DIFFERENT KINDS OF ESTIMATORS OF THE MEAN DENSITY OF RANDOM CLOSED SETS: THEORETICAL RESULTS AND NUMERICAL EXPERIMENTS

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In this talk...

A random set Θ_n in $\mathbb{R}^d$ of locally finite $\mathcal{H}^n$-measure induces a random measure

$$\mu_{\Theta_n}(A) := \mathcal{H}^n(\Theta_n \cap A), \qquad A \in \mathcal{B}_{\mathbb{R}^d},$$

and the corresponding expected measure

$$\mathbb{E}[\mu_{\Theta_n}](A) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \qquad A \in \mathcal{B}_{\mathbb{R}^d}.$$

Whenever $\mathbb{E}[\mu_{\Theta_n}] \ll \mathcal{H}^d$ on $\mathbb{R}^d$, its density, say λ_{Θ_n} is called *mean density* of Θ_n

A crucial problem is the pointwise estimation of $\lambda_{\Theta_n}(x)$.

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- *Kernel estimator* $\widehat{\lambda}^{\kappa,N}_{\Theta_n}(x)$
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- *"Minkowski content"-based estimator* $\widehat{\lambda}^{\mu,N}_{\Theta_n}(x)$
  It will follow by a local approximation of λ_{Θ_n} based on a stochastic version of the n-dimensional Minkowski content of Θ_n.

We remind that...

- A compact set $S \subset \mathbb{R}^d$ is called
  - ***n*-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;
  - **countably $\mathcal{H}^n$-rectifiable** if there exist countably many Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\Big(S \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n)\Big) = 0.$$

- A **random closed set** $(r.c.s.)\Theta$ in $\mathbb{R}^d$ is a measurable map

$$\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ is the class of the closed subsets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the σ-algebra generated by the so called *Fell topology*, or *hit-or-miss topology*.

IN WHAT FOLLOWS:

- Θ_n is a countably $\mathcal{H}^n$-rectifiable r.c.s. of locally finite $\mathcal{H}^n$-measure
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NOTE THAT: if $n = 0$ and $\Theta_0 = X$ **random vector with pdf** f_X, then

$$\mathbb{E}[\mathcal{H}^0(X \cap A)] = \mathbb{P}(X \in A) = \int_A f_X(x)\mathrm{d}x \quad \forall A \in \mathcal{B}_{\mathbb{R}^d}$$

and so $\lambda_X(x) = f_X(x)$.

The natural estimator $\widehat{\lambda}^{\nu,N}_{\Theta_n}(x)$

The Besicovitch derivation theorem implies that if

$$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x, \quad \forall A \in \mathcal{B}_{\mathbb{R}^d},$$

then

$$\lambda_{\Theta_n}(x) = \lim_{r\downarrow 0} \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d} \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

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This suggests the following **natural estimator** for the mean density $\lambda_{\Theta_n}(x)$ of Θ_n,

$$\widehat{\lambda}^{\nu,N}_{\Theta_n}(x) := \frac{1}{N b_d r_N^d} \sum_{i=1}^{N} \mathcal{H}^n(\Theta_n^i \cap B_{r_N}(x)).$$

Here and in the following r_N is called the **bandwidth** associated with the sample size N, as usual in literature.

The kernel estimator $\widehat{\lambda}^{\kappa,N}_{\Theta_n}(x)$

We remind that

- A measurable function $k : \mathbb{R}^d \to \mathbb{R}$ is said to be a **multivariate kernel** if it satisfies the following conditions:
  - $0 \leq k(z) \leq M$ for all $z \in \mathbb{R}^d$, for some $M > 0$;
  - k is radially symmetric;
  - $\int_{\mathbb{R}^d} k(z)dz = 1$.
- Given $X_1, \ldots, X_N$ i.i.d. random sample for X random vector with p.d.f f_X, the multivariate kernel density estimator of f_X based on a chosen kernel k, and scaling parameter $r_N \in (0, +\infty)$, is defined by

$$\widehat{f}^N_X(x) := \frac{1}{N}\sum_{i=1}^{N} k_{r_N} * \mathcal{H}^0_{|X_i}(x) = \frac{1}{Nr_N^d}\sum_{i=1}^{N} k\left(\frac{x - X_i}{r_N}\right)$$

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As a natural extension to the n-dimensional r.c.s, we define the following **kernel estimator for the mean density of Θ_n**:

$$\widehat{\lambda}^{\kappa,N}_{\Theta_n}(x) := \frac{1}{N}\sum_{i=1}^{N} k_{r_N} * \mathcal{H}^n_{|\Theta^i_n}(x) = \frac{1}{N r_N^d}\sum_{i=1}^{N}\int_{\Theta^i_n} k\left(\frac{x-y}{r_N}\right)\mathcal{H}^n(dy)$$

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NOTE THAT: $\widehat{\lambda}^{\nu,N}_{\Theta_n}(x) = \widehat{\lambda}^{\kappa,N}_{\Theta_n}(x)$ by choosing as kernel $k(z) = \frac{1}{b_d}\mathbf{1}_{B_1(0)}(z)$.

The "Minkowski content"-based estimator $\widehat{\lambda}^{\mu,N}_{\Theta_n}(x)$

We remind that

- The ***n*-dimensional Minkowski content** of a closed set $S \subset \mathbb{R}^d$ is the quantity $\mathcal{M}^n(S)$ so defined

$$\mathcal{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}}$$

provided the limit exists finite, where $A_{\oplus r} := \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) \leq r\}$.

