Quantum Wiener chaos expansion

Mateusz Jurczyński

Lancaster University, United Kingdom

(joint work with Martin Lindsay)

Workshop on Operator Spaces,
Locally Compact Quantum Groups and Amenability
Toronto, 29th May 2014

Integration - product formulae

Classical integration by parts

Let $x_t = \int_0^t \alpha(s)ds, y_t = \int_0^t \beta(s)ds$. Then

$$\begin{aligned} x_t y_t &= \int_0^t (\alpha(s)y_s + x_s\beta(s))ds \\ &= \int_0^t \int_0^s (\alpha(s)\beta(r) + \alpha(r)\beta(s))drds \end{aligned}$$

Integration - product formulae

Wiener integration by parts

Let $X_t = \int_0^t f(s)d{\color{red}W}_s$, $Y_t = \int_0^t g(s)d{\color{red}W}_s$. Then

$$\begin{aligned} x_t y_t &= \int_0^t (\alpha(s)y_s + x_s\beta(s))ds \\ &= \int_0^t \int_0^s (\alpha(s)\beta(r) + \alpha(r)\beta(s))drds \end{aligned}$$

Integration - product formulae

Wiener integration by parts

Let $X_t = \int_0^t f(s)dW_s$, $Y_t = \int_0^t g(s)dW_s$. Then

$$\begin{aligned} X_t Y_t &= \int_0^t (f(s)Y_s + X_s g(s))dW_s + \int_0^t f(s)g(s)ds \\ &= \int_0^t \int_0^s (f(s)g(r) + f(r)g(s))dW_r dW_s + \int_0^t f(s)g(s)ds \end{aligned}$$

Integration - product formulae

Quantum stochastic integration by parts

Let $X_t = \int_0^t F(s)d\Lambda_s$, $Y_t = \int_0^t G(s)d\Lambda_s$. Then

$$\begin{aligned} X_t Y_t &= \int_0^t (f(s)Y_s + X_s g(s))dW_s + \int_0^t f(s)g(s)ds \\ &= \int_0^t \int_0^s (f(s)g(r) + f(r)g(s))dW_r dW_s + \int_0^t f(s)g(s)ds \end{aligned}$$

Integration - product formulae

Quantum stochastic integration by parts

Let $X_t = \int_0^t F(s)d\Lambda_s$, $Y_t = \int_0^t G(s)d\Lambda_s$. Then

$$\begin{aligned} X_t Y_t &= \int_0^t (F_s Y_s + X_s G_s + F_s \Delta G_s) d\Lambda_s \\ &= \int_0^t \int_0^s (F_r G_s + F_s G_r) d\Lambda_r d\Lambda_s + \int_0^t F_s \Delta G_s d\Lambda_s. \end{aligned}$$

Motivations

1. Fock space provides a universal arena for stochastic calculus.

Motivations

1. Fock space provides a universal arena for stochastic calculus.
2. 'Duality transforms' induce algebraic structure on Fock space.

Motivations

1. Fock space provides a universal arena for stochastic calculus.
2. 'Duality transforms' induce algebraic structure on Fock space.
3. Products of stochastic integrals are expressible in closed-form formulas.

Motivations

1. Fock space provides a universal arena for stochastic calculus.
2. 'Duality transforms' induce algebraic structure on Fock space.
3. Products of stochastic integrals are expressible in closed-form formulas.
4. Known classical examples are particular cases of a general quantum picture.

Multiple Wiener integrals

Let $(W_t)_{t \geq 0}$ be the canonical one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$.

Multiple Wiener integrals

Let $(W_t)_{t\geq 0}$ be the canonical one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. One can form multiple Wiener integrals:

$$I_n(f) = \int \cdots \int_{\Delta^n} f(t_1, \cdots, t_n) dW_{t_1} ... dW_{t_n} \in L^2(\Omega, \mathcal{F}, P)$$

$f \in L^2(\Delta^n)$, $\Delta^n = \{(t_1, \cdots, t_n) \in [0, \infty)^n \colon t_1 \leq ... \leq t_n\}$.

