INTERPRETER FOR TOPOLOGISTS

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Topological interpretations.

If $M \models$ "$\langle X, \tau \rangle$ is a topological space" and $\langle \hat{X}, \hat{\tau} \rangle$ is a topological space then $\pi\colon X \to \hat{X}$ and $\pi\colon \tau \to \hat{\tau}$ is a *topological preinterpretation* if

- $x \in O \leftrightarrow \pi(x) \in \pi(O)$;
- $\pi(0) = 0$, $\pi(X) = \hat{X}$;
- π commutes with finite intersections and arbitrary unions in M.

An interpretation $\pi_0\colon X \to \widehat{X}_0$ is *reducible* to $\pi_1\colon X \to \widehat{X}_1$ if there is a function $h\colon \widehat{X}_0 \to \widehat{X}_1$ such that $\pi_1 = h \circ \pi_0$ and for every $O \in \tau$, $h^{-1}\pi_1(O) = \pi_0(O)$.

A *topological interpretation* of X is the preinterpretation largest in the sense of reducibility.

Theorem. Topological interpretation exists for every regular Hausdorff space and it is unique.

Borel interpretations.

If $M \models$ "$\langle X, \tau, \mathcal{B} \rangle$ is a topological space" and $\langle \hat{X}, \hat{\tau}, \hat{\mathcal{B}} \rangle$ is a topological space then $\pi \colon X \to \hat{X}$ and $\pi \colon \tau \to \hat{\tau}$ and $\pi \colon \mathcal{B} \to \hat{\mathcal{B}}$ is a *Borel-topological preinterpretation* if

- $x \in O \leftrightarrow \pi(x) \in \pi(O)$;
- $\pi(0) = 0$, $\pi(X) = \hat{X}$;
- π commutes with finite intersections and arbitrary unions of open sets in M;
- π commutes with complements, countable unions and intersections of Borel sets in M.

A Borel-topological interpretation is a preinterpretation which is largest in the reducibility order.

Theorem. Borel-topological interpretation exists for every regular Hausdorff space and it is unique.

Čech complete and Borel complete spaces

Definition. A space is Čech complete if it is a G_δ subspace of a compact Hausdorff space.

Example. Every completely metrizable space is Čech complete.

Definition. A space is Borel complete if it is a Borel subspace of a compact Hausdorff space.

Example. The space of continuous functions from reals to reals with pointwise convergence is Borel complete and not Čech complete.

Comparison

Theorem. A topological interpretation of a Čech complete space can be uniquely extended to a Borel-topological interpretation.

Theorem. If V does not contain an unbounded real over M and every countable subset of M is a subset of a set countable in M then a topological interpretation of every regular Hausdorff space can be uniquely extended to a Borel-topological interpretation.

First computations.

Theorem. Every compact Hausdorff space has a unique compact Hausdorff preinterpretation which is its interpretation.

Theorem. The interpretation of a complete metric space is its completion in the larger model.

Subspaces.

Theorem. If $\pi\colon X \to \hat{X}$ is a topological interpretation and $A \subset X$ is open or closed, then $\pi \restriction A\colon A \to \pi(A)$ is a topological interpretation.

Theorem. If $\pi\colon X \to \hat{X}$ is a Borel-topological interpretation and $A \subset X$ is Borel, then $\pi \restriction A\colon A \to \pi(A)$ is a Borel-topological interpretation.

Corollary. An interpretation of a Čech complete space is Čech complete.

Products.

Theorem. A product of any collection of compact Hausdorff spaces is interpreted as product of interpretations.

Theorem. A product of countable collection of Borel-complete spaces is interpreted as product of interpretations.

Example. It does not work for product of two Sorgenfrey lines or for product of Baire space with the space of well-founded trees.

Continuous functions.

Theorem. Total continuous functions between Borel-complete spaces are interpreted as total continuous functions between interpretations.

Theorem. Open continuous functions between Čech complete spaces are interpreted as open continuous functions.

Hyperspaces.

Theorem. If X is Čech complete and $\pi\colon X \to \hat{X}$ is an interpretation then $\pi\colon K(X) \to K(\hat{X})$ is an interpretation.

Theorem. Suppose that X is Čech complete, $K \subset X$ is compact, and Y obtains from X by gluing all points in K. If $\pi\colon X \to \hat{X}$ is an interpretation then Y is interpreted as $\hat{X}$ with the set $\pi(K)$ glued together.

Čech structures.

Definition. A Čech structure is a tuple $\mathfrak{X} = \langle \vec{X}, \vec{R}, \vec{f} \rangle$ where $\vec{X}$ are Čech complete spaces, $\vec{R}$ are finitary Borel relations and $\vec{f}$ are finitary continuous functions with Borel domains.

Theorem. (Analytic absoluteness) The interpretation map between Čech structures is a Σ_1-elementary embedding.

Question. If a closed set is definable in a Čech structure by a Π_1 formula, is its interpretation definable by the same formula?

Examples.

- the real line with addition and multiplication;
- topological groups;
- normed topological vector spaces;
- Banach algebras.

Functional analysis.

Theorem. If N is a closed vector subspace of X, then the quotient vector space is interpreted as the quotient of interpretations.

Theorem. The unit ball in the weak* topology of a Banach space is interpreted as the unit ball in the weak* topology of the interpretation.

Theorem. The normed dual of a uniformly convex X is interpreted as the normed dual of the interpretation of X.

Theorem. If X is compact and Y is metrizable, then $C(X,Y)$ with the compact-open topology is interpreted as $C(\hat{X},\hat{Y})$.

Theorem. If μ is a regular Borel measure on a locally compact space X and $\pi\colon X \to \hat{X}$ is an interpretation then there is a unique regular Borel measure $\hat{\mu}$ on $\hat{X}$ such that for every Borel set $B \subset X$, $\mu(B) = \hat{\mu}(\pi(B))$.

Haar measures on locally compact groups are interpreted as Haar measures again.

Faithfulness.

Theorem. If $M_0 \subset M_1 \subset M_2$ are transitive models, $M_0 \models X_0$ is Čech complete, $\pi_0 \colon X_0 \to X_1$ is an interpretation of X_0 in M_1 and $\pi_1 \colon X_1 \to X_2$ is an interpretation of X_1 in M_2 then $\pi_1 \circ \pi_0$ is an interpretation of X_0 in M_2.

Theorem. If $M \prec H_\theta$ is an elementary submodel containing Čech complete X and a basis for X as an element and subset, then the elementary embedding from $X \cap M$ to X is an interpretation.

Similarly for Borel complete spaces.

Example.

Theorem. Let $X = \omega^{\omega_1}$. Then faithfulness fails for X.

In a σ-closed extension, the interpretation of X^V is $X^{V[G]}$. On the other hand, if a ladder system is uniformized then the interpretation of X^V is not $X^{V[G]}$. So find $V \subset V[G] \subset V[H]$ so that

- both $V[G]$ and $V[H]$ are σ-closed extensions of V;
- $V[H]$ uniformizes a ladder from $V[G]$.

Preservation theorems.

The following properties of Čech complete spaces are preserved under interpretations:

- compactness;
- local compactness;
- complete metrizability;
- local connectedness;
- local metacompactness;
- local pseudocompactness.