Locally compact quantum groups
4. Locally compact quantum groups, amenability, cohomological properties

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Locally compact quantum groups

Definition (Kustermans, Vaes)

A locally compact quantum group $\mathbb{G}$ is a Hopf von Neumann algebra (M, Δ) with invariant weights φ, ψ

$$(\mathrm{id} \otimes \varphi)\Delta(x) = \varphi(x)1, \qquad (\psi \otimes \mathrm{id})\Delta(x) = \psi(x)1.$$

- Means e.g. that if $x \in M^+$ with $\varphi(x) < \infty$, and $\omega \in M_*^+$, then $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \varphi(x)\langle 1, \omega \rangle$.
- Write $M = L^\infty(\mathbb{G})$; let $L^2(\mathbb{G})$ be the GNS space of φ.
- Let $\mathfrak{n}_\varphi = \{x \in L^\infty(\mathbb{G}) : \varphi(x^*x) < \infty\}$ and $\Lambda : \mathfrak{n}_\varphi \to L^2(\mathbb{G})$ be the GNS map: $(\Lambda(x)|\Lambda(y)) = \varphi(y^*x)$.

Constructions

- Define W^* on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$, as before, by
$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)).$$
- φ (left-)invariant implies W^* is an isometry.
- More subtle argument using ψ shows W is unitary.
- W is a corepresentation, $(\Delta \otimes \mathrm{id})(W) = W_{13} W_{23}$.
- $\Delta(x) = W^*(1 \otimes x)W$ for $x \in L^\infty(\mathbb{G})$.
- $L^\infty(\mathbb{G})$ is the weak*-closure of $\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$.
- There is an unbounded antipode S defined by/ which satisfies
$$S\big((\mathrm{id} \otimes \omega)(W)\big) = (\mathrm{id} \otimes \omega)(W^*), \qquad S(S(x)^*)^* = x \quad (x \in D(S)).$$
- Decompose S as $S = R\tau_{-i/2}$ where R is an anti-$*$-isomorphism and (τ_t) a continuous one-parameter group.

Duality

$$L^\infty(\widehat{\mathbb{G}}) = \{(\omega \otimes \mathrm{id})(W) : \omega \in L^1(\mathbb{G})\}''$$

- W is multiplicative; $\widehat{W} = \sigma W^* \sigma$, $\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}$.
- $W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$.
- Can construct invariant weights $\widehat{\varphi}, \widehat{\psi}$ so that $L^\infty(\widehat{\mathbb{G}})$ becomes a locally compact quantum group.
- Same duality interactions: e.g. $\widehat{J} x^* \widehat{J} = R(x)$ for $x \in L^\infty(\mathbb{G})$.
- $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$ canonically.
- Becomes a category (Ng, and Meyer–Roy–Woronowicz).

C^*-algebra considerations

$$C_0(\mathbb{G}) = \{(\mathrm{id} \otimes \omega)(W) : \omega \in L^1(\widehat{\mathbb{G}})\}^{\|\cdot\|}.$$

- This is a C^*-algebra, and $R, (\tau_t)$ restrict to it, and S becomes a norm-closed operator.
- The weights restrict to densely defined, faithful, KMS weights.
- $C_0(\mathbb{G})$ satisfies the cancellation laws.
- Can analogously axiomatise a C^*-algebraic version of the theory.
- This is a "reduced" theory: $C_r^*(G)$ is the cocommutative example.
- There is a procedure to form the "full" or "universal" C^*-completion, leading to $C_0^u(\mathbb{G})$: everything holds, but weights are no longer faithful.

Coamenability

Definition

$\mathbb{G}$ is *coamenable* if $C_0(\mathbb{G})^*$ is a unital Banach algebra.