Multiple Wiener integrals

Let $(W_t)_{t\geq 0}$ be the canonical one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. One can form multiple Wiener integrals:

$$I_n(f) = \int \cdots \int_{\Delta^n} f(t_1, \cdots, t_n) dW_{t_1} \ldots dW_{t_n} \in L^2(\Omega, \mathcal{F}, P)$$

$f \in L^2(\Delta^n)$, $\Delta^n = \{(t_1, \cdots, t_n) \in [0, \infty)^n \colon t_1 \leq \ldots \leq t_n\}$.

Theorem (Wiener-Segal-Itô)

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n\geq 0} I_n(L^2(\Delta_n)).$$

Multiple Wiener integrals

Let $(W_t)_{t\geq 0}$ be the canonical one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. One can form multiple Wiener integrals:

$$I_n(f) = \int \cdots \int_{\Delta^n} f(t_1, \cdots, t_n) dW_{t_1} ... dW_{t_n} \in L^2(\Omega, \mathcal{F}, P)$$

$f \in L^2(\Delta^n)$, $\Delta^n = \{(t_1, \cdots, t_n) \in [0, \infty)^n \colon t_1 \leq ... \leq t_n\}$.

Theorem (Wiener-Segal-Itô)

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n\geq 0} I_n(L^2(\Delta_n)).$$

Also: compensated Poisson process, certain Azéma martingales.

Noncommutative probability

Probability	Quantum probability

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \omega), \mathcal{A}$ v.N.a., ω normal state

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \omega)$, $\mathcal{A}$ v.N.a., ω normal state
$1_A, A \in \mathcal{F}$	

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \omega)$, $\mathcal{A}$ v.N.a., ω normal state
$1_A, A \in \mathcal{F}$	$p = p^* = p^2 \in \mathcal{A}$

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \omega)$, $\mathcal{A}$ v.N.a., ω normal state
$1_A, A \in \mathcal{F}$	$p = p^* = p^2 \in \mathcal{A}$
$f : \Omega \to X$ r.v.	$F : \mathcal{B} \to \mathcal{A}$ unital $* -$homomorphism

Noncommutative probability

Probability	Quantum probability
$L^\infty(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \omega)$, $\mathcal{A}$ v.N.a., ω normal state
$1_A, A \in \mathcal{F}$	$p = p^* = p^2 \in \mathcal{A}$
$f : \Omega \to X$ r.v.	$F : \mathcal{B} \to \mathcal{A}$ unital $*$-homomorphism
$P \circ f^{-1}$	$\omega \circ F.$

Fock space

Fix Hilbert space k.
Set $\mathsf{K} = L^2(\mathbb{R}_+; \mathsf{k})$, $\mathsf{k}^{\otimes n} = \mathsf{k} \otimes \ldots \otimes \mathsf{k}$ ($\mathsf{k}^{\otimes 0} = \mathbb{C}$) and

$$\mathsf{K}^{\vee n} = \overline{\mathrm{Lin}}\{f^{\otimes n} : f \in \mathsf{K}\}.$$

Symmetric Fock space over K:

$$\mathcal{F}^{\mathsf{k}} = \Gamma(\mathsf{K}) = \bigoplus_{n \geq 0} \mathsf{K}^{\vee n}.$$

Fock space

Fix Hilbert space k.
Set $\mathsf{K} = L^2(\mathbb{R}_+; \mathsf{k})$, $\mathsf{k}^{\otimes n} = \mathsf{k} \otimes \ldots \otimes \mathsf{k}$ ($\mathsf{k}^{\otimes 0} = \mathbb{C}$) and

$$\mathsf{K}^{\vee n} = \overline{\mathrm{Lin}}\{f^{\otimes n} : f \in \mathsf{K}\}.$$

Symmetric Fock space over K:

$$\mathcal{F}^{\mathsf{k}} = \Gamma(\mathsf{K}) = \bigoplus_{n \geq 0} \mathsf{K}^{\vee n}.$$

