Complex Vector Bundles over Higher-dimensional Connes-Landi Spheres

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The θ-deformed $C(S^n_\theta)$

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Definition

Let $(\theta)_{ij}$ be an $m \times m$ real-valued skew-symmetric matrix, and let $\rho_{ij} = \exp(2\pi i \theta_{ij})$.
The θ-deformed $2m-1$-sphere $C(S^{2m-1}_\theta)$ is the universal C*-algebra generated by m normal elements $z_1, ..., z_m$ satisfying the relations

$$z_1 z_1^* + ... + z_m z_m^* = 1, \quad z_i z_j = \rho_{ji} z_j z_i.$$

The θ-deformed $2m$-sphere $C(S^{2m}_\theta)$ is the universal C*-algebra generated by m normal elements $z_1, ..., z_m$ and a hermetian element x satisfying the relations

$$z_1 z_1^* + ... + z_m z_m^* + x^2 = 1, \quad z_i z_j = \rho_{ji} z_j z_i, \quad [x, z_i] = 0.$$

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We note that $C(S^{2m-1}_\theta)$ is obviously a quotient of $C(S^{2m}_\theta)$ (so that S^{2m-1}_θ is the "equator" of S^{2m}_θ), but that $C(S^{2m-2}_\theta)$ is apparently not a quotient of $C(S^{2m-1}_\theta)$.

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The $C(S^n_\theta)$ are (strict) deformation quantizations of S^n by actions of the appropriate T^m(periodic actions of $\mathbb{R}^m$), and so have the same K-groups as $C(S^n)$ [22, 27, 25, 23]. The $C(S^n_\theta)$ are intimately related to the noncommutative tori $C(T^m_\theta)$ [20], being continuous fields of noncommutative tori (with some degenerate fibers) in exactly the same way that S^n decomposes as an orbit space for the action of T^m [16, 18].

Each $C(S^n_\theta)$ admits the structure of a spectral triple, and satisfies the (tentative) axioms [4, 8] of a "noncommutative *Spin*$^{\mathbb{C}}$ manifold".

The $C(S^n_\theta)$ are (completions) of solutions of homological equations satisfied by (the coordinate algebras of) ordinary spheres, but not by, for example, the q-deformed spheres $C(S^n_q)$ of Podleś [19, 10].

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The moduli space of solutions of the homological equations for the case $n = 3$ has been determined by Connes and Dubois-Violette. Critical values of the moduli space are the full polynomial $*$-subalgebras of the $C(S^3_\theta)$'s, while generic values are (quotients of) the Sklyanin algebras of noncommutative algebraic geometry [26].

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Instanton solutions of the Euclidean Yang-Mills equations for S^4_θ and their moduli have been extensively studied (e.g.[2, 3, 13]), inspired both by the classical work of Atiyah, Ward, Donaldson, etc. [1, 11], and also by investigations of the gauge theories of the noncommutative tori [9, 5, 15] and of the Moyal-deformed 4-plane [17, 24, 12]. Despite this, numerous fundamental questions remain open or unexplored.

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If $k \geq [\frac{n}{2}]$, then the map $\pi_{n-1}(GL_k(\mathbb{C})) \to K^{-n \mod 2}(S^{n-1})$ is an isomorphism, but as n increases, these homotopy groups become difficult to compute for $k < [\frac{n}{2}]$. Cancellation fails for the semigroup of isomorphism classes of complex vector bundles over S^n for $n \geq 5$. For example, S^5 has only one nontrivial bundle over it, coming from the fact that $\pi_4(S^3) \cong \mathbb{Z}_2$.

if $n \neq 2$, then S^n has no nontrivial line bundles.

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Theorem

If θ is totally irrational, then all finitely-generated projective $C(S^{2m-1}_\theta)$-modules are free,, i.e. all "complex vector bundles" over S^{2m-1}_θ are trivial, and $V(S^{2m-1}_\theta)$ satisfies cancellation.

