Fields Institute Tutorial

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Mechanism

$N =$ set of agents.
$\Gamma =$ finite set of at least three outcomes.
$T \subseteq \Re^{|\Gamma|}$ set of (multi-dimensional) types.
$T^n =$ set of all n-agent profiles of types.
Allocation rule is a function

$$f : T^n \to \Gamma.$$

For each $\alpha \in \Gamma$ there is a $\mathbf{t} \in T^n$ such that $f(\mathbf{t}) = \alpha$.

Mechanism

Payment rule is a function P such that

$$P : T^n \rightarrow \Re^n.$$

In profile $(t^1, \ldots t^n)$ agent i has type t^i she makes a payment of $P_i(t^1, \ldots, t^n)$.

Value agent i with type $t \in T$ assigns to allocation $\alpha \in \Gamma$ is $v^i(\alpha|t) = t_\alpha$.

Incentive Compatability

For all agents i and all types $s^i \neq t^i$:

$$v^i(f(t^i, t^{-i})|t^i) - P_i(t^i, t^{-i})$$

$$\geq v^i(f(s^i, t^{-i})|t) - P_i(s^i, t^{-i}) \ \forall \ t^{-i}.$$

Suppress dependence on i, t^{-i}

$$v(f(t)|t) - P(t) \geq v(f(s)|t) - P(s)$$

$$t_{f(t)} - P(t) \geq t_{f(s)} - P(s)$$

Incentive Compatability

$$t_{f(t)} - P(t) \geq t_{f(s)} - P(s) \quad (1)$$

$$s_{f(s)} - P(s) \geq s_{f(t)} - P(t). \quad (2)$$

Add (1) and (2)

$$t_{f(t)} + s_{f(s)} \geq t_{f(s)} + s_{f(t)}.$$

$$t_{f(t)} - t_{f(s)} \geq -[s_{f(s)} - s_{f(t)}].$$

2-cycle inequality

$$[t_{f(t)} - t_{f(s)}] + [s_{f(s)} - s_{f(t)}] \geq 0.$$

Incentive Compatability

f is dominant strategy IC if $\exists$ P such that:

$$t_{f(t)} - P(t) \geq t_{f(s)} - P(s)$$

Fix f, find P such that

$$P(t) - P(s) \leq t_{f(t)} - t_{f(s)}. \quad (3)$$

Incentive Graph

$$P(t) - P(s) \leq t_{f(t)} - t_{f(s)}.$$

A vertex for each type t

From vertex s to vertex t an edge of length $t_{f(t)} - t_{f(s)}$

From vertex t to vertex s an edge of length $s_{f(s)} - s_{f(t)}$

System 3 is feasible iff Incentive graph has no (-)ve cycles.

Incentive Compatability

2-cycle inequality

$$[t_{f(t)} - t_{f(s)}] + [s_{f(s)} - s_{f(t)}] \geq 0.$$

All 2-cycles in network are of non-negative length.

For many preference domains, 2-cycles non (-)ve $\Rightarrow$ all cycles are non (-)ve

T is convex

Roberts' Theorem

$|\Gamma| \geq 3$, $T = \Re^{|\Gamma|}$, if f is onto and DSIC $\exists$ non-negative weights $\{w_i\}_{i \in N}$ and weights $\{D_\alpha\}_{\alpha \in \Gamma}$ such that

$$f(t) \in \arg\max_{\alpha \in \Gamma} \sum_i w_i t^i_\alpha - D_\alpha$$

(equivalent) There is a solution $w, \{D_\gamma\}_{\gamma \in \Gamma}$ to the following:

$$D_\alpha - D_\gamma \leq \sum_{i=1}^{n} w_i (t^i_\alpha - t^i_\gamma) \ \forall \gamma, \ \mathbf{t} \text{ s.t. } f(\mathbf{t}) = \alpha$$

Roberts' Theorem

Fix a non-zero and nonnegative vector w.

Network Γ_w will have one node for each $\gamma \in \Gamma$.

For each ordered pair (β, α) introduce a directed arc from β to α of length

$$l_w(\beta, \alpha) = \inf_{\mathbf{t}: f(\mathbf{t}) = \alpha} \sum_{i=1}^{n} w_i (t^i_\alpha - t^i_\beta).$$

Is there a choice of w for which Γ_w has no negative length cycles?

