Universally Baire subsets of 2^κ

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Outline

1. Topology
2. B-names for reals as continuous functions
3. Universally Baire sets of reals
4. $\mathsf{UB}^{\Gamma}_{\kappa}$-sets
   - Different characterizations of $\mathsf{UB}^{\Gamma}_{\kappa}$-sets
   - Forcing axioms and $\mathsf{UB}^{\Gamma}_{\omega_1}$-sets
   - Generic absoluteness for $\mathsf{UB}^{\Gamma}_{\omega_1}$-sets
5. Final comments

Definition

Let (X, τ) be a topological space.

- $A \subseteq X$ is a regular open set if $A = \overset{\circ}{\overline{A}}$.
- (X, τ) is an *interesting for this talk* (IFTT) topological space if it has a base of regular open sets.
- $\mathrm{RO}(X, \tau)$ denotes the family of regular open sets in τ.

Fact

Most topological spaces are IFTT, among which:

- *T_3 spaces (thus all compact Haussdorff spaces),*
- *Spaces given by the order topology induced by a partial order: Let $(P, \leq)$ be a separative partial order, set for $A \subseteq P$*

$$\downarrow A = \{q : \exists p \in A \, q \leq p\}$$

and let

$$\tau = \{\downarrow A : A \subseteq P\}.$$

Then (P, τ) is an IFTT space.

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Recall the following:

Proposition

Assume (X, τ) is any topological space, then $\mathrm{RO}(X, \tau)$ with operations given by

- $\neg A = X \setminus \overline{A}$,
- $\bigwedge_{i \in I} A_i = \mathring{\overbrace{\bigcap\{A_i : i \in I\}}}$,
- $\bigvee_{i \in I} A_i = \mathring{\overline{\bigcup\{A_i : i \in I\}}}$

is a complete boolean algebra.

Given (X, τ) a topological space, $\mathrm{CL}(X, \tau)$ denotes the clopen subsets of (X, τ).
$\mathrm{CL}(X, \tau)$ is a subalgebra of $\mathrm{RO}(X, \tau)$.

Definition

Let (X, τ) be a topological space.

- X is 0-dimensional if $\mathrm{CL}(X, \tau)$ is a base for τ,
- X is *extremely disconnected* if $\mathrm{CL}(X, \tau) = \mathrm{RO}(X, \tau)$.

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Stone duality

Definition

Let B be a boolean algebra. $\mathtt{St}(\mathrm{B})$ is its space of ultrafilters with topology τ_{B} given by the basis of clopen sets

$$N_b = \{G \in \mathtt{St}(\mathrm{B}) : b \in G\}.$$

Theorem (Stone Duality)

The following holds:

- $(St(\mathrm{B}), \tau_{\mathrm{B}}))$ *is a* 0*-dimensional compact Haussdorff space and* B *can be identified with the clopen sets of this space.*
- $(St(\mathrm{B}), \tau_{\mathrm{B}}))$ *is extremely disconnected iff* B *is complete.*
- *A compact space* (X, τ) *is extremely disconnected iff* $CL(\tau) = RO(\tau)$ *is a complete boolean algebra.*

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Weak universality of extremely disconnected compact spaces

Let (X, τ) be an IFTT space (X has a base of regular open sets). Then we can define a (partial) projection map

$$\pi_X : \mathbb{S}\mathrm{t}(\mathrm{RO}(X, \tau)) \to X$$
$$G \mapsto x_G$$

where x_G (if defined) is the unique point in X such that the neighboorhood filter of regular open sets around x_G is contained in G.
(If $X = \mathbb{R}$ and $G \supseteq \{(a, +\infty) : a \in \mathbb{R}\}$, $x_G = +\infty \notin \mathbb{R}$).

Theorem

Let (X, τ) be an IFTT space.
Then for every compact topological space (Z, σ) and any continuous

$$f : X \to Z$$

there is a continuous $\bar{f} : \mathbb{S}\mathrm{t}(\mathrm{RO}(X, \tau)) \to Z$ such that $f \circ \pi_X = \bar{f}$.

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In particular if our aim is to study all possible preimages of continuous function with range in say 2^ω, we can just focus on continuous functions whose domain is a compact and extremely disconnected topological space.

