ON ZETA FUNCTIONS OF BORCEA-VOISIN THREEFOLDS OVER $\mathbb{Q}$

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Abstract. We discuss a possible method for computing the Hasse-Weil zeta functions of Calabi-Yau threefolds of Borcea-Voisin type over $\mathbb{Q}$, up to finitely many Euler factors. Taking inspiration from the work by Batyrev [1] on birational invariance of Betti numbers for Calabi-Yau manifolds, we speculate that the zeta functions of a Borcea-Voisin threefold over $\mathbb{Q}$ may be obtained in certain cases by computing those of a simpler Calabi-Yau threefold constructed using the "twist map," which is amenable to explicit counting and toric methods as shown by by Goto-Kloosterman-Yui [7].

1. Introduction

This is a brief report on aspects of Borcea-Voisin type Calabi-Yau threefolds, studied by the authors as part of the Fields Undergraduate Research Program under the supervision of Noriko Yui. Collected here are basic background and observations made by the authors in still-ongoing investigation.

Originating from string theory, the mirror symmetry conjecture for Calabi-Yau manifolds predicts an intimate relationship between the symplectic geometry of a Calabi-Yau manifold and the complex geometry of its "mirror" Calabi-Yau. Apart from the rich geometric ramifications, arithmetic aspects of this phenomenon have also been studied in special cases, with interesting results (see for example [11]).

In general, given a Calabi-Yau manifold, it is a nontrivial task to construct a mirror manifold and to establish properties of the mirror pair. In this respect, the Borcea-Voisin construction of Calabi-Yau threefolds is remarkable in that the family of Calabi-Yau threefolds obtained via this method is closed under the mirror correspondence. Therefore, a detailed arithmetic study of the Borcea-Voisin threefolds would provide clues towards the general form of arithmetic mirror symmetry.

With this in mind, we describe a possible method of computing the Hasse-Weil zeta functions of the Borcea-Voisin threefolds defined over $\mathbb{Q}$, at least up to finitely many Euler factors. Following the ideas of Batyrev's paper [1] on birational invariance of Betti numbers of Calabi-Yau manifolds, we speculate that we may reduce the problem to computing the zeta functions of certain Calabi-Yau threefolds constructed via "twist maps." Varieties of this latter type were studied extensively by Goto-Kloosterman-Yui [7] using methods of Jacobi sums and toric desingularization.

In section 2, we give a brief review of Calabi-Yau manifolds and the mirror symmetry conjecture in the classical geometric setting. In section 3, we describe the Borcea-Voisin and the twist map construction. In section 4, we review some notions from arithmetic geometry, in particular the Hasse-Weil zeta function. We also discuss how the methods employed in the paper of Batyrev [1] may be used in our computation of the zeta functions for Bocrea-Voisin threefolds.

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2. Calabi-Yau manifolds and (topological) mirror symmetry

In this section, we recall the definition of a Calabi-Yau manifold and describe its Hodge diamond in the case of low dimension. We then give a simplified account of mirror symmetry and consider its implications on the Hodge diamonds of Calabi-Yau manifolds.

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Throughout this paper, all manifolds and varieties will be assumed connected unless otherwise stated. Given a complex manifold X, we shall denote by T_X (resp. Ω_X) the holomorphic tangent bundle (resp. holomorphic cotangent bundle) of X. We keep the same notation for the algebraic counterparts of the bundles if X is a smooth projective variety over a field k.

Definition 1. Let X be a smooth projective variety over a field k (or a compact Kähler complex manifold) of dimension d. We say that X is a *Calabi-Yau* manifold if its canonical bundle $K_X = \Omega_X^d$ is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.

In this section, we look at the transcendental (Hodge-theoretic) aspects of algebraic varieties, and focus on the case where X is a compact Kähler manifold. From Hodge theory, we know that each of the de Rham cohomology groups $H^k(X, \mathbb{C})$ of any compact Kähler manifold admits a direct sum decomposition as follows:

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p+q=k \\ 0 \le p,q \le d}} H^{p,q}(X),$$

where each $H^{p,q}(X)$ is the subspace spanned by the de Rham classes of closed (p,q)-forms on X, and is canonically isomorphic to $H^q(X, \Omega_X^p)$. As the spaces $H^k(X, \mathbb{C})$ are finite-dimensional, so are $H^{p,q}(X)$. We record the *Hodge numbers* $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(X)$ of X conveniently in the *Hodge diamond* of X as follows:

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & \vdots & h^{0,1} & & & \\
 & & \iddots & \vdots & & \vdots & \ddots & & \\
h^{d,0} & \cdots & & & \vdots & & & \cdots & h^{0,d} \\
 & & \ddots & \vdots & & \vdots & \iddots & & \\
 & & & h^{d,d-1} & \vdots & h^{d-1,d} & & & \\
 & & & & h^{d,d} & & & &
\end{array}$$

The Hodge diamond of a compact Kähler manifold satisfies horizontal and vertical symmetries $h^{p,q} = h^{q,p}$ and $h^{p,q} = h^{d-q,d-p}$, for all integers p, q. Horizontal symmetry follows from the complex conjugation property $H^{p,q}(X) = \overline{H^{q,p}(X)}$ of the Hodge decomposition, and vertical symmetry follows from the Hard Lefschetz theorem.

We describe the general shape of the Hodge diamond of a Calabi-Yau manifold in low dimension. Let X be a Calabi-Yau manifold. If $\dim X = 1$, then X is an *elliptic curve*, and its Hodge diamond is given as

$$\begin{array}{ccc}
 & 1 & \\
1 & & 1 \\
 & 1 &
\end{array}$$

If $\dim X = 2$, then X is a *K3 surface*, with Hodge diamond

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

If $\dim X = 3$, then X is called a *Calabi-Yau threefold*. Its Hodge diamond takes the following form

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

We now give a brief account of mirror symmetry. Let X be a Calabi-Yau threefold, where we assume that $H^0(X, T_X) = 0$; this implies that the (holomorphic) automorphism group of X is discrete. For simplicity, we also assume that $H^{2,0}(X) = 0$ (this excludes the case of $K3$ surfaces only). By the Bogomolov-Tian-Todorov theorem, the complex deformation theory of X is unobstructed, and hence the moduli space $\mathrm{Def}(X)$ of complex deformations of X is a smooth complex manifold, with tangent space at the moduli point $[X]$ isomorphic to $H^1(X, T_X)$ via the Kodaira-Spencer map.

There is another type of moduli attached to a Calabi-Yau manifold X (with fixed complex structure), called the *complexified Kähler moduli.* We define the moduli space $K(X)$ to be the open subset of $H^2(X, \mathbb{C})/H^2(X, \mathbb{Z})$ consisting of classes of the form

$$\alpha = \beta + i\omega,$$

where $\beta, \omega \in H^2(X, \mathbb{R})$ and ω is a Kähler class for the complex structure on X. (The form β arises from considerations in physics.) It is obvious that the tangent space to $K(X)$ at any point is canonically isomorphic to $H^2(X, \mathbb{C}) = H^1(X, \Omega_X)$; recall that we assumed $H^{2,0}(X) = 0$.

In its most basic form, the mirror symmetry conjecture can be stated as follows:

Conjecture 2. *Let X be a Calabi-Yau manifold. Then there exists another Calabi-Yau manifold $X^\vee$, called the* mirror *of X, of the same dimension and such that we have local isomorphisms*

$$K(X) \longleftrightarrow \mathrm{Def}(X^\vee) \quad \text{and} \quad \mathrm{Def}(X) \longleftrightarrow K(X^\vee).$$

In particular, comparing tangent spaces to the moduli spaces, we must have $H^1(X, \Omega_X) \cong H^1(X^\vee, T_{X^\vee})$. More generally, one expects in mirror symmetry that

$$H^q(X, \Omega_X^p) \cong H^q(X^\vee, \Lambda^p T_{X^\vee}) \cong H^q(X^\vee, \Omega_{X^\vee}^{d-p}), \quad \forall\, 0 \le p \le d,$$

where $d = \dim X = \dim X^\vee$ and the last isomorphism follows by interior product with a holomorphic volume form on $X^\vee$. This indicates that the Hodge diamond of $X^\vee$ is obtained by a "flip" of the Hodge diamond of X along a certain diagonal, as shown in the following example for Calabi-Yau threefolds. If the Hodge diamond of a Calabi-Yau threefold X is given by

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

then the Hodge diamond of $X^\vee$ should take the form

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Remark. Note that the conjecture as stated above cannot be true for all Calabi-Yau manifolds. Indeed, if X is a *rigid* Calabi-Yau threefold, meaning that $h^{2,1}(X) = 0$, then its mirror in the above sense would have to satisfy $h^{1,1}(X^\vee) = h^{2,1}(X) = 0$. But this is impossible if $X^\vee$ is to be a compact Kähler manifold. Indeed, the exterior power $\frac{1}{d!}\omega^d$ of a Kähler form ω is the volume form of X for the Riemannian metric it induces, and therefore we must have

$$\frac{1}{d!}\int \omega^d = \mathrm{vol}(X) > 0,$$

which is impossible by Stokes' theorem if ω is exact.