Roberts' Theorem

$U(\beta, \alpha) = \{d \in \mathbb{R}^n : \exists \mathbf{t} \in T^n \text{ s.t. } f(\mathbf{t}) = \alpha, \text{ s.t. } d^i = t^i_\alpha - t^i_\beta \ \forall i\}.$

$l_w(\beta, \alpha) = \inf_{d \in U(\beta,\alpha)} w \cdot d.$

Roberts' Theorem

Suppose a cycle $C = \alpha_1 \rightarrow \ldots \rightarrow \alpha_k \rightarrow \alpha_1$ through elements of Γ.

From each α_j pick a profile $\mathbf{t}[j]$ such that $f(\mathbf{t}[j]) = \alpha_j$.

Associate with the cycle C a vector b whose i^{th} component is

$$b^i = (t^i_{\alpha_1}[1] - t^i_{\alpha_k}[1]) + (t^i_{\alpha_2}[2] - t^i_{\alpha_1}[2]) + \ldots + (t^i_{\alpha_k}[k] - t^i_{\alpha_{k-1}}[k]).$$

Let $K \subseteq \mathbb{R}^n$ be the set of vectors that can be associated with some cycle through the elements of Γ.

Roberts' Theorem

Asserts the existence of a feasible w such that $w \cdot b \geq 0$ for all $b \in K$.

1. If $b \in K$ is associated with cycle $\alpha_1 \to \ldots \to \alpha_k \to \alpha_1$, then b is associated with the cycle $\alpha_1 \to \alpha_k \to \alpha_1$.
2. If $b \in K$ is associated with a cycle through (α, β), then b is associated with a cycle through (γ, θ) for all $(\gamma, \theta) \neq (\alpha, \beta)$. So, restrict to just one cycle.
3. The set K is convex.
4. K is disjoint from the negative orthant, invoke separating hyperplane theorem.

Roberts' Theorem

Lemma

Suppose $f(\mathbf{t}) = \alpha$ *and* $s \in T^n$ *such that* $s^i_\alpha - s^i_\beta > t^i_\alpha - t^i_\beta$ *for all* i. *Then* $g(\mathbf{s}) \neq \beta$.

Consider the profile $(s^1, \mathbf{t}^{-1})$ and suppose that $s^1_\alpha - s^1_\beta > t^1_\alpha - t^1_\beta$ and $g(s^1, \mathbf{t}^{-1}) = \beta$. This violates 2-cycle.

Roberts' Theorem

For every pair $\alpha, \beta \in \Gamma$ define

$$h(\beta, \alpha) = \inf_{t \in T^n : g(\mathbf{t}) = \alpha} \max_i t^i_\alpha - t^i_\beta = \inf_{d \in U(\beta, \alpha)} \max_i d^i.$$

Lemma

For every pair $\alpha, \beta \in \Gamma$, $h(\beta, \alpha)$ is finite.

Roberts' Theorem

Lemma

For all $\alpha, \beta \in \Gamma$, $h(\alpha, \beta) + h(\beta, \alpha) = 0$.

Roberts' Theorem

Suppose $h(\alpha, \beta) + h(\beta, \alpha) > 0$.

Choose $\mathbf{t} \in T^n$ to satisfy

$$t^i_\alpha - t^i_\beta < h(\beta, \alpha)\ \forall i \tag{4}$$

$$t^i_\beta - t^i_\alpha < h(\alpha, \beta)\ \forall i \tag{5}$$

$$t^i_\gamma - t^i_\alpha < h(\alpha, \gamma)\ \forall i\ \forall \gamma \neq \alpha, \beta \tag{6}$$

(4) implies that $g(\mathbf{t}) \neq \alpha$. (5) implies that $g(\mathbf{t}) \neq \beta$. Together with (6) we deduce that $g(\mathbf{t}) \notin \Gamma$ a contradiction.