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- If $S \subset \mathbb{R}^d$ is countably $\mathcal{H}^n$-rectifiable and compact, and

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \, , \, \forall r \in (0,1)$$

holds for some $\gamma > 0$ and some Radon measure η in $\mathbb{R}^d$, $\eta \ll \mathcal{H}^n$, then

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  for all $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathbb{E}[\mu_{\Theta_n}](\partial A) = 0$

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It can be proved, *under suitable regularity assumptions on Θ_n*, that [EV, *Bernoulli*, 2014]

$$\lambda_{\Theta_n}(x) = \lim_{r\downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}}, \qquad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d$$

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PROBLEM 1: regularity properties on Θ_n such that the 3 proposed estimators are asymptotically unbiased and consistent.

PROBLEM 2: optimal bandwidth r_N.

Assumptions and notation

Every **random closed set** Θ in $\mathbb{R}^d$ can be represented **as "particle process" (or "germ-grain process")**, and so described by a **marked point process** $\Phi = \{(X_i, S_i)\}_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ with marks in a suitable **mark space K** so that

$$\Theta(\omega) = \bigcup_{(x_i, s_i)\in\Phi(\omega)} x_i + Z(s_i), \qquad \omega \in \Omega,$$

where $Z_i = Z(S_i)$, $i \in \mathbb{N}$ is a random set containing the origin.
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We assume that Φ has **intensity measure**

$$\Lambda(\mathrm{d}(x,s)) = f(x,s)\mathrm{d}xQ(\mathrm{d}s)$$

and **second factorial moment measure**

$$\nu_{[2]}(\mathrm{d}(x,s,y,t)) = g(x,s,y,t)\mathrm{d}x\mathrm{d}yQ_{[2]}(\mathrm{d}(s,t))$$

It follows that $\lambda_{\Theta_n}(x) = \int_{\mathbf{K}}\int_{x-Z(s)} f(y,s)\mathcal{H}^n(\mathrm{d}y)Q(\mathrm{d}s)$

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Note that:

- if Θ_n Boolean model: $g(x,s,y,t) = f(x,s)f(y,t)$, and $Q_{[2]}(\mathrm{d}(s,t)) = Q(\mathrm{d}s)Q(\mathrm{d}t)$
- if $\Theta_0 = X$ random point with pdf f_X : $\Phi = (X,s)$, $Z(s) := s \in \mathbb{R}^d$, and

$$\Lambda(\mathrm{d}(y,s)) = f_X(y)\mathrm{d}y\delta_0(s)\mathrm{d}s, \qquad \nu_{[2]}(\mathrm{d}(x,s,y,t)) \equiv 0$$

In what follows:

$\alpha := (\alpha_1, ..., \alpha_d)$ multi-index of $\mathbb{N}_0^d$; $|\alpha| := \alpha_1 + \cdots + \alpha_d$;

$\alpha! := \alpha_1! \cdots \alpha_d!$ $y^\alpha := y_1^{\alpha_1} \cdots y_d^{\alpha_d}$;

$D_y^\alpha f(y,s) := \dfrac{\partial^{|\alpha|} f(y,s)}{\partial y_1^{\alpha_1} \cdots \partial y_d^{\alpha_d}}$; $\mathcal{D}^{(\alpha)}(s) := \mathrm{disc}(D_y^\alpha f(y,s))$.

k will be a kernel with compact support

Moreover, "under regularity assumptions" will mean that **some of the following assumptions are satisfied:**

(A1) for any $(y,s) \in \mathbb{R}^d \times \mathbf{K}$, $y + Z(s)$ is a countably $\mathcal{H}^n$-rectifiable and compact subset of $\mathbb{R}^d$, such that there exists a closed set $\Xi(s) \supseteq Z(s)$ such that $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s))Q(\mathrm{d}s) < \infty$ and

$$\mathcal{H}^n(\Xi(s) \cap B_r(x)) \geq \gamma r^n \quad \forall x \in Z(s),\ \forall r \in (0,1) \tag{1}$$

for some $\gamma > 0$ independent of s;

($\overline{A1}$) as (A1), replacing (1) with

$$\gamma r^n \leq \mathcal{H}^n(\Xi(s) \cap B_r(x)) \leq \widetilde{\gamma} r^n \quad \forall x \in Z(s),\ r \in (0,1)$$

for some $\gamma, \widetilde{\gamma} > 0$ independent of s;

(A1) for $\widehat{\lambda}_{\Theta_n}^{\mu,N}(x)$; ($\overline{A1}$) for $\widehat{\lambda}_{\Theta_n}^{\kappa,N}(x)$ and $\widehat{\lambda}_{\Theta_n}^{\nu,N}(x)$
in proving asymptotical unbiasedness and consistency

(A2) for any $s \in \mathbf{K}$, $\mathcal{H}^n(\mathrm{disc}(f(\cdot,s))) = 0$ and $f(\cdot,s)$ is locally bounded such that for any compact $K \subset \mathbb{R}^d$

$$\sup_{x \in K_{\oplus \mathrm{diam}(Z(s))}} f(x,s) \leq \widetilde{\xi}_K(s)$$

for some $\widetilde{\xi}_K(s)$ with

$$\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s))\widetilde{\xi}_K(s) Q(\mathrm{d}s) < \infty$$

