Totally Disconnected L.C. Groups: Subgroups associated with an automorphism

George Willis
The University of Newcastle

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Lecture 1: The scale and minimizing subgroups for an endomorphism

Lecture 2: Tidy subgroups and the scale

Lecture 3: Subgroups associated with an automorphism

- Contraction groups
- The structure of closed contraction groups
- The nub of α

Lecture 4: Flat groups of automorphisms

The contraction group for α

Definition

Let $\alpha \in \mathrm{Aut}(G)$. The *contraction group for α* is

$$\mathrm{con}(\alpha) := \{x \in G \mid \alpha^n(x) \to 1 \text{ as } n \to \infty\}.$$

Then $\mathrm{con}(\alpha)$ is an α-stable subgroup of G.
Examples show that it need not be a closed subgroup.

Examples of contraction groups

Examples

1. $F^{\mathbb{Z}}$, where F is a finite group, with the product topology. Let α be the shift: $\alpha(g)_n = g_{n+1}$.
2. $(\mathbb{F}_p((t)), +)$, the additive group of the field of formal Laurent series over the field of order p. Let α be multiplication by t.
3. Aut(T_q), the automorphism group of the regular tree with every vertex having valency q. Let α be the inner automorphism α_g, g a translation of T.
4. $SL(n, \mathbb{Q}_p)$, the special linear group over the field of p-adic numbers. Let α be conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

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Contraction groups in representation theory

Proposition (Mautner phenomenon)

Let $\rho : G \to \mathcal{L}(X)$ be a bounded, strongly continuous representation of G on the Banach space X. Suppose, for some $g \in G$ and $x \in X$, that $\rho(g)x = x$.
Then $\rho(h)x = x$ for every $h \in \mathrm{con}(g)$.

Non-triviality of con(α)

The following were shown by U. Baumgartner & W. in the case when G is metrizable. The metrizability condition was removed by W. Jaworski.

Theorem

Suppose that $s(\alpha^{-1}) > 1$. Then con(α) *is not trivial.*

The converse does not hold.

Theorem

Let $\alpha \in Aut(G)$ and $V \in \mathcal{B}(G)$ be tidy for α. Then

$$V_{--} = V_0 \mathrm{con}(\alpha). \tag{1}$$

Moreover,

$$\bigcap \{U_{--} \mid U \text{ tidy for } \alpha\} = \overline{\mathrm{con}(\alpha)}. \tag{2}$$

Normal closures

Proposition

Let $\alpha \in \mathrm{Aut}(G)$. Then the map

$$\eta : \overline{\mathrm{con}(\alpha)} \to \overline{\mathrm{con}(\alpha)} \text{ defined by } \eta(x) = x\alpha(x^{-1})$$

is surjective.

Proposition

Let $g \in G$. Then $\overline{\mathrm{con}(g)}$ is contained in every (abstractly) normal subgroup of G that contains g.

The Tits core

Definition

The *Tits core* of the t.d.l.c. group G is

$$G^\dagger = \langle \overline{\mathrm{con}(g)} \mid g \in G \rangle.$$

Theorem (Caprace, Reid & W.)

Let D be a dense subgroup of the t.d.l.c. group G. If $G^\dagger$ normalises D, then $G^\dagger \leq D$.

Corollary (Caprace, Reid & W.)

Suppose that G belongs to $\mathcal{S}$, that is, G is compactly generated and topologically simple. Then $G^\dagger$ is either trivial or it is the smallest non-trivial normal subgroup of G.

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Closed contraction groups

Theorem (Glöckner & W.)