In reality, mirror symmetry is much more extensive than the "topological" version described above. For example, the local isomorphism between $K(X)$ and $\mathrm{Def}(X^\vee)$ should identify certain trilinear forms defined on these spaces, called the *Yukawa couplings*. Classically, this allowed Candelas et al. [3] to make predictions about the number of rational curves (defined using Gromov-Witten invariants) on a quintic threefold X by studying the structure (period integrals) of the deformation space $\mathrm{Def}(X^\vee)$ of its mirror. This prediction was confirmed rigorously and in greater generality by Givental [6]. See [4] for an introductory account of classcial mirror symmetry.

3. The Borcea-Voisin Construction

In general, it is very difficult to find and work with explicit mirror pairs. The mirror correspondence is closed in the family of Borcea-Voisin threefolds, and so these varieties are a great testing ground for the conjectures in mirror symmetry on Calabi-Yau threefolds. We briefly describe the Borcea-Voisin construction now.

Let E be an elliptic curve with canonical involution ι (we are working in characteristic 0), and let S be a $K3$ surface with (non-symplectic) involution σ acting by -1 on $H^{2,0}(S)$. Then a crepant resolution

$$X_{BV} = \widetilde{E \times S/\iota \times \sigma}$$

of the diagonal quotient $E \times S/\iota \times \sigma$ yields a Calabi-Yau threefold, which we call a *Borcea-Voisin threefold*. (See [2], [12] for details of the construction.)

To motivate the study of Borcea-Voisin threefolds, we mention a result due to Voisin below (Proposition 3), which gives explicit formulas for their middle Hodge numbers using information about the fixed locus S^σ of S under σ. To begin with, in order to resolve the quotient threefold $E \times S/\iota \times \sigma$ we must find the fixed locus of $\iota \times \sigma$ on $E \times S$. It is well known that an involution has four fixed points on an elliptic curve, say $P_1, \dots, P_4$. It can also be shown that the fixed locus of a non-symplectic involution on S is a disjoint union of curves. Hence, we may write

$$S^\sigma = C_1 \cup C_2 \cup \cdots \cup C_N,$$

where each curve C_i has genus g_i. Thus, the singularities on the threefold are the products $P_j \times C_i$ where $1 \le j \le 4$ and $1 \le i \le N$. Using this information, one can look for classes in the respective cohomology groups and show the following.

Proposition 3 (Voisin [12]). *Let $X_{BV} = \widetilde{E \times S/\iota \times \sigma}$ be a Borcea-Voisin threefold. Then*

$$h^{1,1}(X_{BV}) = 11 + 5N - N',$$
$$h^{2,1}(X_{BV}) = 11 + 5N' - N,$$

where N is the number of curves in the fixed locus S^σ and $N' = \sum g_i$ is the sum of their genera.

Remark. Given a Borcea-Voisin threefold X_{BV} constructed from elliptic curve E and $K3$ surface S, the (topological) mirror threefold $X_{BV}^\vee$ is obtained by applying the same Borcea-Voisin construction to a suitable "mirror" $K3$ surface $S^\vee$ (see [12] for details).

Considerations of [8] show that the Borcea-Voisin threefolds may be defined over $\mathbb{Q}$, at least for certain examples whose constituent varieties E and S have defining equations with rational coefficients and involutions are given as multiplication by -1 on one of the variables.

We now consider a different construction of Calabi-Yau threefolds, using the so-called "twist map." This construction will be useful for finding a birational model of the Borcea-Voisin threefold over $\mathbb{Q}$ which is computationally more feasible. For this, we first review the notion of weighted projective space.

Definition 4. Let $(w_0, \dots, w_n)$ be an $(n+1)$-tuple of positive integers, and let k be a field. Assume that each w_i is coprime to the characteristic of k. We define the *weighted projective space* $\mathbb{P}(w_0, \dots, w_n)$ over k with weights $(w_0, \dots, w_n)$ to be the variety $\mathrm{Proj}\, k[x_0, \cdots, x_n]$, where we set $\deg x_i = w_i$.