(A2*) for $|\alpha = 2|$, for any $s \in \mathbf{K}$, $\mathcal{H}^n(\mathcal{D}^{(\alpha)}(s)) = 0$ and $D_y^\alpha f(y,s)$ is locally bounded such that, for any compact $C \subset \mathbb{R}^d$,

$$\sup_{y \in C_{\oplus \mathrm{diam} Z(s)}} |D_y^\alpha f(y,s)| \leq \widetilde{\xi}_C^{(\alpha)}(s)$$

for some $\widetilde{\xi}_C^{(\alpha)}(s)$ with

$$\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s))\widetilde{\xi}_C^{(\alpha)}(s) Q(\mathrm{d}s) < \infty$$

(A2) for all the 3 estimators in proving asymptotical unbiasedness and consistency; (A2*) for $\widehat{\lambda}_{\Theta_n}^{\kappa,N}(x)$ and $\widehat{\lambda}_{\Theta_n}^{\nu,N}(x)$ in finding the optimal bandwidth

(A3) for any $(s,y,t)\in \mathbf{K}\times\mathbb{R}^d\times\mathbf{K}$, $\mathcal{H}^n(\mathrm{disc}(g(\cdot,s,y,t)))=0$ and $g(\cdot,s,y,t)$ is locally bounded such that for any compact $K\subset\mathbb{R}^d$ and $a\in\mathbb{R}^d$,

$$\mathbf{1}_{(a-Z(t))_{\oplus 1}}(y)\sup_{x\in K_{\oplus \mathrm{diam}(Z(s))}} g(x,s,y,t)\le \xi_{a,K}(s,y,t)$$

for some $\xi_{a,K}(s,y,t)$ with

$$\int_{\mathbb{R}^d\times\mathbf{K}^2}\mathcal{H}^n(\Xi(s))\xi_{a,K}(s,y,t)\mathrm{d}yQ_{[2]}(\mathrm{d}s,\mathrm{d}t)<\infty. \tag{2}$$

($\overline{A3}$) for any $s,t\in\mathbf{K}$, $g(\cdot,s,\cdot,t)$ is locally bounded such that, for any $C,\overline{C}\subset\mathbb{R}^d$ compact sets:

$$\sup_{y\in\overline{C}_{\oplus \mathrm{diam} Z(t)}}\ \sup_{x\in C_{\oplus \mathrm{diam} Z(s)}} g(x,s,y,t)\le \xi_{C,\overline{C}}(s,t)$$

for some $\xi_{C,\overline{C}}(s,t)$ with

$$\int_{\mathbf{K}^2}\mathcal{H}^n(\Xi(s))\mathcal{H}^n(\Xi(t))\xi_{C,\overline{C}}(s,t)Q_{[2]}(\mathrm{d}s,\mathrm{d}t)<\infty. \tag{3}$$

(A3) for $\widehat{\lambda}^{\mu,N}_{\Theta_n}(x)$; ($\overline{A3}$) for $\widehat{\lambda}^{\kappa,N}_{\Theta_n}(x)$ and $\widehat{\lambda}^{\nu,N}_{\Theta_n}(x)$
in proving asymptotical unbiasedness and consistency

Under regularity assumptions on Θ, **the 3 proposed estimators** of $\lambda_{\Theta_n}(x)$ are **asymptotically unbiased and weakly consistent** for $\mathcal{H}^d$ a.e. $x \in \mathbb{R}^d$.

In order to find the optimal bandwidth of the 3 proposed estimators, we proceed along the same lines as what is commonly done for the kernel density estimator $\widehat{f}_X^N(x)$ of the pdf $f_X(x)$ of a random variable X; that is

$$r_N^{\mathrm{o,AMSE}}(x) := \arg\min_{r_N} AMSE(\widehat{\lambda}^N_{\Theta_n}(x)),$$

where

$$MSE(\widehat{\lambda}^N_{\Theta_n}(x)) := \mathbb{E}[(\widehat{\lambda}^N_{\Theta_n}(x) - \lambda_{\Theta_n}(x))^2] = Bias(\widehat{\lambda}^N_{\Theta_n}(x))^2 + Var(\widehat{\lambda}^N_{\Theta_n}(x))$$

is the **mean square error** of $\widehat{\lambda}^N_{\Theta_n}(x)$, and *AMSE* is the **asymptotic MSE**, obtained by Taylor series expansion of Bias and Variance.

Under regularity assumptions on Θ_n we obtain

- for $\widehat{\lambda}^{\kappa,N}_{\Theta_n}$ [Camerlenghi F, Capasso V, EV, *J. Multivariate Anal.*, 2014]

$$r_N^{\mathrm{o,AMSE}}(x) = \sqrt[4+d-n]{\frac{(d-n)C_{Var}(x)}{4NC^2_{Bias}(x)}}, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d$$

with

$$C_{Bias}(x) := \sum_{|\alpha|=2} \frac{1}{\alpha!} \int_{\mathbb{R}^d} k(z) z^\alpha \mathrm{d}z \int_{\mathbf{K}} \int_{x-Z(s)} D_y^\alpha f(y,s) \mathcal{H}^n(\mathrm{d}y) Q(\mathrm{d}s)$$

$$C_{Var}(x) := \int_{\mathbf{K}} \int_{\mathbb{R}^d} \int_{x-Z(s)} \int_{\pi_y^{x,s}} k(z)k(z+w) f(y,s) \mathcal{H}^n(\mathrm{d}w) \mathcal{H}^n(\mathrm{d}y) \mathrm{d}z Q(\mathrm{d}s)$$

where $\pi_y^{x,s} \in \mathbf{G}_n$ is the **approximate tangent space** to $x - Z(s)$ at $y \in x - Z(s)$.