K and $\mathcal{F}$ decompose as follows:

$$\mathsf{K} = \mathsf{K}_{[0,t[} \oplus \mathsf{K}_{[t,\infty[}, \quad \mathcal{F}^{\mathsf{k}} = \mathcal{F}^{\mathsf{k}}_{[0,t[} \otimes \mathcal{F}^{\mathsf{k}}_{[t,\infty[},$$

where $\mathsf{K}_I := L^2(I; \mathsf{k})$ and $\mathcal{F}^{\mathsf{k}}_I := \Gamma(\mathsf{K}_I)$ for $I \subset \mathbb{R}_+$.

Exponential vectors

$$\varepsilon(f) := \left(1, f, \frac{f^{\otimes 2}}{\sqrt{2}}, \cdots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots\right).$$

Exponential vectors

$$\varepsilon(f) := \left(1, f, \frac{f^{\otimes 2}}{\sqrt{2}}, \cdots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots\right).$$

$$Exp := \mathrm{Lin}\{\varepsilon(f) \colon f \in L^2(\mathbb{R}_+; \mathsf{k})\}.$$

Exponential vectors

$$\varepsilon(f) := \left(1, f, \frac{f^{\otimes 2}}{\sqrt{2}}, \cdots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots\right).$$

$$Exp := \mathrm{Lin}\{\varepsilon(f) \colon f \in L^2(\mathbb{R}_+; \mathrm{k})\}.$$

$$\Omega := \varepsilon(0) = (1, 0, 0, \ldots).$$

Exponential vectors

$$\varepsilon(f) := (1, f, \frac{f^{\otimes 2}}{\sqrt{2}}, \cdots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots).$$

$$Exp := \mathrm{Lin}\{\varepsilon(f) \colon f \in L^2(\mathbb{R}_+; \mathrm{k})\}.$$

$$\Omega := \varepsilon(0) = (1, 0, 0, \ldots).$$

$$\omega_\Omega(A) = \langle \Omega, A\Omega \rangle.$$

Topological version of Schürmann Reconstruction Theorem

Theorem (Das–Lindsay 2012, Lindsay–Skalski 2004)

Every quantum Lévy process is (equivalent to) a Fock space quantum Lévy process, i.e. a process $(j_{s,t} \colon \mathsf{A} \to (B(\mathcal{F}^{\mathsf{k}}), \omega_\Omega))_{0 \leq s \leq t}$ *for some Hilbert space* k.

Topological version of Schürmann Reconstruction Theorem

Theorem (Das–Lindsay 2012, Lindsay–Skalski 2004)

Every quantum Lévy process is (equivalent to) a Fock space quantum Lévy process, i.e. a process $(j_{s,t} \colon \mathsf{A} \to (B(\mathcal{F}^{\mathsf{k}}), \omega_\Omega))_{0 \le s \le t}$ *for some Hilbert space* k*. Moreover, this process is expressible in terms of multiple quantum Wiener integrals.*

Guichardet space 1: Definition

Definition

$$\Gamma = \mathrm{Fin}(\mathbb{R}_+); \quad \Gamma_{s,t} = \mathrm{Fin}([s, t[)$$

Guichardet space 1: Definition

Definition

$$\Gamma = \mathrm{Fin}(\mathbb{R}_+); \quad \Gamma_{s,t} = \mathrm{Fin}([s,t[)$$

$$\{s_1 < \ldots < s_n\} \mapsto (s_1, \cdots, s_n) \in \mathbb{R}_+^n;$$

Guichardet space 1: Definition

Definition

$$\Gamma = \mathrm{Fin}(\mathbb{R}_+); \quad \Gamma_{s,t} = \mathrm{Fin}([s,t[)$$