In case $k = \mathbb{C}$, we may informally consider $\mathbb{P}(w_0, \dots, w_n)$ as the quotient $\mathbb{C}^{n+1}/\sim$, where the equivalence $\sim$ is generated by $(x_0, x_1, \dots, x_n) \sim (\lambda^{w_0} x_0, \lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n)$ for all $\lambda \in \mathbb{C}^*$. From the above definition, we automatically have

$$\mathbb{P}^n(1, 1, \dots, 1) \simeq \mathbb{P}^n$$

for all n. Without loss of generality, we can always assume our weights are *normalized*, in that

$$\gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) = 1$$

for all i, $0 \le i \le n$. This is follows from the observation that $\mathbb{P}^n(aw_0, aw_1, \ldots, aw_n) \simeq \mathbb{P}^n(w_0, w_1, \ldots, w_n)$ for any positive integer a, and that moreover, setting

$$d_i = \gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n),$$

$$a_i = \operatorname{lcm}(d_0, \ldots, d_{i-1}, d_{i+1}, d_n),$$

we have the isomorphism

$$\mathbb{P}^n(w_0, w_1, \ldots, w_n) \simeq \mathbb{P}^n(w_0/a_0, w_1/a_1, \ldots, w_n/a_n).$$

See Dolgachev [5] for more details, including proofs of the above claims.

Yonemura [15] classified $K3$ surfaces S with non-symplectic involutions in (normalized) weighted projective space, defined over $\mathbb{Q}$. It is not too difficult to find defining equations for elliptic curves in weighted projective spaces, also defined over $\mathbb{Q}$ (see [7]). Now, suppose our elliptic curve and $K3$ surface have defining equations

$$E : x_0^2 + f(x_1, x_2) = 0 \subset \mathbb{P}^2(w_0, w_1, w_2),$$

$$S : y_0^2 + g(y_1, y_2, y_3) = 0 \subset \mathbb{P}^3(v_0, v_1, v_2, v_3),$$

with $\gcd(w_0, v_0) = 1$. Then there exist integers s_0 and t_0 such that $s_0w_0 + t_0v_0 = -1$. Moreover, we can assume $0 \le s_0 < v_0$ and $0 \le t_0 < w_0$. Then $s = (s_0w_0 + 1)/v_0$ and $t = (t_0v_0 + 1)/w_0$ are both non-zero integers by assumption. The *twist map* is the rational map $\Phi : \mathbb{P}^2(w_0, w_1, w_2) \times \mathbb{P}^3(v_0, v_1, v_2, v_3) \dashrightarrow \mathbb{P}^4(v_0w_1, v_0w_2, w_0v_1, w_0v_2, w_0v_3)$ given by

$$((x_0, x_1, x_2), (y_0, y_1, y_2, y_3)) \mapsto (x_0^{s_0w_1}y_0^{tw_1}x_1, \ldots, x_0^{s_0w_2}y_0^{tw_2}x_2, x_0^{sv_1}y_0^{t_0v_1}y_1, \ldots, x_0^{sv_3}y_0^{t_0v_3}y_3).$$

Restricted to $E \times S$, the map is generically 2-to-1 onto the variety

$$X = \{f(z_1, z_2) - g(u_1, u_2, u_3) = 0\} \subset \mathbb{P}^4(v_0w_1, v_0w_2, w_0v_1, w_0v_2, w_0v_3).$$

This variety is defined over $\mathbb{Q}$ if both f and g (i.e., E and S) are, and moreover, X is birational to the quotient $E \times S/\iota \times \sigma$ over $\mathbb{Q}$. (See [8, Section 7] for details.) Let $\widetilde{X}$ be a smooth resolution of X. Combining this with the Borcea-Voisin construction described above, we have a commutative diagram

$$\begin{tikzcd} & E \times S \arrow[d] \arrow[dr] & & \\ X_{BV} = \widetilde{E \times S/\iota \times \sigma} \arrow[r] & E \times S/\iota \times \sigma \arrow[r, dashed] & X & \widetilde{X} \arrow[l] \end{tikzcd}$$

in which the bottom horizontal arrows are birational maps (defined over $\mathbb{Q}$ under certain conditions), and hence $\widetilde{X}$ is birationally equivalent to the Borcea-Voisin threefold X_{BV} defined by E and S. Now, in weighted projective space, a sufficient condition for the resolution $\widetilde{X}$ to be Calabi-Yau is that (cf. [7, Proposition 3.2]):

$$2w_0v_0 = w_0 \sum_{i=1}^{3} v_i + v_0 \sum_{j=1}^{2} w_j.$$

Therefore, under the appropriate conditions mentioned above, one obtains a Calabi-Yau threefold over $\mathbb{Q}$ which is obtained by resolving singularities of a weighted projective hypersurface and which is birational over $\mathbb{Q}$ to the Borcea-Voisin threefold under consideration.