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- for $\widehat{\lambda}^{\kappa,N}_{\Theta_n}$ [Camerlenghi F, Capasso V, EV, *J. Multivariate Anal.*, 2014]

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  with

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$$C_{Var}(x) := \int_{\mathbf{K}} \int_{\mathbb{R}^d} \int_{x-Z(s)} \int_{\pi^{x,s}_y} k(z)k(z+w) f(y,s) \mathcal{H}^n(\mathrm{d}w) \mathcal{H}^n(\mathrm{d}y) \mathrm{d}z Q(\mathrm{d}s)$$

  where $\pi^{x,s}_y \in \mathbf{G}_n$ is the **approximate tangent space** to $x - Z(s)$ at $y \in x - Z(s)$.

- for $\widehat{\lambda}^{\nu,N}_{\Theta_n}$
  the same as above with $k(z) = \frac{1}{b_d} \mathbf{1}_{B_1(0)}(z)$

- for $\widehat{\lambda}^{\mu,N}_{\Theta_n}$, if $\forall s \in \mathbf{K}$, $\mathrm{reach}(Z(s)) > R$ for some $R > 0$, [Camerlenghi F, EV, *work in progress*, 2014]

$$r_N^{\mathrm{o,AMSE}}(x) = \begin{cases} \left(\dfrac{(d-n)\lambda_{\Theta_n}(x)}{2Nb_{d-n}(\mathcal{A}_1(x)-\mathcal{A}_2(x))^2}\right)^{\frac{1}{d-n+2}} & \text{if } d-n>1 \\ \left(\dfrac{\lambda_{\Theta_{d-1}}(x)}{4N\big(\mathcal{A}_1(x)-\mathcal{A}_3(x)\big)^2}\right)^{\frac{1}{3}} & \text{if } d-n=1 \end{cases}$$

with

$$\mathcal{A}_1(x) := \frac{b_{d-n+1}}{b_{d-n}} \int_{\mathbf{K}} \int_{Z(s)} f(x-y,s)\Phi_{n-1}(Z(s);\mathrm{d}y)Q(\mathrm{d}s)$$

$$\mathcal{A}_2(x) := \frac{d-n}{d-n+1} \sum_{|\alpha|=1} \int_{\mathbf{K}} \int_{N(Z(s))} D_x^\alpha f(x-y,s)u^\alpha \mu_n(Z(s);\mathrm{d}(y,u))Q(\mathrm{d}s)$$

$$\mathcal{A}_3(x) := \int_{\mathbf{K}^2} \int_{(x-Z(s_1))} \int_{(x-Z(s_2))} g(y_1,s_1,y_2,s_2)\mathcal{H}^{d-1}(\mathrm{d}y_2)\mathcal{H}^{d-1}(\mathrm{d}y_1)Q_{[2]}(\mathrm{d}(s_1,s_2))$$

where $\Phi_n(Z(s),\cdot)$ is the ***n*-dimensional curvature measure of** $Z(s)$ and $\mu_n(Z(s),\cdot)$ is the ***n*-dimensional support measure of** $Z(s)$.

Particular cases

- $d = 1$, $n = 0$, $\Theta_0 = X$ **random variable with pdf** $f_X \in C^2$
  We reobtain the well known results for kernel density estimates of f_X

$$r_N^{\mathrm{o,AMSE}}(x) = \begin{cases} \sqrt[5]{\dfrac{f_X(x)\displaystyle\int k(z)^2 \mathrm{d}z}{N\left(f_X''(x)\displaystyle\int_{\mathbb{R}} z^2 k(z) dz\right)^2}}, & \text{for } \widehat{\lambda}_X^{\kappa,N} \\ \sqrt[5]{\dfrac{9 f_X(x)}{2N(f_X''(x))^2}}, & \text{for } \widehat{\lambda}_X^{\nu,N}(x) = \widehat{\lambda}_X^{\mu,N}(x) \end{cases}$$

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- $\Theta_0 = \Psi$ **point process in** $\mathbb{R}^d$ **with intensity** $\lambda_\Psi \in C^2$
  If $N = 1$, $\widehat{\lambda}_\Psi^{\nu,N}(x)$ coincides with the well-known classic and widely used Berman-Diggle estimator

$$\widehat{\lambda}_\Psi^{\kappa,N}(x) = \frac{\Psi(B_r(x))}{b_d r^d}$$

- **Θ_n stationary**
  Φ with intensity measure $\Lambda(\mathrm{d}(x,s)) = c\mathrm{d}xQ(\mathrm{d}s)$.
  It follows that:
  - $\lambda_{\Theta_n}(x) \equiv c\mathbb{E}[\mathcal{H}^n(Z)] \quad \mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,
  - $\widehat{\lambda}^{\kappa,N}_{\Theta_n}$ and $\widehat{\lambda}^{\nu,N}_{\Theta_n}$ are unbiased for any bandwidth $r > 0$, and any sample size N;
  - $\widehat{\lambda}^{\kappa,N}_{\Theta_n}$ and $\widehat{\lambda}^{\nu,N}_{\Theta_n}$ are strongly consistent for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$, as $N \to \infty$.
  - if Θ_n **Boolean model** with $\mathbb{E}[(\mathcal{H}^n(Z))^2] < \infty$, then