Theorem

Let θ be totally irrational. Then

$$V(S_\theta^{2m}) \cong \{0\} \cup (\mathbb{N} \times K_1(C(S_\theta^{2m-1}))) \cong \{0\} \cup (\mathbb{N} \times \mathbb{Z}).$$

Thus every complex vector bundle over S_θ^{2m} decomposes as the direct sum of a "line bundle" and a trivial bundle, and cancellation holds.

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If θ contains a mix of rational and irrational terms, then somewhat surprising phenomena can occur. For instance if $n = 5$ and θ consists of one irrational entry (besides its negative) and all other entries are zero, then S_θ^5 has $\mathbb{Z} \times \mathbb{Z}$-many nontrivial "line bundles" over it, but all bundles of higher rank are trivial. For higher n torsion phenomena can occur. Also, for $n \geq 7$ it is possible for θ to contain certain mixes of rational and irrational terms and for cancellation to still hold, though for generic mixed θ cancellation fails.

Idea of the Proofs of the Theorems.

As a generalization of the genus-1 Heegaard splitting of S^3, one sees that

$$S^{2m+1} = (D^{2m} \times S^1) \cup_{S^{2m-1} \times S^1} (S^{2m-1} \times D^2).$$

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Thus we can view S_θ^{2m+1} as consisting of $(D^{2n} \times S^1)_\theta$ and $(S^{2n-1} \times D^2)_\theta$ "hemispheres" glued together over a $(S^{2n-1} \times S^1)_\theta = C(S_{\theta'}^{2n-1}) \times_\alpha \mathbb{Z}$ "equator". We can view S_θ^{2m} as two D_θ^{2m} hemispheres over a S_θ^{2m-1} equator. We prove the theorems simultaneously with obtaining the homotopy-theoretic results that

$$\pi_0(GL_k(C(S_{\theta'}^{2n-1}))) \cong \mathbb{Z}$$

and

$$\pi_0(GL_k(C(S_{\theta'}^{2n-1}) \times_\alpha \mathbb{Z})) \cong \mathbb{Z} \times \mathbb{Z}.$$

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Actually to do this our first move is to show that the map

$$\pi_j(GL_k(C(S_\theta^{2n-1})\times_{\alpha_1}\mathbb{Z}...\times_{\alpha_r}\mathbb{Z})) \to K_{1-j \mod 2}(C(S_\theta^{2n-1})\times_{\alpha_1}\mathbb{Z}...\times_{\alpha_r}\mathbb{Z})$$

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The argument uses Rieffel's [21] result that, so long as θ contains at least one irrationaly entry, then

$$\pi_j(GL_k(C(T_\theta^m))) \cong \mathbb{Z}^{2^{m-1}},$$

along with using the Pimsner-Voiculescu sequence and K-theory and unstabilized homotopy versions of Mayer-Vietoris.

The case n=4.

We can give a very explicit description of the modules $M(k,s)$ in this case. The Rieffel projection [20] $p = M_g V + M_f + V^* M_g$ of trace $|\theta| \mod 1$ plays a central role in the construction.

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Viewing $C(S^3_\theta)$ as a continuous field of noncommutative 2-tori over $[0,1]$, we consider the invertible

$$X = \exp(2\pi i t)p + 1 - p \in C(S^3_\theta),$$

where p is a Rieffel projection with trace $|\theta|$ mod 1. (note that X corresponds to the image of p under the Bott map $K_0(C(T^2_\theta)) \to K_1(SC(T^2_\theta)))$.

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Lemma

Let θ be irrational. Then the natural map $\pi_0(GL_j(C(S^3_\theta))) \to K_1(C(S^3_\theta)) \cong \mathbb{Z}$ is an isomorphism for all $j \geq 1$. The invertible X is a generator of $\pi_0(GL_1(C(S^3_\theta)))$.

It follows that one can take the representative $M(k,s)$ to be the result of using the image of X^s in $GL_k(S^3_\theta)$ as a clutching element.