Remark. The twist map can be defined more generally, for varieties

$$V_1 : \{x_0^\ell + f(x_1, \ldots, x_n) = 0\} \subset \mathbb{P}^n(w_0, \ldots, w_n),$$

$$V_2 : \{y_0^\ell + g(y_1, \ldots, y_m) = 0\} \subset \mathbb{P}^m(v_0, \ldots, v_m),$$

similar to the work above, giving a rational map

$$\mathbb{P}^n(w_0, \ldots, w_n) \times \mathbb{P}^m(v_0, \ldots, v_m) \dashrightarrow \mathbb{P}^{n+m-1}(v_0w_1, \ldots, v_0w_n, w_0v_1, \ldots, w_0v_m).$$

4. Arithmetic Geometry and Batyrev's theorem

In this section, we recall the notions of the (local and global) zeta functions defined on an algebraic variety over $\mathbb{Q}$. We then proceed to state and sketch the proof of Batyrev's result on the birational invariance of Betti numbers for Calabi-Yau varieties under birational equivalence, and comment on how the method may be used in the computation of zeta functions for Borcea-Voisin threefolds over $\mathbb{Q}$.

Note that, while the background below is presented for schemes over $\mathbb{Q}$ and $\mathbb{Z}$, the theory is indeed valid over any number field K and its ring of integers $\mathcal{O}_K$.

Let $\mathcal{X}$ be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. For any $x \in \mathcal{X}$, one can show that x is a closed point if and only if the residue field $k(x)$ of x is finite. We shall denote by $\overline{\mathcal{X}}$ the set of closed points of $\mathcal{X}$, and write $N(x) = |k(x)|$ for any $x \in \overline{\mathcal{X}}$.

Definition 5. The *zeta function* of $\mathcal{X}/\mathbb{Z}$ is defined by the infinite product

$$\zeta(\mathcal{X}, s) = \prod_{x \in \overline{\mathcal{X}}} \left(1 - \frac{1}{N(x)^s}\right)^{-1}.$$

One can show that there are only a finite number of points $x \in \overline{\mathcal{X}}$ of any given norm, and hence the above product is given by a formal Dirichlet series $\sum a_n/n^s$. In fact, denoting by $\dim \mathcal{X}$ the dimension of $\mathcal{X}$ as a noetherian topological space, we have the following convergence result (cf. [10]):

Theorem 6. *The product $\zeta(\mathcal{X}, s)$ converges absolutely for $\Re(s) > \dim \mathcal{X}$.*

Remark. It is conjectured (and unknown in general) that, for any $\mathcal{X}$ of finite type, the function $\zeta(\mathcal{X}, s)$ can be analytically continued as a meromorphic function to the entire complex s-plane.

Since $\overline{\mathcal{X}}$ can be written as a disjoint union $\overline{\mathcal{X}} = \coprod_p \overline{\mathcal{X}}_p$ where each $\mathcal{X}_p = \mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_p$ is the fibre over $p \in \operatorname{Spec}\mathbb{Z}$, it follows that

$$\zeta(\mathcal{X}, s) = \prod_p \zeta(\mathcal{X}_p, s).$$

The factors in the righthand product are called the *Euler factors* of the zeta function $\zeta(\mathcal{X}, s)$. Defining

$$Z_p(\mathcal{X}_p, T) = \exp\left(\sum_{k=1}^{\infty} \frac{|\mathcal{X}_p(\mathbb{F}_{p^k})|}{k} T^k\right),$$

a simple computation shows that we have $\zeta(\mathcal{X}_p, s) = Z_p(\mathcal{X}_p, p^{-s})$.