$$r^{\mathrm{o,MSE}} = \begin{cases} +\infty & \text{for } \widehat{\lambda}^{\kappa,N}_{\Theta_n} \text{ and } \widehat{\lambda}^{\nu,N}_{\Theta_n} \\ \sqrt[3]{\dfrac{c\mathbb{E}[\mathcal{H}^n(Z)]}{N\left(\pi c\mathbb{E}[\Phi_{n-1}(Z)] - 2(c\mathbb{E}[\mathcal{H}^n(Z)])^2\right)^2}} & \text{for } \widehat{\lambda}^{\kappa,N}_{\Theta_n} \text{ if } d-n=1 \\ \sqrt[d-n+2]{\dfrac{(d-n)b_{d-n}c\mathbb{E}[\mathcal{H}^n(Z)]}{2N\left(cb_{d-n+1}\mathbb{E}[\Phi_{n-1}(Z)]\right)^2}} & \text{for } \widehat{\lambda}^{\kappa,N}_{\Theta_n} \text{ if } d-n>1 \end{cases}$$

Summarizing:

kernel estimator

$$\widehat{\lambda}^{\kappa,N}_{\Theta_n}(x) = \frac{1}{Nr_N^d}\sum_{i=1}^{N}\int_{\Theta_n^i} k\Big(\frac{x-y}{r_N}\Big)\mathcal{H}^n(dy)$$

$\xrightarrow{\Theta_0 = X}$

kernel density estimator

$$\widehat{f}^N_X(x) = \frac{1}{Nr_N^d}\sum_{i=1}^{N} k\Big(\frac{x-X_i}{r_N}\Big)$$

$\downarrow$ $k(z) = \frac{1}{b_d}\mathbf{1}_{B_1(0)}(z)$

$\downarrow$ $k(z) = \frac{1}{b_d}\mathbf{1}_{B_1(0)}(z)$

natural estimator

$$\widehat{\lambda}^{\nu,N}_{\Theta_n}(x) = \frac{1}{Nb_d r_N^d}\sum_{i=1}^{N}\mathcal{H}^n(\Theta_n^i \cap B_{r_N}(x))$$

"Minkowski content"-based estimator

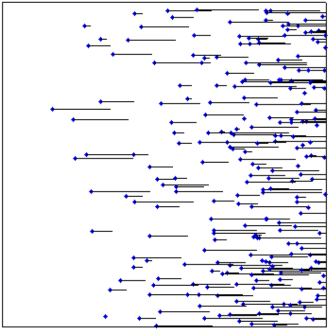
$$\widehat{\lambda}^{\mu,N}_{\Theta_n}(x) = \frac{\#\{i : x \in \Theta^i_{n_{\oplus r_N}}\}}{Nb_{d-n} r_N^{d-n}}$$

$\xrightarrow{\Theta_0 = X}$

naive estimator

$$\widehat{f}^N_X(x) = \frac{\#\{i : X_i \in B_{r_N}(x)\}}{Nr_N^d}$$

- Θ_1 = **inhomogeneous Boolean model of segments** of the type $[0, l] \times \{0\}$ in $\mathbb{R}^2$, with random length $l \sim U(0, 0.2)$ in the compact window $W = [0, 1]^2$, where the underlying Poisson point process has intensity $f(x_1, x_2) = 700x_1^2$.



$$\lambda_{\Theta_1}(x_1, x_2) = \frac{175}{3}(0.2)^3 - \frac{700}{3}(0.2)^2 x_1 + 70x_1^2$$

Natural estimator and **“Minkowski content”-based estimator** at point $x = (0.5, 0.5)$ as function of the **bandwidth expressed in pixels** $(1\text{pixel} = 0.0029)$, for $N = 10$ and $N = 100$

$\lambda_{\Theta_1}(0.5, 0.5) = 13.30$

Natural estimator and **“Minkowski content”-based estimator** at point $x = (0.5, 0.5)$ as function of the **bandwidth expressed in pixels** ($1\text{pixel} = 0.0029$), for $N = 10$ and $N = 100$

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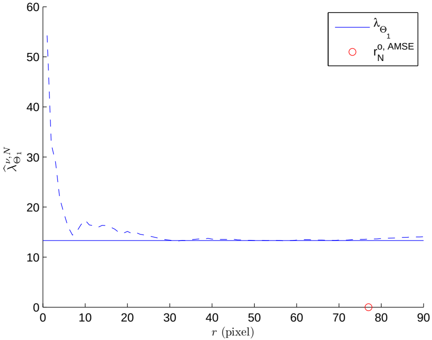
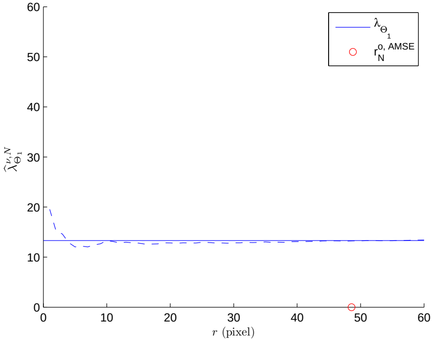
Natural estimator