Rationalizability (quasi-linear Afriat)

Set of purchase decisions $\{p_i, x_i\}_{i=1}^n$ is **rationalizable** by

- locally non-satiated,
- quasi-linear,
- concave utility function $u : \mathbb{R}_+^m \mapsto \mathbb{R}$
- for some budget B

if for all i,

$$x_i \in \arg\max\{u(x) + s : p_i \cdot x + s = B, x \in \mathbb{R}_+^m\}.$$

Rationalizability

If at price p_i, $p_i \cdot x_j \leq B$, it must be that x_j delivers less utility than x_i.

$$u(x_i) + B - p_i \cdot x_i \geq u(x_j) + B - p_i \cdot x_j$$

$$\Rightarrow u(x_j) - u(x_i) \leq p_i \cdot (x_j - x_i)$$

Given set $\{(p_i, x_i)\}_{i=1}^{n}$ we formulate the system:

$$y_j - y_i \leq p_i \cdot (x_j - x_i), \ \forall i, j \ \text{ s.t. } \ p_i \cdot x_j \leq B$$

Rationalizability

$$y_j - y_i \leq p_i \cdot (x_j - x_i), \ \forall i, j \ \text{ s.t. } \ p_i \cdot x_j \leq B \qquad (7)$$

1. One node for each i.
2. For each ordered pair (i, j) such that $p_i \cdot x_j \leq B$, an arc with length $p_i \cdot (x_j - x_i)$.
3. The system (7) is feasible iff. associated network has no negative length cycles.

Rationalizability

Use any feasible choice of $\{y_j\}_{j=1}^{n}$ to construct a concave utility.

Set $u(x_i) = y_i$.

For any other $x \in \mathbb{R}^n_+$ set

$$u(x) = \min_{i=1,\ldots,n} \{u(x_i) + p_i \cdot (x - x_i)\}.$$

Cardinal Matching (TU)

Cardinal Matching

Given a graph $G = (V, E)$, find a matching that maximizes a weighted sum of the edges.

Bipartite: Poly time, natural LP formulation has integral extreme points

Non-bipartite: Poly time, natural LP formulation is $1/2$ fractional, exact formulation exponential

Ordinal Matching (NTU)

Given $G(V, E)$ and 'preferences over edges' find a matching that 'respects' preferences.

Bipartite Stable Matching: $(D \cup H, E)$, $D =$ doctors and $H =$ hospitals (unit capacity)

Each $d \in D$ has a strict preference ordering $\succ_d$ over H and each $h \in H$ has a strict $\succ_h$ over D.

Stable Matching

A matching $\mu : D \rightarrow H$ is *blocked* by the pair (d, h) if

1. $\mu(d) \neq h$
2. $h \succ_d \mu(d)$
3. $d \succ_h \mu^{-1}(h)$

A matching μ is stable if it is not blocked.

Stable Matching

Bipartite Graph

$D \cup H =$ set of vertices (doctors and hospitals)

$E =$ set of edges

$\delta(v) \subseteq E$ set of edges incident to $v \in D \cup H$

Each $v \in D \cup H$ has a strict ordering $\succ_v$ over edges in $\delta(v)$

Stable Matching

$$\sum_{e \in \delta(v)} x_e \leq 1 \; \forall v \in D \cup H$$

For all $e \in E$ there is a $v \in D \cup H$ such that $e \in \delta(v)$ and

$$\sum_{f \succ_v e} x_f + x_e = 1$$

Scarf's Lemma

$Q =$ an $n \times m$ nonnegative matrix and $r \in \mathbb{R}^n_+$.

$Q_i =$ the i^{th} row of matrix Q.

$\mathcal{P} = \{x \in \mathbb{R}^m_+ : Qx \leq r\}$.

Each row $i \in [n]$ of Q has a strict order $\succ_i$ over the set of columns j for which $q_{i,j} > 0$ (the columns that intersect it).

A vector $x \in \mathcal{P}$ **dominates** column j if there exists a row i such that $Q_i x = r_i$ and $k \succeq_i j$ for all $k \in [m]$ such that $q_{i,k} > 0$ and $x_k > 0$.

We say x **dominates column** j **at row** i.

Scarf's Lemma

Kiralyi & Pap version

Let Q be an $n \times m$ nonnegative matrix, $r \in \mathbb{R}^n_+$ and $\mathcal{P} = \{x \in \mathbb{R}^m_+ : Qx \leq r\}$. Then, $\mathcal{P}$ has a vertex that dominates every column of Q.