Now, let X be an algebraic variety (i.e. reduced, separated scheme of finite type) over $\mathbb{Q}$. Recall that an *integral model* $\mathcal{X}/\mathbb{Z}$ of $X/\mathbb{Q}$ is a scheme $\mathcal{X}$ flat, surjective, and of finite type over $\operatorname{Spec}\mathbb{Z}$ having generic fibre $\mathcal{X}_\eta \simeq X$ (isomorphism over $\mathbb{Q}$). Naïvely, choosing an integral model amounts to choosing defining polynomials with integer coefficients for the variety.

Lemma 7. *If $\mathcal{X}$ and $\mathcal{X}'$ are two integral models of an algebraic variety $X/\mathbb{Q}$, then*

$$\zeta(\mathcal{X}, s) \sim \zeta(\mathcal{X}', s),$$

where the symbol $\sim$ indicates equality up to finitely many Euler factors.

Proof. Since we have an isomorphism $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Q} \cong X \cong \mathcal{X}' \times_{\mathbb{Z}} \mathbb{Q}$ where $\mathbb{Q} = \mathcal{O}_{\operatorname{Spec}\mathbb{Z},(0)}$, there exists an open $U \subseteq \operatorname{Spec}\mathbb{Z}$ such that $\mathcal{X} \times_{\mathbb{Z}} U \cong \mathcal{X}' \times_{\mathbb{Z}} U$ as schemes over U (cf. [9, Exercise 3.2.5]). Since U contains all but finitely many primes, we have the desired result. □

In view of the above lemma, we may define, up to finitely many Euler factors, the *Hasse-Weil zeta function* $\zeta(X, s)$ of the variety $X/\mathbb{Q}$ by

$$\zeta(X, s) \sim \zeta(\mathcal{X}, s)$$

for any integral model $\mathcal{X}/\mathbb{Z}$ of X.

Assume now that $X/\mathbb{Q}$ is a smooth projective variety of dimension d, and let $\mathcal{X}/\mathbb{Z}$ be a smooth, projective integral model of X. Then the projective variety $\mathcal{X}_p$ is smooth over $\mathbb{F}_p$ for almost all primes p. If $\mathcal{X}_p$ is smooth, then the Weil conjectures imply that the corresponding Euler factor $\zeta(\mathcal{X}_p, s) = Z_p(\mathcal{X}_p, p^{-s})$ of the Hasse-Weil zeta function has a particularly nice form. More precisely,

$$Z_p(\mathcal{X}_p, T) = \prod_{k=0}^{2d} P_{k,p}(\mathcal{X}_p, T)^{(-1)^{k+1}},$$

where each

$$P_{k,p}(\mathcal{X}_p, T) = \det(1 - TF_p^{-1} | H^k_{\text{ét}}(\mathcal{X}_p \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)) \quad (\ell \neq p)$$

is the characteristic polynomial of the Frobenius $F_p = (x \mapsto x^p) \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acting on $\mathcal{X}_p \otimes \overline{\mathbb{F}}_p$ by $\mathrm{id}_{\mathcal{X}_p} \times F_p^*$and hence on $H^k_{\text{ét}}(\mathcal{X}_p \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)$. It is a consequence of the Weil conjectures that each $P_{k,p}(\mathcal{X}_p, T)$ is a polynomial with integer coefficients and its reciprocal roots have absolute value $p^{k/2}$.

We now state Batyrev's theorem [1] and sketch its proof.

Theorem 8 (Batyrev [1])**.** *Let X and Y be birationally equivalent projective Calabi-Yau threefolds over $\mathbb{C}$. Then they have the same Betti numbers, i.e., $\dim H^n(X, \mathbb{C}) = \dim H^n(Y, \mathbb{C})$ for all integers n.*

Sketch of proof. By a standard procedure, there exists a finitely generated $\mathbb{Z}$-subalgebra $\mathcal{R}$ of $\mathbb{C}$ and regular projective schemes $\mathcal{X}$ and $\mathcal{Y}$ over $\mathrm{Spec}\,\mathcal{R}$ such that $X = \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{R}} \mathrm{Spec}\,\mathbb{C}$ and $Y = \mathcal{Y} \times_{\mathrm{Spec}\,\mathcal{R}} \mathrm{Spec}\,\mathbb{C}$. One may further impose additional conditions on these schemes $\mathcal{X}$ and $\mathcal{Y}$, special to Calabi-Yau varieties; in particular, they imply certain identification of p-adic measures on $\mathcal{X}$ and $\mathcal{Y}$.