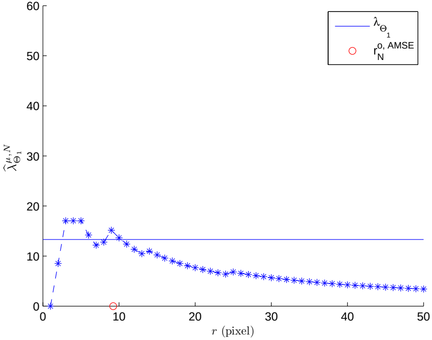


$N = 10$

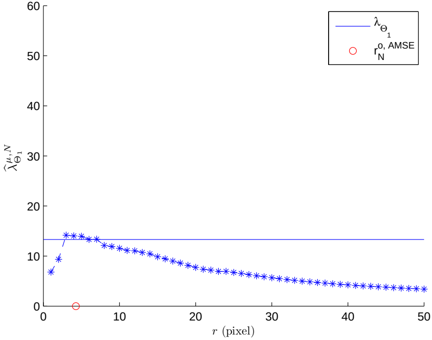


$N = 100$

‘Minkowski content”-based estimator



$N = 10$



$N = 100$

$\lambda_{\Theta_1}(0.5, 0.5) = 13.30$ theoretical value

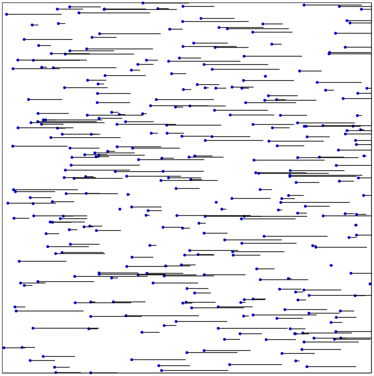
$|\widehat{\lambda}^{\nu,N}_{\Theta_1}(0.5, 0.5) - \lambda_{\Theta_1}(0.5, 0.5)| = 0.2973$ for $N = 10$, $r^{\mathrm{o,AMSE}}_{10} \approx 77 pixel (0.2973)$

$|\widehat{\lambda}^{\nu,N}_{\Theta_1}(0.5, 0.5) - \lambda_{\Theta_1}(0.5, 0.5)| = 0.0614$ for $N = 100$, $r^{\mathrm{o,AMSE}}_{100} \approx 49 pixel (0.1425)$

$|\widehat{\lambda}^{\mu,N}_{\Theta_1}(0.5, 0.5) - \lambda_{\Theta_1}(0.5, 0.5)| = 1.8556$ for $N = 10$, $r^{\mathrm{o,AMSE}}_{10} \approx 9 pixel (0.0271)$

$|\widehat{\lambda}^{\mu,N}_{\Theta_1}(0.5, 0.5) - \lambda_{\Theta_1}(0.5, 0.5)| = 0.70$ for $N = 100$, $r^{\mathrm{o,AMSE}}_{100} \approx 4 pixel (0.0126)$

- **Θ_1 = homogeneous Boolean model of segments** of the type $[0, l] \times \{0\}$ in $\mathbb{R}^2$, with random length $l \sim U(0, 0.2)$ in the compact window $W = [0, 1]^2$, where the underlying Poisson point process has intensity $f(x_1, x_2) = 300$.



$$\lambda_{\Theta_1}(x_1, x_2) = 30$$

Natural estimator and **“Minkowski content”-based estimator** at point $x = (0.5, 0.5)$ as function of the **bandwidth expressed in pixel ($1\text{pixel} = 0.0029$)**, for $N = 10$ and $N = 100$

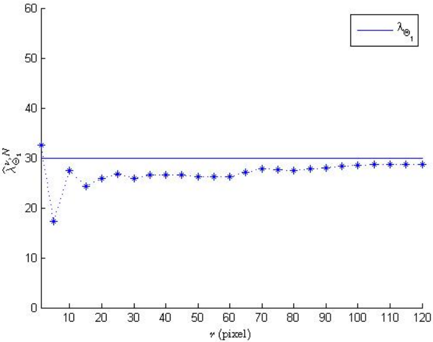
$\lambda_{\Theta_1}(x) \equiv 30$; we choose $x = (0.5, 0.5)$.

Natural estimator and **"Minkowski content"-based estimator** at point $x = (0.5, 0.5)$ as function of the **bandwidth expressed in pixel ($1\text{pixel} = 0.0029$)**, for $N = 10$ and $N = 100$

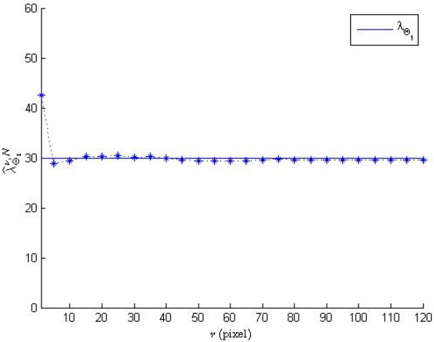
$\lambda_{\Theta_1}(x) \equiv 30$; we choose $x = (0.5, 0.5)$.