We obtain $M(1,s) \cong PC(S^4_\theta)^2$, where

$$P = \frac{1}{2}\begin{pmatrix} 1+x & (1-x^2)^{1/2}X \\ (1-x^2)^{1/2}X^* & 1-x \end{pmatrix}$$

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(here x is the hermetian generator x from the definition of $C(S^4_\theta)$.).

The first example of an $C(S^4_\theta)$-module to appear in the literature is the "instanton bundle of charge-1" $eC(S^4_\theta)^4$ discovered by Connes and Landi, is given by

$$e := \frac{1}{2}\begin{pmatrix} 1+x & 0 & z_2 & z_1 \\ 0 & 1+x & -\rho z_1^* & z_2^* \\ z_2^* & -\bar{\rho} z_1 & 1-x & 0 \\ z_1^* & z_2 & 0 & 1-x \end{pmatrix},$$

where $\rho = \exp(2\pi i\theta)$, and the z_i and x are the generators of $C(S^4_\theta)$. The Levi-Civita connection *ede* gives an instanton solution to the Euclidean Yang-Mills equations for S^4_θ. In the case $\theta = 0$, the projection e corresponds to the complex rank-2 vector bundle E_1 over S^4 with second Chern number (charge) 1. The Levi-Civita connection *ede* is then a charge-1 instanton on E_1.

Proposition

Let e be Connes and Landi's instanton projection, and let θ be irrational. Then the corresponding module $eC(S^4_\theta)^4$ is isomorphic to $M(1,-1) \oplus C(S^4_\theta)$.

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The Proposition follows from first showing that $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} z_2 & z_1 \\ -\rho z_1^* & z_2^* \end{pmatrix}$ are path-connected in $GL_2(C(S^3_\theta))$, and then seeing that $eC(S^4_\theta)^4$ results from clutching using $\begin{pmatrix} z_2^* & -\bar{\rho} z_1 \\ z_1^* & z_2 \end{pmatrix}$.

Thus the basic rank-2 instanton bundle for S^4_θ splits as the sum of a nontrivial line bundle and a trivial line bundle!

The invertible $X \in C(S^3_\theta)$ generates a C*-subalgebra $C^*(X) \cong C(S^1)$. One may "suspend" $C^*(X)$ by coning it twice, unitizing the cones, and then gluing them together to obtain a C*-subalgebra of $C(S^4_\theta)$ isomorphic to $C(S^2)$.

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Theorem

Suppose that θ is irrational. Then

$$M(k,s) \cong E(k,s) \otimes_{C(S^2)} C(S^4_\theta),$$

where $E(k,s)$ is the module of continuous sections of a rank-k complex vector bundle over S^2 with Chern number $-s$, and the inclusion $C(S^2) \hookrightarrow C(S^4_\theta)$ is as described above.

Thus every complex vector bundle over S^4_θ is the pullback of a complex vector bundle over S^2 via a certain fixed quotient map $S^4_\theta \to S^2$. The basic instanton bundle e of charge 1 over S^4_θ is just the pullback of the direct sum of the Bott bundle over S^2 with Chern number 1 and a trivial line bundle! This is intriguing as it provides a link between the classical Bott bundle on S^2 and a deformation of the charge-1 instanton bundle on S^4.

Further Directions

I have managed to calculate certain higher homotopy groups $\pi_k(GL_j(C(S^n_\theta)))$ for various k, j, n and θ and have obtained interesting values in many cases (e.g $\pi_0(GL_1(C(S^4_\theta))) \cong \mathbb{Z} \times \mathbb{Z}$, while replacing 1 with $j \geq 2$ yields zero).

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I am also investigating the gauge theory of the $C(S^n_\theta)$ as part of a larger project. It seems to me that $U(1)$ instantons for $C(S^4_\theta)$ should probably exist. There should also be a nontrivial monopole theory. The gauge theory for higher $C(S^n_\theta)$ could potentially be simpler than that for classical spheres.

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