Theorem

The following are equivalent to $\mathbb{G}$ being coamenable:

1. $L^1(\mathbb{G})$ *has a bounded approximate identity.*
2. *there is a net of unit vectors* (ξ_i) *with* $\|W(\xi_i \otimes \xi) - \xi_i \otimes \xi\| \to 0$ *for each* $\xi \in H$.
3. $C_0(\mathbb{G}) = C_0^u(\mathbb{G})$.

Sketch proof of (2)⇒(1).

For $\omega_{\xi,\eta} \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$,

$$\langle x, \omega_{\xi_i,\xi_i} \star \omega_{\xi,\eta} \rangle = \langle W^*(1 \otimes x)W, \omega\xi_i, \xi_i \otimes \omega_{\xi,\eta} \rangle = ((1 \otimes x)W(\xi_i \otimes \xi)|W(\xi_i \otimes \eta))$$
$$\approx ((1 \otimes x)(\xi_i \otimes \xi)|\xi_i \otimes \eta) = (x\xi|\eta) = \langle x, \omega_{\xi,\eta} \rangle.$$

$\square$

Amenability

Definition

$\mathbb{G}$ is *amenable* if there is a state $M \in L^\infty(\mathbb{G})^*$ with $(\mathrm{id} \otimes M)\Delta(x) = \langle M, x\rangle 1$ for $x \in L^\infty(\mathbb{G})$.

Theorem

$\widehat{\mathbb{G}}$ coamenable implies that $\mathbb{G}$ is amenable.

Proof.

If $\|\widehat{W}(\xi_i \otimes \xi) - \xi_i \otimes \xi\| \to 0$ then W unitary, $\widehat{W} = \sigma W^* \sigma$ implies $\|W(\xi \otimes \xi_i) - \xi \otimes \xi_i\| \to 0$. If M is a weak*-limit point of the net (ω_{ξ_i,ξ_i}) in $L^1(\mathbb{G})$ then for $x \in L^\infty(\mathbb{G})$,

$$\langle(\mathrm{id} \otimes M)\Delta(x), \omega_{\xi,\eta}\rangle = \lim_i \langle W^*(1 \otimes x)W, \omega_{\xi,\eta} \otimes \omega_{\xi_i,\xi_i}\rangle = \cdots = \langle M, x\rangle\langle 1, \omega_{\xi,\eta}\rangle.$$

How do you "reverse" the argument? □

See Bédos–Tuset, Int. J. Math, 2003.

Amenability 2

Theorem

Let $\mathbb{G}$ be compact with $\widehat{\mathbb{G}}$ amenable. Then $\mathbb{G}$ is coamenable.

Proof.

See Tomatsu, J. Math. Soc. Japan, 2006 (or for Kac algebras, Ruan, JFA, 1996). $\square$

Open outside the compact/discrete case.

Cohomological condition: biprojectivity

Definition

A Banach algebra A is *biprojective* if the multiplication map $\Delta_* : A\widehat{\otimes}A \to A$ has a right inverse which is an A-bimodule map: i.e. $\rho : A \to A\widehat{\otimes}A$ with $\Delta_* \circ \rho = \mathrm{id}_A$.

Can also ask in the category of operator spaces.

Theorem (Helemskii)

A is amenable if and only if it has a bounded approximate identity and is biflat ($\Leftarrow$ biprojective).

Theorem (Ruan/Xu, Aristov)

If $L^1(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is compact. If $\mathbb{G}$ is compact of Kac type, then $L^1(\mathbb{G})$ is operator biprojective.

Theorem (Caspers–Lee–Ricard)

If $L^1(\mathbb{G})$ is operator biprojective, then $\mathbb{G}$ is compact of Kac type.

Proof: diagonalisation

Fix $\mathbb{G}$ a compact quantum group.