Natural estimator

$$r_N^{o,\mathrm{AMSE}} = +\infty$$

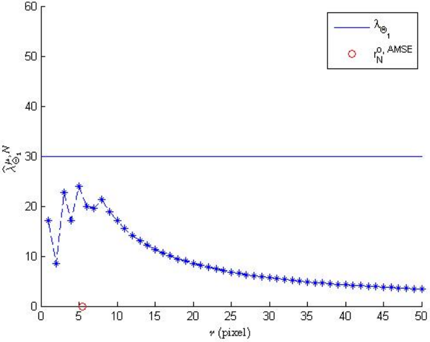


$N = 10$

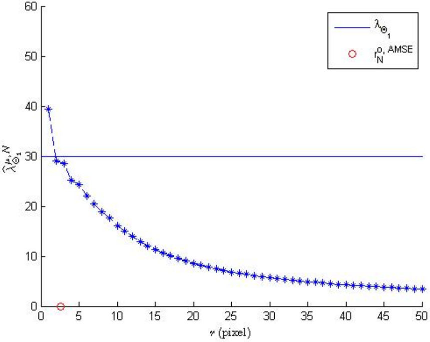


$N = 100$

“Minkowski content”-based estimator



$N = 10$



$N = 100$

$\lambda_{\Theta_1} \equiv 30$ theoretical value

$|\widehat{\lambda}^{\nu,N}_{\Theta_1} - \lambda_{\Theta_1}| = 1.2647$ with $r = 105 pixel$ for $N = 10$, $r^{\mathrm{o,AMSE}}_{10} = +\infty$

$|\widehat{\lambda}^{\nu,N}_{\Theta_1} - \lambda_{\Theta_1}| = 0.4082$ with $r = 105 pixel$ for $N = 100$, $r^{\mathrm{o,AMSE}}_{100} = +\infty$

$|\widehat{\lambda}^{\mu,N}_{\Theta_1} - \lambda_{\Theta_1}| = 6.13$ for $N = 10$, $r^{\mathrm{o,AMSE}}_{10} \approx 5 pixel (0.016)$

$|\widehat{\lambda}^{\mu,N}_{\Theta_1} - \lambda_{\Theta_1}| = 1.50$ for $N = 100$, $r^{\mathrm{o,AMSE}}_{100} \approx 3 pixel (0.0074)$

- **$\Theta_0 = \Psi$ inhomogeneous Poisson point process** with intensity $f(x_1, x_2) = x_1^2 + x_2^2$ in the compact window $W = [-2, 2]^2$.

$\Psi^1, \dots, \Psi^N$ i.i.d. random sample for Ψ

- *kernel estimator*

$$\widehat{\lambda}_{\Psi}^{\kappa,N}(x) = \frac{1}{N r_N^2} \sum_{i=1}^{N} \sum_{y_j \in \Psi^i} k\Big(\frac{x - y_j}{r_N}\Big)$$

with *kernel of Epanechnikov*:

$$k(t) = \begin{cases} \frac{2}{\pi}(1 - x_1^2 - x_2^2), & \text{if } (x_1, x_2) \in B_1(0) \\ 0, & \text{otherwise} \end{cases}$$

- *natural estimator*

$$\widehat{\lambda}_{\Psi}^{\nu,N}(x) = \frac{1}{N \pi r_N^2} \sum_{i=1}^{N} \mathcal{H}^0(\Psi^i \cap B_{r_N}(x)) = \frac{1}{N \pi r_N^2} \sum_{i=1}^{N} \sum_{y_j \in \Psi^i} \mathbf{1}_{B_{r_N}(x)}(y_j)$$

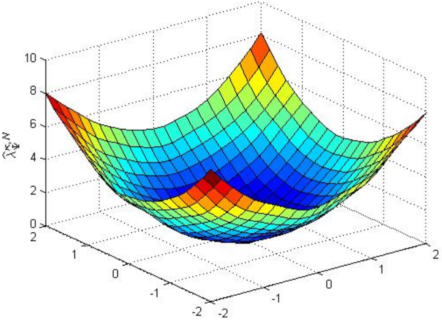
- *"Minkowski content"-based estimator*

$$\widehat{\lambda}_{\Psi}^{\mu,N}(x) = \frac{\#\{i : x \in \Psi^i_{\oplus r_N}\}}{N \pi r_N^2} = \frac{1}{N \pi r_N^2} \sum_{i=1}^{N} \mathbf{1}_{\{\Psi^i(B_{r_N}(x)) > 0\}}$$

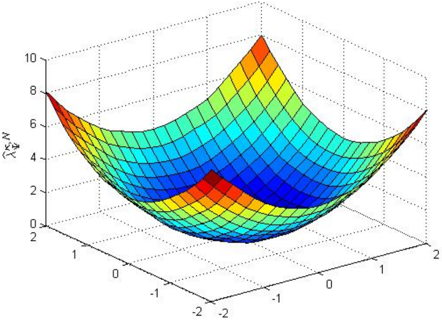
Epanechnikov-kernel estimator, Natural estimator and **"Minkowski content"-based estimator** in $W = [-2, 2]^2$ with **grid step size=0.2**, for $N = 1000$ and $N = 10000$

Epanechnikov-kernel estimator, Natural estimator and **"Minkowski content"-based estimator** in $W = [-2, 2]^2$ with **grid step size=0.2**, for $N = 1000$ and $N = 10000$