Let G be a t.d.l.c. group and suppose that $\alpha \in \mathrm{Aut}(G)$ *is such that* $\alpha^n(g) \to 1$ *as* $n \to \infty$ *for every* $g \in G$*. Then the set* $tor(G)$ *of torsion elements and the set* $div(G)$ *of divisible groups are* α*-stable closed subgroups of G and*

$$G = tor(G) \times div(G).$$

Furthermore $div(G)$ *is a direct product*

$$div(G) = G_{p_1} \times \cdots \times G_{p_n},$$

where each G_{p_j} *is a nilpotent* p_i*-adic Lie group.*

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Closed contraction groups 2

Every group G with a contractive automorphism α has a composition series of closed α-stable subgroups where each of the composition factors is a *simple* contraction group in the sense that it has no closed, proper, non-trivial α-stable subgroups.

Theorem (Glöckner & W.)

Let G be a t.d.l.c. group, $\alpha \in \mathrm{Aut}(G)$ and suppose that (G, α) is simple. Then G is either:

1. *a torsion group and isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with F a finite simple group and α the shift; or*
2. *torsion free and isomorphic to a p-adic vector space with α a contractive linear transformation.*

Ergodic actions by automorphisms

Conjecture (Halmos)

Let G be a l.c. group and suppose that there is $\alpha \in \mathrm{Aut}(G)$ that acts ergodically on G. Then G is compact.

Proved for G connected in the 1960's and for G totally disconnected in the 1980's. Short proof by Previts & Wu uses the scale.

S. G. Dani, N. Shah & W. show that, if G has a finitely generated abelian group of automorphisms that acts ergodically, then G is, modulo a compact normal subgroup, a direct product of vector groups over $\mathbb{R}$ and $\mathbb{Q}_p$.

The largest subgroup on which α acts ergodically

Definition

The *nub* of $\alpha \in \mathrm{Aut}(G)$ is the subgroup

$$\mathrm{nub}(\alpha) = \bigcap \{ V \mid V \text{ is tidy for } \alpha \} \ \ (= \mathrm{nub}(\alpha^{-1})).$$

The nub of α is trivial if and only if $\mathrm{con}(\alpha)$ is closed.

Theorem

$\mathrm{nub}(\alpha)$ *is the largest closed* α*-stable subgroup of* G *on which* α *acts ergodically.*

Theorem

The compact open subgroup V *is tidy below for* $\alpha \in \mathrm{Aut}(G)$ *if and only if* $\mathrm{nub}(\alpha) \leq V$*.*

The structure of nub(α)

(B. Kitchens & K. Schmidt.   W. Jaworski)

Theorem

The nub of α is isomorphic to an inverse limit

$$(\mathrm{nub}(\alpha), \alpha) \cong \varprojlim (G_i, \alpha_i),$$

where G_i is a compact t.d. group, $\alpha_i \in \mathrm{Aut}(G_i)$ and G_i has a composition series

$$\{1\} = H_0 < H_1 < \cdots < H_r = G_i,$$

of α_i stable subgroups, with the composition factors H_{j+1}/H_j isomorphic to $F_j^{\mathbb{Z}}$, for a finite simple group F_j and the induced automorphism the shift.

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The nub and contraction groups

Theorem

Let $\alpha \in \mathrm{Aut}(G)$. *Then*

$$\mathrm{nub}(\alpha) = \overline{\mathrm{con}(\alpha)} \cap \overline{\mathrm{con}(\alpha^{-1})}$$

and

$$\mathrm{nub}(\alpha) \cap \mathrm{con}(\alpha) = \{g \in \mathrm{con}(\alpha) \mid \{\alpha^n(g)\}_{n\in\mathbb{Z}} \text{ is bounded}\}$$

is dense in $\mathrm{nub}(\alpha)$. *Denote this set by* $\mathrm{bcon}(\alpha)$.
The intersection $\mathrm{bcon}(\alpha) \cap \mathrm{bcon}(\alpha^{-1})$ *need not be dense in* $\mathrm{nub}(\alpha)$ *but*

$$\mathrm{nub}(\alpha)/\overline{\mathrm{bcon}(\alpha) \cap \mathrm{bcon}(\alpha^{-1})}$$

is abelian.

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