Stable Matching with Couples

D^1 = set of single doctors

D^2 = set of couples, each couple $c \in D^2$ is denoted $c = (f, m)$

$D = D^1 \cup \{m_c | c \in D^2\} \cup \{f_c | c \in D^2\}$.

Each $s \in D^1$ has a strict preference relation $\succ_s$ over $H \cup \{\emptyset\}$

Each $c \in D^2$ has a strict preference relation $\succ_c$ over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$

Stable Matching with Couples

Hospital $h \in H$ has a capacity $k_h > 0$

Preference of hospital h over subsets of D is summarized by choice function $ch_h(.) : 2^D \rightarrow 2^D$.

$ch_h(.)$ is responsive

h has a strict priority ordering $\succ_h$ over elements of $D \cup \{\emptyset\}$.

$ch_h(D^*)$, consists of the (upto) k_h highest priority doctors among the feasible doctors in D^*.

Blocking

μ = matching

μ_h = the subset of doctors matched to h

μ_s position that single doctor s receives

μ_{f_c}, μ_{m_c} are the positions that the female member, the male member of the couple c obtain in the matching

Blocking

μ is individual rational if

- $ch_h(\mu_h) = \mu_h$ for any hospital h
- $\mu_s \succeq_s \emptyset$ for any single doctor s
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \mu_{m_c})$
  $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\mu_{f_c}, \emptyset)$
  $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \emptyset)$
  for any couple c

Blocking

Matching μ can be blocked as follows

1. A pair $s \in D^1$ and $h \in H$ can block μ if $h \succ_s \mu(s)$ and $s \in ch_h(\mu(h) \cup s)$.
2. A triple $(c, h, h') \in D^2 \times (H \cup \{\emptyset\}) \times (H \cup \{\emptyset\})$ with $h \neq h'$ can block μ if $(h, h') \succ_c \mu(c)$, $f_c \in ch_h(\mu(h) \cup f_c)$ when $h \neq \emptyset$ and $m_c \in ch_{h'}(\mu(h') \cup m_c)$ when $h' \neq \emptyset$.
3. A pair $(c, h) \in D^2 \times H$ can block μ if $(h, h) \succ_c \mu(c)$ and $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$.

Matching with Couples

Each doctor in D^1 has a strict preference ordering over the elements of $H \cup \{\emptyset\}$

Each couple in D^2 has a strict preference ordering over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$

Each hospital has responsive preferences

(Nguyen & Vohra) For any capacity vector k, there exists a k' and a stable matching with respect to k', such that $\max_{h \in H} |k_h - k'_h| \leq 4$. Furthermore, $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h \leq \sum_{h \in H} k_h + 9$.

Matching with Couples

Apply Scarf's Lemma to get a 'fractionally' stable solution

$Q =$ constraint matrix of a 'generalized' transportation problem

Rows correspond to $D^1 \cup D^2$ and H

Column corresponds to an assignment of a single doctor to a hospital or a couple to a pair of slots

Each row has an ordering over the columns that intersect it

Generalized Transportation Problem

$x_d(S) = 1$ if $S \subseteq H$ is assigned to agent $d \in D$ and zero otherwise.

$x_d(S) = 0$ for all $|S| > \alpha$

$$\sum_{S \subseteq H} x_d(S) \leq 1 \; \forall d \in D \; (dem)$$

$$\sum_{i \in D} \sum_{S \ni h} x_d(S) \leq k_h \; \forall h \in H \; (supp)$$

Iterative Rounding

Solve the LP to get a fractional extreme point solution x^*.

If every variable is 0 or fractional, there must exist a $h \in H$ such that

$$\sum_{d \in D} \sum_{S \ni h} \lceil x_d^*(S) \rceil \leq k_h + \alpha - 1$$

Rounding

For every extreme point x^* and u optimized at x^*, there is an integer y such that $u \cdot y \geq u \cdot x^*$ and

$$\sum_{S \subseteq H} y_d(S) \leq 1 \ \forall d \in D$$

$$\sum_{d \in D} \sum_{S \ni h} y_d(S) \leq k_h + \alpha - 1 \ \forall h \in H$$