Epanechnikov-kernel estimator

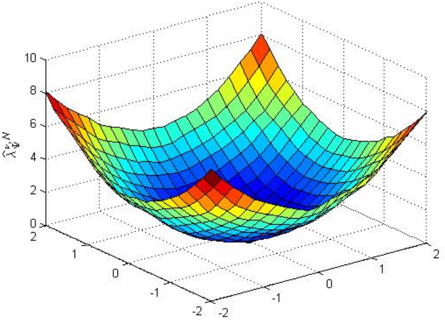


$N = 1000$

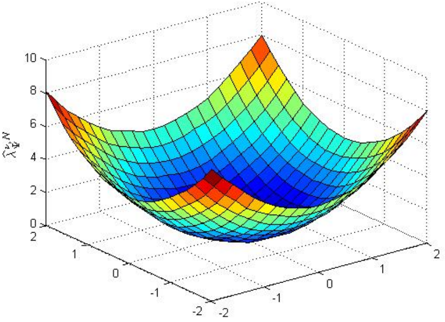


$N = 10000$

Natural estimator

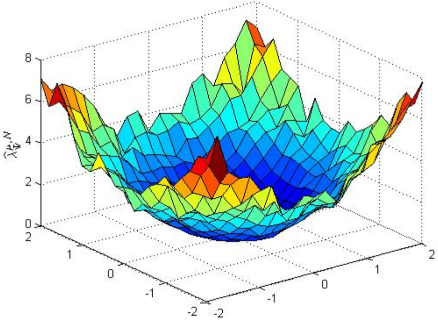


$N = 1000$

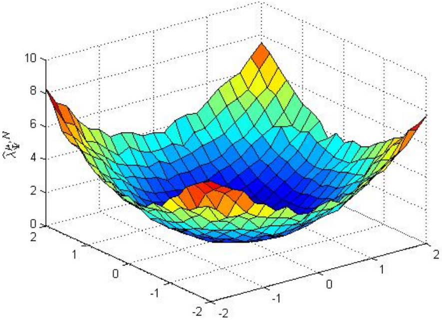


$N = 10000$

"Minkowski content"-based estimator



$N = 1000$



$N = 10000$

$\lambda_\Psi(1.8, 1.8) = 6.48$ theoretical value

$|\widehat{\lambda}^{\kappa,N}_{\Theta_1}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.0878$ for $N = 1000$, $r^{\mathrm{o,AMSE}}_{1000} \approx 166\,pixel(0.4809)$

$|\widehat{\lambda}^{\kappa,N}_{\Psi}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.0434$ for $N = 10000$, $r^{\mathrm{o,AMSE}}_{10000} \approx 113\,pixel(0.3277)$

$|\widehat{\lambda}^{\nu,N}_{\Psi}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.0336$ for $N = 1000$, $r^{\mathrm{o,AMSE}}_{1000} \approx 138\,pixel(0.4005)$

$|\widehat{\lambda}^{\nu,N}_{\Psi}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.0379$ for $N = 10000$, $r^{\mathrm{o,AMSE}}_{10000} \approx 94\,pixel(0.2728)$

$|\widehat{\lambda}^{\mu,N}_{\Psi}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.2371$ for $N = 1000$, $r^{\mathrm{o,AMSE}}_{1000} \approx 27\,pixel(0.0789)$

$|\widehat{\lambda}^{\mu,N}_{\Psi}(1.8, 1.8) - \lambda_\Psi(1.8, 1.8)| = 0.2291$ for $N = 10000$, $r^{\mathrm{o,AMSE}}_{10000} \approx 18\,pixel(0.0537)$

$\max_{x\in W} |\widehat{\lambda}^{\kappa,N}_{\Psi}(x) - \lambda_\Psi(x)| = 0.2938$ for $N = 1000$

$\max_{x\in W} |\widehat{\lambda}^{\kappa,N}_{\Psi}(x) - \lambda_\Psi(x)| = 0.1618$ for $N = 10000$

$\max_{x\in W} |\widehat{\lambda}^{\nu,N}_{\Psi}(x) - \lambda_\Psi(x)| = 0.3247$ for $N = 1000$

$\max_{x\in W} |\widehat{\lambda}^{\nu,N}_{\Psi}(x) - \lambda_\Psi(x)| = 0.1698$ for $N = 10000$

$\max_{x\in W} |\widehat{\lambda}^{\mu,N}_{\Psi}(x) - \lambda_\Psi(x)| = 1.8202$ for $N = 1000$

$\max_{x\in W} |\widehat{\lambda}^{\mu,N}_{\Psi}(x) - \lambda_\Psi(x)| = 0.7601$ for $N = 10000$

Concluding Remarks

- **Kernel estimators:**
  - **pro:** they extend in a natural way the corresponding kernel estimators for random objects of dimension $n = 0$ (random variables - univariate and multivariate, point processes) to random closed sets of any integer Hausdorff dimension $n < d$, in $\mathbb{R}^d$;
  - **cons:** practical applicability; non-trivial computation of integrals is required

Concluding Remarks

- **Kernel estimators:**
  - **pro:** they extend in a natural way the corresponding kernel estimators for random objects of dimension $n = 0$ (random variables - univariate and multivariate, point processes) to random closed sets of any integer Hausdorff dimension $n < d$, in $\mathbb{R}^d$;
  - **cons:** practical applicability; non-trivial computation of integrals is required
- **Natural estimators:**
  - **pro:** direct derivation from the Besicovitch Theorem;
    generalization of the notion of histogram estimators for the case $\Theta_0 = X$ random variable;
    more stable with respect to the choice of the bandwidth, than the "Minkowski content"-based estimators
  - **cons:** include the nontrivial evaluation of $\mathcal{H}^n(\Theta_n^i \cap B_{r_N}(x))$ for any element Θ_n^i of the sample; for segment processes ($n = 1$) it seems more feasible, but for other sets of dimension $n \geq 1$ it results of higher computational complexity.

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- **"Minkowski content"-based estimators:**
  - **pro:** easy computational evaluation;
  - **cons:** quite sensitive to the choice of the bandwidth;
    low rate of convergence

Essential Bibliography:

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...and references therein