$$\{s_1 < \ldots < s_n\} \mapsto (s_1, \cdots, s_n) \in \mathbb{R}_+^n;$$

$$\mu(\{\emptyset\}) = 1.$$

Guichardet space 1: Definition

Definition

$$\Gamma = \mathrm{Fin}(\mathbb{R}_+); \quad \Gamma_{s,t} = \mathrm{Fin}([s,t[)$$
$$\{s_1 < \ldots < s_n\} \mapsto (s_1, \cdots, s_n) \in \mathbb{R}_+^n;$$
$$\mu(\{\emptyset\}) = 1.$$

$$\mu(\Gamma_{0,t}) = e^t.$$

Guichardet space 1: Definition

Definition

$$\Gamma = \mathrm{Fin}(\mathbb{R}_+); \quad \Gamma_{s,t} = \mathrm{Fin}([s,t[)$$
$$\{s_1 < \ldots < s_n\} \mapsto (s_1, \cdots, s_n) \in \mathbb{R}_+^n;$$
$$\mu(\{\emptyset\}) = 1.$$

$$\mu(\Gamma_{0,t}) = e^t.$$

We denote the sets by Greek letters - α, β, γ.

Guichardet space 1: Definition

Definition

$$\varGamma = \mathrm{Fin}(\mathbb{R}_+); \quad \varGamma_{s,t} = \mathrm{Fin}([s,t[)$$
$$\{s_1 < \ldots < s_n\} \mapsto (s_1, \cdots, s_n) \in \mathbb{R}_+^n;$$
$$\mu(\{\emptyset\}) = 1.$$

$$\mu(\varGamma_{0,t}) = e^t.$$

We denote the sets by Greek letters - α, β, γ.

Definition

$$\mathcal{F}^{\mathrm{k}} \cong \{F \in L^2(\varGamma; \bigoplus_{n \geq 0} \mathrm{k}^{\otimes n}) : \forall_{\sigma \in \varGamma}\ F(\sigma) \in \mathrm{k}^{\otimes \# \sigma}\}.$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+; \mathsf{k}) = \mathsf{K}$. Then

$$x \qquad \in (\mathbb{C}, \mathsf{K}, \mathsf{K}^{\otimes 2}, \ldots, \qquad)$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+; \mathsf{k}) = \mathsf{K}$. Then

$$x(\sigma) \quad \in (\mathbb{C}, \mathsf{k}, \mathsf{k}^{\otimes 2}, \ldots, \qquad)$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+; \mathsf{k}) = \mathsf{K}$. Then

$$x(\sigma) \quad \in (\mathbb{C}, \mathsf{k}, \mathsf{k}^{\otimes 2}, \ldots, \qquad)$$

$$\downarrow$$

$$\sigma$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+; \mathsf{k}) = \mathsf{K}$. Then

$$\begin{array}{cl} x(\sigma) & \in (\mathbb{C}, \mathsf{k}, \mathsf{k}^{\otimes 2}, \ldots, \qquad\qquad) \\ \downarrow & \\ \sigma & = \{s_1 < \ldots < s_n\} \end{array}$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+; \mathsf{k}) = \mathsf{K}$. Then

$$\begin{array}{ccc} x(\sigma) & \in (\mathbb{C}, \mathsf{k}, \mathsf{k}^{\otimes 2}, \ldots, & \mathsf{k}^{\otimes n}, \ldots) \\ \downarrow & & \nearrow \\ \sigma & = \{s_1 < \ldots < s_n\} & \end{array}$$

Guichardet space 2: Explanation

Let x be a vector from the Fock space over $L^2(\mathbb{R}_+;\mathsf{k}) = \mathsf{K}$. Then

$$\begin{array}{ccc} x(\sigma) & \in (\mathbb{C}, \mathsf{k}, \mathsf{k}^{\otimes 2}, \ldots, & \mathsf{k}^{\otimes n}, \ldots) \\ \downarrow & & \nearrow \\ \sigma & = \{s_1 < \ldots < s_n\} & \end{array}$$

$$f \in L^2(\mathbb{R}_+;\mathsf{k}) :$$

$$\pi_f(\emptyset) = 1_{\mathbb{C}},\ \pi_f(\sigma) = \vec{\bigotimes}_{s\in\sigma} f(s), \quad \pi_f \mapsto \varepsilon(f).$$

Guichardet space 3: quantum Wiener integrands

Let $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$.