We work with the boolean valued models approach to forcing.
Let B be a complete boolean algebra and $\sigma \in V^{\mathsf{B}}$ be such that

$$[\![\sigma \in 2^\omega]\!] = 1_{\mathsf{B}}.$$

Then

$$\sigma \approx \{\langle (n,i), a_{n,i}\rangle : n \in \omega, i < 2\}$$

with $a_{n,0} = \neg a_{n,1}$ for all n, i.

We naturally identify σ with a continuous $f_\sigma : \mathrm{St}(\mathrm{B}) \to 2^\omega$ (where 2^ω is given the product topology) letting

$$\sigma \mapsto f_\sigma = \{\langle G, x\rangle : G \in \mathtt{St}(\mathrm{B}),\, x \in 2^\omega,\, x(n) = i \Leftrightarrow a_{n,i} \in G\}.$$

Conversely let $f : \mathrm{St}(\mathrm{B}) \to 2^\omega$ be a continuous function. Then we identify it with the B-name

$$\sigma_f = \{\langle (n, i), f^{-1}[N_{n,i}]\rangle : n \in \omega, i < 2\}$$

where

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Then

- $[\![\sigma_{f_\sigma} = \sigma]\!] = 1_{\mathsf{B}}$,
- $f_{\sigma_f} = f$.

We get that the for any cba B the new name for elements in 2^ω correspond to continuous functions $f_\sigma : \mathrm{St(B)} \to 2^\omega$.

We chose 2^ω but we could have chosen any Polish space to implement this translation (some extra care is needed for this general case).

If we chose $\mathbb{C}$ we would be looking at commutative unital C^*-algebras of the form $C(X)$ with X extremely disconnected.

More precisely we would be looking at the space

$$\{f : X \to \mathbb{C} \cup \{\infty\} = \mathbb{S}^2 : \ f \text{ is continuous and } f^{-1}[\{\infty\}] \text{ is closed nowhere dense}\}.$$

This correspondence is functorial and extends to many combinatorial (first order, Borel,....) structures by which we can endow 2^ω.

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Let (X, τ) be a topological space.

- $A \subseteq X$ is nowhere dense if $\overline{A}$ has a dense open complement.
- $A \subseteq X$ is *meager* if it is contained in the countable union of nowhere dense sets.
- $A \subseteq X$ has the Baire property if there is a unique regular open set $U \in \tau$ such that $A \Delta U$ is meager.

Definition (Feng, Magidor, Woodin)

$A \subseteq 2^{\omega}$ is universally Baire UB if for any cba B and any continuous function $f : \mathtt{St}(\mathsf{B}) \to 2^{\omega}$. $f^{-1}[A]$ has the Baire property in $\mathtt{St}(\mathsf{B})$.

Equivalently A is UB if $f^{-1}[A]$ has the Baire property in X for any continuous $f : X \to 2^{\omega}$ with X compact Hausdorff.

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Universally Baire sets design properties of the real numbers whose meaning cannot be affected by forcing.

It is immediate to see that open and closed sets are universally Baire and slightly less immediate to check that Borel sets are also universally Baire.
Σ^1_1-sets are also universally Baire.

There are in L Δ^1_2-sets which are not universally Baire in L.

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Set

$$\dot{A} = \{\langle \tau, U_{f_\tau^{-1}[A]}\rangle : [\![\tau \in 2^\omega]\!] = 1_\mathsf{B}\}.$$

We can easily check that for all V-generic filters G $\dot{A}_G \cap V = A$. But much more is true about $\dot{A}$.

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Given a cba B and $G \in \mathtt{St}(\mathsf{B})$, $\tau, \sigma \in V^{\mathsf{B}}$, set for R among $=, \in$:

- $\tau R_G \sigma$ iff $[\![\tau R \sigma]\!] \in G$,
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- $V^{\mathsf{B}}/G = \{[\tau]_G : \tau \in V^{\mathsf{B}}\}$.

