the full surplus $(10/3 + 100/6 + 100/6 + 100/3 = 70)$:

$$\pi = (-15 - 5)/3 + (100 + 30)/3 + 100/3 = 70$$

The Mechanism Is Feasible

- IR constraints:
  - $\theta_1 = 10 : U_1(\theta_1) = (15)\,2/3 + (-30)/3 = 0;$
  - $\theta_1 = 100 : U_1(\theta_1) = (0)/3 + (0)\,2/3 = 0.$
- IC constraints:
  - $\theta_1 = 10, \theta_1' = 100 :$

$$U_1(\theta_1, \theta_1') = \frac{2}{3}(10 - 100) + \frac{1}{3}\left(\frac{1}{2}(10 - 100)\right) = -75 < 0.$$

  - $\theta_1 = 100, \theta_1' = 10 :$

$$U_1(\theta_1, \theta_1') = \frac{1}{3}\left(\frac{1}{2}(15) + \frac{1}{2}(5 + 100)\right) + \frac{2}{3}(-30) = 0.$$

Decomposition of the Mechanism

- We can decompose the mechanism into two parts
  - sell the object to one of the highest bidders at the highest bidders' valuations.
  - if a bidder reports value 10, invite the bidder to accept a side-bet: pay 30 if the other bidder's value is 100, get 15 if the other bidder's value is 10.
- The side-bet has zero expected payoff if the bidder's true value is 10, but if he lies then this side-bet would have negative value.
- What's wrong?
  - one-to-one mapping between beliefs and (payoff) types.

Generalization

- Cremer and McLean (1985, 1988): finite type space
  - if types are statistically correlated, seller can fully extract the surplus
  - can be implemented in dominant strategies
- McAfee and Reny (1992): infinite type space
  - extend it to a more general mechanism design setting
- Solution:
  - Neeman (2004): beliefs determines preferences (BDP) property
  - Bergemann and Morris (2005): robust mechanism design goback

Information Externality: Example

- Single object auction with n agents
  - valuation functions $v_i\left(\theta^i, \theta^{-i}\right) = g^i(\theta^i) + h^i(\theta^{-i})$.
  - $\theta^k = \left(\theta_1^k, \theta_2^k\right)$ for some agent k, and all other agent signals are one-dimensional
  - suppose private marginal rate of substitution of bidder's information differ from social rate of substitution:

$$\frac{\sum_j \partial v_j / \partial \theta_1^k}{\sum_j \partial v_j / \partial \theta_2^k} \neq \frac{\partial v_k / \partial \theta_1^k}{\partial v_k / \partial \theta_2^k}$$

  - solution concept: Bayesian Nash equilibrium
  - two agent (k and j) example: $u_k = \theta_1^k + 2\theta_2^k$ and $u_j = 2\theta_1^k + \theta_2^k$.
- No efficient auction exists
  - consider $\theta^k, \widehat{\theta}^k$ such that $g^k(\theta^k) = g^k(\widehat{\theta}^k)$.
  - agent k indifferent but not efficient allocation. goback