Guichardet space 3: quantum Wiener integrands

Let $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$. For $f \in L^2(\mathbb{R}_+; \mathsf{k})$ let $\widehat{f}(s) = \binom{1}{f(s)} \in \widehat{\mathsf{k}}$.

Guichardet space 3: quantum Wiener integrands

Let $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$. For $f \in L^2(\mathbb{R}_+; \mathsf{k})$ let $\widehat{f}(s) = \binom{1}{f(s)} \in \widehat{\mathsf{k}}$.

Consider $\mathbb{F} \colon \Gamma \to \bigoplus_{n \in \mathbb{N}} B(\widehat{\mathsf{k}}^{\otimes n})$ such that $\mathbb{F}(\sigma) \in B(\widehat{\mathsf{k}}^{\otimes \# \sigma})$.

Guichardet space 3: quantum Wiener integrands

Let $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$. For $f \in L^2(\mathbb{R}_+; \mathsf{k})$ let $\widehat{f}(s) = \binom{1}{f(s)} \in \widehat{\mathsf{k}}$.
Consider $\mathbb{F}\colon \Gamma \to \bigoplus_{n \in \mathbb{N}} B(\widehat{\mathsf{k}}^{\otimes n})$ such that $\mathbb{F}(\sigma) \in B(\widehat{\mathsf{k}}^{\otimes \# \sigma})$.

Example

For $F\colon \mathbb{R}_+ \to B(\widehat{\mathsf{k}})$ we can define $\pi_F\colon \Gamma \to \bigoplus_{n \in \mathbb{N}} B(\widehat{\mathsf{k}}^{\otimes n})$ by

$$\pi_F(\{s_1 < \ldots < s_n\}) = F_{s_1} \otimes \ldots \otimes F_{s_n}.$$

Guichardet space 4: quantum Wiener integrals

Implicit definition: Multiple quantum Wiener integral

$$\langle \pi_f, \Lambda^{\mathbb{N}}_{s,t}(\mathbb{F})\pi_g\rangle = e^{\langle f,g\rangle} \int_{\Gamma_{s,t}} d\sigma \langle \pi_{\widehat{f}}(\sigma), \mathbb{F}(\sigma)\pi_{\widehat{g}}(\sigma)\rangle.$$

$$\Lambda^{\mathbb{N}}_t := \Lambda^{\mathbb{N}}_{0,t} \text{ "="} \sum_{n\geq 0} \Lambda^n_t.$$

Guichardet space 4: quantum Wiener integrals

Implicit definition: Multiple quantum Wiener integral

$$\langle \pi_f, \Lambda^{\mathbb{N}}_{s,t}(\mathbb{F})\pi_g\rangle = e^{\langle f,g\rangle} \int_{\Gamma_{s,t}} d\sigma \langle \pi_{\widehat{f}}(\sigma), \mathbb{F}(\sigma)\pi_{\widehat{g}}(\sigma)\rangle.$$

$$\Lambda^{\mathbb{N}}_t := \Lambda^{\mathbb{N}}_{0,t} \text{ "=" } \sum_{n\geq 0} \Lambda^n_t.$$

Bi-adaptedness property:

$$\Lambda^{\mathbb{N}}_{s,t}(\cdot) = I_{\mathcal{F}_{[0,s[}} \otimes (*) \otimes I_{\mathcal{F}_{[t,\infty[}}.$$

Guichardet space 4: quantum Wiener integrals

Implicit definition: Multiple quantum Wiener integral

$$\langle \pi_f, \Lambda^{\mathbb{N}}_{s,t}(\mathbb{F})\pi_g\rangle = e^{\langle f,g\rangle} \int_{\Gamma_{s,t}} d\sigma \langle \pi_{\widehat{f}}(\sigma), \mathbb{F}(\sigma)\pi_{\widehat{g}}(\sigma)\rangle.$$

$$\Lambda^{\mathbb{N}}_t := \Lambda^{\mathbb{N}}_{0,t} \text{ "="} \sum_{n\geq 0} \Lambda^n_t.$$