Theorem (Cohen's forcing theorem)

Assume V is amodel of ZFC, B is a cba in V, $G \in St(\mathsf{B})$. Then $(V^{\mathsf{B}}/G, \in_G)$ is a Tarski model of ZFC and

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Theorem (Shoenfield's absoluteness)

Let $A \in \mathrm{UB}$, B be a cba, $G \in \mathcal{S}t(\mathrm{B})$.
Then

$$(H_{\omega_1}, \in, A) \prec_{\Sigma_1} (H_{\omega_1}^{V^{\mathrm{B}}/G}, \in_G, [\dot{A}]_G).$$

Theorem (Woodin's absoluteness)

Let V be any model of ZFC+ there exists class many Woodin limit of Woodins.
Then

- UB^V *is a boolean algebra contained in* $P(2^\omega)$ *containing all projective sets and much more.*
- *For any* $\mathrm{B} \in V$ *which is a cba for* V *and any* $A \in \mathrm{UB}^V$ *and any* $G \in \mathcal{S}t(\mathrm{B})$

$$(L(Ord^\omega, A), \in) \prec ((L(Ord^\omega, [\dot{A}]_G)^{V^{\mathrm{B}}/G}, \in_G).$$

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We aim to find:

- The correct topological counterpart of the notion of B-name for a subset of κ.
- The correct notion of a property defining a subset of κ which has a definition whose meaning is unaffected by forcing.

We have already set up a language which will make the first task simple.

We have already seen that the extension of the class UB depends very much on the set theory we work in: in L there are few such sets, assuming large cardinals there are plenty of such sets.

In particular we expect that our notion of universally Baire subset of $P(\kappa)$ will be very much dependent on our set theoretic assumptions and may eventually stabilize when we are assuming "maximal extensions" of ZFC.

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- The correct notion of a property defining a subset of κ which has a definition whose meaning is unaffected by forcing.

We have already set up a language which will make the first task simple.

We have already seen that the extension of the class UB depends very much on the set theory we work in: in L there are few such sets, assuming large cardinals there are plenty of such sets.

In particular we expect that our notion of universally Baire subset of $P(\kappa)$ will be very much dependent on our set theoretic assumptions and may eventually stabilize when we are assuming "maximal extensions" of ZFC.

Fact

Let B *be a complete boolean algebra. There is a natural correspondence between* B*-names for elements of* 2^κ *and continuous functions* $f : \mathcal{S}t(\mathrm{B}) \to 2^\kappa$*, where* 2^κ *is endowed with the PRODUCT topology.*

Definition

Let (X, τ) be a topological space.

- $A \subseteq X$ is κ-meager if it is contained in the union of κ-many closed nowhere dense sets.
- $A \subseteq X$ has the κ-Baire property if it has a κ-meager difference with a regular open set in τ.

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Definition (V., Ikegami)

Let Γ be a class of complete boolean algebras.
$A \subseteq 2^{\kappa}$ is in $\mathrm{UB}^{\Gamma}_{\kappa}$ if $f^{-1}[A]$ has the κ-Baire property for all $\mathrm{B} \in \Gamma$ and all continuous $f : \mathtt{St}(\mathrm{B}) \to 2^{\kappa}$.

We will show that (almost) all equivalent characterizations of the notion of universal baireness can be stepped up to this new general setting.

Definition

Let B be a complete boolean algebra.
$\mathsf{FA}_{\kappa}(\mathsf{B})$ holds if

$$\bigcap\{D_\alpha : \alpha < \kappa\} \neq \emptyset$$

for all collections $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of $\mathtt{St}(\mathsf{B})$.
Γ_κ is the class of all B satisfying $\mathsf{FA}_{\kappa}(\mathsf{B})$.

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The restriction on cbas in Γ_κ is *natural, necessary and harmless*:

Fact

Assume $\mathsf{FA}_\kappa(\mathsf{B})$ fails, then $\mathcal{St}(\mathsf{B})$ is a κ-meager topological space, in particular any subset of $\mathcal{St}(\mathsf{B})$ vacuously has the κ-Baire property.

In particular for any class of cbas Γ all cbas such that $\mathsf{FA}_\kappa(\mathsf{B})$ fails will not be able to detect a subset of 2^κ which may not be in in $\mathrm{UB}^{\Gamma}_{\kappa}$.

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Definition

Let $\mathtt{Fun}(\kappa, X)$ denote the family of finite partial functions from κ to X.