For all but finitely many primes p, there exists a closed point $\pi \in \mathrm{Spec}(\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ coming from a prime of $\mathrm{Spec}\,\mathcal{R}$ such that reduction of $\mathcal{X}$ and $\mathcal{Y}$ modulo π yields smooth varieties over some finite field F of characteristic p. Using p-adic analysis and Weil's results [14] relating p-adic integrals fo the number of rational points in these reductions, one obtains that $|\mathcal{X}(F)| = |\mathcal{Y}(F)|$. Applying this procedure to certain extensions of $\mathcal{R}$, one can show that in fact $\mathcal{X}$ and $\mathcal{Y}$ have the same local zeta function. From the Weil conjectures and the comparison theorem in cohomology, one deduces by comparing polynomials with reciprocal roots of equal absolute value in the decomposition of the zeta functions that X and Y have the same Betti numbers, as desired. □

It is speculated that, in the case where X and Y are Calabi-Yau varieties defined over $\mathbb{Q}$ and are birationally equivalent over $\mathbb{Q}$, the $\mathbb{Z}$-algebra $\mathcal{R}$ in the above proof may be taken to be a suitable localization $\mathbb{Z}[1/N]$ of $\mathbb{Z}$ itself for some nonzero integer N. Applying the above argument for each prime $p \in \mathrm{Spec}\,\mathbb{Z}[1/N]$ of good reduction, it may thus be possible to show that X and Y have the same Hasse-Weil zeta function, at least up to finitely many Euler factors.

In the previous section, for each Calabi-Yau threefold X_{BV} of Borcea-Voisin type over $\mathbb{Q}$ we associated a Calabi-Yau threefold $\widetilde{X}$ arising as a resolution of a weighted projective hypersurface. Applying the above speculation to this situation, we see that the zeta function of X_{BV} over $\mathbb{Q}$ may be obtained by computing that of $\widetilde{X}$, which can be done using Weil's method and toric geometry methods, see [7].

References

[1] V.V. Batyrev. *Birational Calabi-Yau n-folds have equal Betti numbers*, in New trends in algebraic geometry (Warwick, 1996). London Math. Soc. Lecture Note Ser., 264. Cambridge Univ. Press, Cambridge (1999). 1–11

[2] C. Borcea. *K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds*, in Mirror Symmetry. AMS/IP Stud. Adv. Math. 1, American Mathematical Society, Providence, RI (1997) pp. 717-743.

[3] P. Candelas, X.C. de La Ossa, P.S. Green, and L. Parkes *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory.* Nuclear Physics B, Volume 359, Issue 1, p. 21-74 (1991).

[4] D.A. Cox and S. Katz. *Mirror Symmetry and Algebraic Geometry.* Mathematical Surveys and Monographs, Vol. 68. American Mathematical Society (1991).

[5] I. Dolgachev. *Weighted projective varieties*, in Group Actions and Vector Fields. Lecture Notes in Mathematics, 956, Springer-Verlag (1982) pp. 34-72.

[6] A. Givental. *Equivariant Gromov-Witten invariants.* Internat. Math. Res. Notices 1996 (13), 613-663 (1996).

[7] Y. Goto, R. Kloosterman, and N. Yui. *Zeta-functions of certain K3-fibered Calabi-Yau threefolds.* Internat. J. Math. 22, No.1, 67129 (2011).

[8] Y. Goto, R. Livné, and N. Yui. *The automorphy of certain K3 fibered Calabi-Yau threefolds with involution over* $\mathbb{Q}$. arXiv:1212.3399v2 [math.NT]

[9] Q. Liu *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics. Oxford University Press (2002).

[10] J.-P. Serre. *Zeta and L-functions*, in Arithmetical Algebraic Geometry 1963. Collected Papers II, #64.

[11] D. Wan. *Mirror Symmetry For Zeta Functions*, in Mirror Symmetry V. AMS/IP Advanced Studies in Mathematics, Vol. 38 (2007) 159-184.

[12] C. Voisin. *Miroirs et involutions sur les surfaces K3*, Journées de Géométrie Algeébrique d'Orsay, Astérisque 218 (1993), pp. 273-323.

[13] C. Voisin. *Hodge Theory and complex algebraic geometry. I.* Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge (2002).

[14] A. Weil. *Adéles and algebraic groups*, Progr. Math. 23, birkhäuser, Boston (1982).

[15] H. Yonemura. *Hypersurface simple K3 singularities*, Tôhoku J. Math. 42 (1990), pp. 351-380.