Bi-adaptedness property:

$$\Lambda^{\mathbb{N}}_{s,t}(\cdot) = I_{\mathcal{F}_{[0,s[}} \otimes (*) \otimes I_{\mathcal{F}_{[t,\infty[}}.$$

Evolution property:

$$\Lambda^{\mathbb{N}}_{r,s}(\pi_F)\Lambda^{\mathbb{N}}_{s,t}(\pi_F) = \Lambda^{\mathbb{N}}_{r,t}(\pi_F), r < s < t.$$

Well-definedness of the integral

Proposition

Let $F = \begin{pmatrix} K & M \\ L & N \end{pmatrix} : \mathbb{R}_+ \to B(\widehat{\mathsf{k}}) = \begin{pmatrix} B(\mathbb{C}) & B(\mathsf{k}; \mathbb{C}) \\ B(\mathbb{C}; \mathsf{k}) & B(\mathsf{k}) \end{pmatrix}$ *have the property that*

$$\|K_\cdot\| \in L^1_{loc}(\mathbb{R}_+), \|L_\cdot\|, \|M_\cdot\| \in L^2_{loc}(\mathbb{R}_+), \|N_\cdot\| \in L^\infty_{loc}(\mathbb{R}_+).$$

Then $\mathrm{Dom}(\Lambda_t^{\mathbb{N}}(\pi_F)) \supset \mathit{Exp}$.

Well-definedness of the integral

Proposition

Let $F = \begin{pmatrix} K & M \\ L & N \end{pmatrix} : \mathbb{R}_+ \to B(\widehat{\mathsf{k}}) = \begin{pmatrix} B(\mathbb{C}) & B(\mathsf{k};\mathbb{C}) \\ B(\mathbb{C};\mathsf{k}) & B(\mathsf{k}) \end{pmatrix}$ *have the property that*

$$\|K_\cdot\| \in L^1_{loc}(\mathbb{R}_+), \|L_\cdot\|, \|M_\cdot\| \in L^2_{loc}(\mathbb{R}_+), \|N_\cdot\| \in L^\infty_{loc}(\mathbb{R}_+).$$

Then $\mathrm{Dom}(\Lambda_t^{\mathbb{N}}(\pi_F)) \supset Exp$*.*

Proposition

Let $\mathbb{F} \in L^{1,2,2,\infty}$*. Then* $Dom(\Lambda_t^{\mathbb{N}}(\mathbb{F})) \supset Exp$*.*

Quantum Wiener convolution

Definition

For $\mathbb{F}, \mathbb{G} \in L^{1,2,2,\infty}$, define $\mathbb{F} \star \mathbb{G} \colon \Gamma \to \bigoplus_{n\in\mathbb{N}} B(\widehat{\mathsf{k}^{\otimes n}})$ by

$$\mathbb{F} \star \mathbb{G}(\sigma) = \sum_{\alpha\cup\beta=\sigma} \mathbb{F}(\alpha;\sigma)\Delta(\alpha\cap\beta;\sigma)\mathbb{G}(\beta;\sigma).$$

Quantum Wiener convolution

Definition

For $\mathbb{F}, \mathbb{G} \in L^{1,2,2,\infty}$, define $\mathbb{F} \star \mathbb{G} \colon \Gamma \to \bigoplus_{n\in\mathbb{N}} B(\widehat{\mathsf{k}^{\otimes n}})$ by

$$\mathbb{F} \star \mathbb{G}(\sigma) = \sum_{\alpha\cup\beta=\sigma} \mathbb{F}(\alpha;\sigma)\Delta(\alpha\cap\beta;\sigma)\mathbb{G}(\beta;\sigma).$$