- $T \subseteq \mathtt{Fun}(\kappa, X)$ is a tree$^{\kappa}$ on X if for all $s \in T$ and $t \subseteq s$, $t \in T$, and for all z finite subset of κ there is some $s \in T$ with domain z.
- A total $f : \kappa \to X$ is a branch of T if $f \upharpoonright z \in T$ for all z finite subset of κ.
- $[T]$ is the body of T consisting of all its branches.
- Assume T is a tree$^{\kappa}$ on $X \times Y$. $p[T]$ is the set of all $g : \kappa \to Y$ such that $(f, g) \in [T]$ for some $f : \kappa \to X$.

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Theorem (V., Ikegami)

TFAE for $A \subseteq 2^{\kappa}$ and $\Gamma \subseteq \Gamma_{\kappa}$:

1. $A \in \mathrm{UB}^{\Gamma}_{\kappa}$.
2. *There are tree$^{\kappa}$ sets S, U on $2 \times V$ such that:*
   - $p[S] = A$,
   - *for any* $\mathrm{B} \in \Gamma$ *we have that*

$$[\![p[U] = 2^{\kappa} \setminus p[S]]\!] = 1_{\mathrm{B}},$$

Notice that there can be non-universally Baire subsets of 2^{ω} which are in $\mathrm{UB}^{\Gamma}_{\kappa}$, (here we require the projection operation p to behave nicely only for forcings in Γ).

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Notice that there can be non-universally Baire subsets of 2^{ω} which are in $\mathrm{UB}^{\Gamma}_{\kappa}$, (here we require the projection operation p to behave nicely only for forcings in Γ).

For any M let $\pi_M : M \to N_M$ be its Mostowski collapsing map.

$T_{\kappa,\mathrm{B}}$ is the set of all $M \prec H_\theta$ (for some large enough θ) such that

- $|M| = \kappa \subseteq M$ and $\mathrm{B} \in M$,
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Ultrafilters G as above are called M-generic ultrafilters.

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What sets are $\mathrm{UB}^{\Gamma}_{\omega_1}$?

Theorem (Moore, Todorčevič, Veličkovič-Caicedo, Aspero...)

Assume BMM *holds in* V *and* $T = \mathrm{ZFC} + \mathrm{BMM}$. *Then there is a well ordering of* $P(\omega_1)$ *which is* $\mathrm{UB}^{\Gamma}_{\omega_1}$, *where* $\Gamma \subseteq \Gamma_{\omega_1}$ *is the class of forcing notions preserving* T.

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Assume MM^{+++} *and* $\Gamma \subseteq \Gamma_{\omega_1}$ *be the class of all* SSP*-forcings which preserve* MM^{+++}*. Then for all* $\mathsf{B} \in \Gamma$ *and all* $A \in \mathsf{UB}^{\Gamma}_{\omega_1}$ *there is* $\dot{A} \in V^{\mathsf{B}}$ *such that for all* $G \in St(\mathsf{B})$

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In particular combining the two results we get that all sets in $L(P(\omega_1))$ are in $\mathsf{UB}^{\Gamma}_{\omega_1}$ assuming MM^{+++}.

The theory MM^{+++} makes the class of $\mathsf{UB}^{\omega_1}_{\Gamma}$-sets extremely large and generically invariant exactly as it occurs for the Universally Baire sets of 2^{ω} assuming large cardinals.

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Here are some questions and comments:

- Assume MM^{+++}. Let Γ be as before. Can there exists a subset of 2^{ω_1} which is not in $\mathsf{UB}^{\Gamma}_{\omega_1}$?
- We can also give (in the presence of MM^{+++}) the characterization of $\mathsf{UB}^{\Gamma}_{\omega_1}$ in terms of generic elementary embeddings in analogy with what is done by means of generic stationary tower forcing for universally Baire sets (Theorem 3.3.7 of Larson's book on the stationary tower).
- With Giorgio Audrito, Joel Hamkins, and Thomas Johnstone we are isolating a boldface iterated resurrection axioms which strenghten bounded forcing axioms yielding various forms of generic absoluteness results with $(H(\omega_2), A)$ in the place of $L(\mathrm{Ord}^{\omega_1}, A)$ (and at the prize of less large cardinal strength).
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Thanks for your attention!