Quantum Wiener convolution

Definition

For $\mathbb{F}, \mathbb{G} \in L^{1,2,2,\infty}$, define $\mathbb{F} \star \mathbb{G} \colon \Gamma \to \bigoplus_{n \in \mathbb{N}} B(\widehat{\mathsf{k}}^{\otimes n})$ by

$$\mathbb{F} \star \mathbb{G}(\sigma) = \sum_{\alpha \cup \beta = \sigma} \mathbb{F}(\alpha; \sigma) \Delta(\alpha \cap \beta; \sigma) \mathbb{G}(\beta; \sigma).$$

Placement notation

Let $k \leq n \in \mathbb{N}, \sigma = \{s_1, \cdots, s_n\}, \alpha = \{s_{i_1}, \cdots, s_{i_k}\}$.
$X = X_1 \otimes \ldots \otimes X_k, X_i \in B(\widehat{\mathrm{k}})$.

$$X(\alpha;\sigma)(u_1 \otimes \ldots \otimes u_n) = \bigotimes_{i=1}^{n} Y_i u_i,$$

where

$$Y_i = \begin{cases} X_j & i = i_j, j \in \{1, \cdots, k\}; \\ I & i \notin \{i_1, \cdots, i_k\}. \end{cases}$$

Placement notation

Let $k \leq n \in \mathbb{N}, \sigma = \{s_1, \cdots, s_n\}, \alpha = \{s_{i_1}, \cdots, s_{i_k}\}$.
$X = X_1 \otimes \ldots \otimes X_k, X_i \in B(\widehat{\mathrm{k}})$.

$$X(\alpha;\sigma)(u_1 \otimes \ldots \otimes u_n) = \bigotimes_{i=1}^{n} Y_i u_i,$$

where

$$Y_i = \begin{cases} X_j & i = i_j, j \in \{1, \cdots, k\}; \\ I & i \notin \{i_1, \cdots, i_k\}. \end{cases}$$

$$X(\alpha;\sigma) = X_{i_1 \cdots i_k}.$$

Placement notation

Let $k \leq n \in \mathbb{N}, \sigma = \{s_1, \cdots, s_n\}, \alpha = \{s_{i_1}, \cdots, s_{i_k}\}$.
$X = X_1 \otimes \ldots \otimes X_k, X_i \in B(\widehat{\mathrm{k}})$.

$$X(\alpha;\sigma)(u_1 \otimes \ldots \otimes u_n) = \bigotimes_{i=1}^{n} Y_i u_i,$$

where

$$Y_i = \begin{cases} X_j & i = i_j, j \in \{1, \cdots, k\}; \\ I & i \notin \{i_1, \cdots, i_k\}. \end{cases}$$

$$X(\alpha;\sigma) = X_{i_1 \cdots i_k}.$$

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathrm{k}} \end{pmatrix}.$$

Placement notation

Let $k \leq n \in \mathbb{N}, \sigma = \{s_1, \cdots, s_n\}, \alpha = \{s_{i_1}, \cdots, s_{i_k}\}$.
$X = X_1 \otimes ... \otimes X_k, X_i \in B(\widehat{\mathrm{k}})$.

$$X(\alpha;\sigma)(u_1 \otimes ... \otimes u_n) = \bigotimes_{i=1}^{n} Y_i u_i,$$

where

$$Y_i = \begin{cases} X_j & i = i_j, j \in \{1, \cdots, k\}; \\ I & i \notin \{i_1, \cdots, i_k\}. \end{cases}$$

$$X(\alpha;\sigma) = X_{i_1 \cdots i_k}.$$

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathrm{k}} \end{pmatrix}.$$

$$\mathbb{F} \star \mathbb{G}(\sigma) = \sum_{\alpha \cup \beta = \sigma} \mathbb{F}(\alpha;\sigma)\Delta(\alpha \cap \beta;\sigma)\mathbb{G}(\beta;\sigma).$$

Quantum Wiener product formula

Theorem (MJ-Lindsay)

Let $\mathbb{F}, \mathbb{G} \in L^{1,2,2,\infty}$. *Then* $\mathbb{F} \star \mathbb{G} \in L^{1,2,2,\infty}$

Quantum Wiener product formula

Theorem (MJ-Lindsay)

Let $\mathbb{F}, \mathbb{G} \in L^{1,2,2,\infty}$. *Then* $\mathbb{F} \star \mathbb{G} \in L^{1,2,2,\infty}$ *and*

$$\Lambda^{\mathbb{N}}(\mathbb{F})\Lambda^{\mathbb{N}}(\mathbb{G}) = \Lambda^{\mathbb{N}}(\mathbb{F} \star \mathbb{G}) \quad \text{on } Exp.$$

Applications

1. Universal picture of various stochastic products:
   $\mathsf{k} = \mathbb{C}$,

Applications

1. Universal picture of various stochastic products:
   $\mathsf{k} = \mathbb{C}$, $\mathcal{F} = L^2(\Gamma)$, $f, g \in \mathcal{F}$,

Applications

1. Universal picture of various stochastic products:
   $\mathsf{k} = \mathbb{C}$, $\mathcal{F} = L^2(\Gamma)$, $f, g \in \mathcal{F}$,

$$(f \star_W g)(\sigma) = \sum_{\alpha \subset \sigma} \int_\Gamma d\omega f(\alpha \cup \omega) g((\sigma \setminus \alpha) \cup \omega).$$

Applications

1. Universal picture of various stochastic products:
   $\mathrm{k} = \mathbb{C}$, $\mathcal{F} = L^2(\Gamma)$, $f, g \in \mathcal{F}$,

$$(f \star_W g)(\sigma) = \sum_{\alpha \subset \sigma} \int_\Gamma d\omega f(\alpha \cup \omega) g((\sigma \setminus \alpha) \cup \omega).$$

$$(f \star_P g)(\sigma) = \sum_{\alpha \cup \beta = \sigma} \int_\Gamma d\omega f(\alpha \cup \omega) g(\beta \cup \omega).$$

Applications

1. Universal picture of various stochastic products:
   $\mathsf{k} = \mathbb{C}$, $\mathcal{F} = L^2(\Gamma)$, $f, g \in \mathcal{F}$,

$$(f \star_W g)(\sigma) = \sum_{\alpha \subset \sigma} \int_\Gamma d\omega f(\alpha \cup \omega) g((\sigma \setminus \alpha) \cup \omega).$$

$$(f \star_P g)(\sigma) = \sum_{\alpha \cup \beta = \sigma} \int_\Gamma d\omega f(\alpha \cup \omega) g(\beta \cup \omega).$$

2. Lie-Trotter product formulae.

References

- M.Jurczyński, J.M. Lindsay, *Quantum Wiener chaos*, in preparation.

- U. Franz, *Lévy processes on quantum groups and dual groups*, Lect. Notes Math. **1866**, 161-257 (2006), Springer-Verlag Berlin Heidelberg 2006
- A. Guichardet, *Symmetric Hilbert spaces and Related Topics*, Lect. Notes Math., 1972.
- J.M. Lindsay, H. Maassen, *Duality transform as $*$-algebraic isomorphism*, Quantum Probability and Applications V, Lect. Notes Math., Springer-Verlag Berlin Heidelberg 1988
- J.M. Lindsay, A. Skalski, *Quantum stochastic convolution cocycles I*, Ann. de l'Institut H. Poincare, Vol 41, Issue 3, 2005.
- J.M. Lindsay, S.J. Wills, *Homomorphic Feller Cocycles on a C^*-algebra*, JLMS (2) 68 (2003) 255-272.
- I.E. Segal, *Tensor algebras over Hilbert spaces. I*, Trans. AMS, Vol. 81, No. 1 (1956).
- A. Skalski, *Quantum Lévy processes on topological quantum groups*, Lecture series, 3 ILJU School of Mathematics, Gyeong-Ju (Korea), 2011.
- N. Wiener, *The homogeneous chaos*, American Journal of Mathematics, Vol. 60, No. 4, 1938.