- Have $u^\alpha \in M_{n_\alpha}(A) \cong A \otimes M_{n_\alpha}$ and associated "F matrix" F^α.
- By a change of (orthonormal) basis of $\mathbb{C}^{n_\alpha}$, say $u^\alpha \mapsto X^* u^\alpha X$, we can diagonalise F^α.
- Get strictly positive (λ_i^α) with $\sum_i \lambda_i^\alpha = \sum_i (\lambda_i^\alpha)^{-1} = m_\alpha$ the "quantum dimension",

$$\varphi\big((u_{ij}^\beta)^* u_{kl}^\alpha\big) = \delta_{\alpha,\beta}\delta_{j,l}\delta_{k,i}\frac{1}{m_\alpha \lambda_i^\alpha}, \quad \varphi\big(u_{ij}^\beta (u_{kl}^\alpha)^*\big) = \delta_{\alpha,\beta}\delta_{j,l}\delta_{k,i}\frac{\lambda_j^\alpha}{m_\alpha}.$$

- Set $Q^\alpha = t(F^\alpha)^{-1}$ with t chosen so that $\mathrm{Tr}(Q^\alpha) = \mathrm{Tr}((Q^\alpha)^{-1}) = m_\alpha$.
- Cauchy-Schwarz:
  $n_\alpha = \sum_i (\lambda_i^\alpha)^{1/2}(\lambda_i^\alpha)^{-1/2} \leq \Big(\sum_i \lambda_i^\alpha\Big)^{1/2}\Big(\sum_i (\lambda_i^\alpha)^{-1}\Big)^{1/2} = m_\alpha.$
- So $n_\alpha = m_\alpha$ iff $\lambda_i^\alpha = 1$ iff $\mathbb{G}$ is of Kac type.

Structure theory of splitting map

Suppose $\rho : L^1(\mathbb{G}) \to L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G})$ is a completely bounded splitting map, and set $\theta = \rho^* : L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$.

Theorem (D.)

There exist matrices X^α with unit trace with

$$\theta\big(u^\alpha_{ij} \otimes u^\beta_{kl}\big) = \delta_{\alpha,\beta} X^\alpha_{j,k} u^\alpha_{il}.$$

Caspers–Lee–Ricard showed this also works for biflatness (when θ is not assumed weak*-continuous).

Theorem (D.)

If θ is contractive (or completely positive), then $\mathbb{G}$ is of Kac type.

General case

Theorem (Caspers–Lee–Ricard)

Always $\mathbb{G}$ is of Kac type.

- $Q^\alpha \propto (F^\alpha)^{-1}$ is actually an intertwiner:

$$(u^\alpha)^t(1 \otimes \overline{Q^\alpha})\overline{u^\alpha} = 1 \otimes \overline{Q^\alpha}.$$

- Drop the "$1\otimes$" and regard $\mathbb{M}_n$ as a subalgebra of $\mathbb{M}_n(A)$.
- Q^α is diagonal with positive entries.
- Hence $\|(Q^\alpha)^{-1/2}(u^\alpha)^t(Q^\alpha)^{1/2}\| = \|(Q^\alpha)^{-1/2}(u^\alpha)^t Q^\alpha \overline{u^\alpha}(Q^\alpha)^{-1/2}\|^{1/2} = 1$.

$$(Q^\alpha)^{-1/2}(u^\alpha)^t(Q^\alpha)^{1/2} = \sum_{i,j} \sqrt{\frac{\lambda_j^\alpha}{\lambda_i^\alpha}}\, u_{ji}^\alpha \otimes e_{ij}.$$

Step II

Using that $M_n(L^\infty)\overline{\otimes}M_n(L^\infty) \cong M_n \otimes M_n \otimes L^\infty\overline{\otimes}L^\infty$ and that u^α unitary,

$$\begin{aligned} 1 &= \|(Q^\alpha)^{-1/2}(u^\alpha)^t(Q^\alpha)^{1/2} \otimes u^\alpha\| \\ &= \Big\| \sum e_{ij} \otimes e_{kl} \otimes \sqrt{\frac{\lambda_j^\alpha}{\lambda_i^\alpha}} u_{ji}^\alpha \otimes u_{kl}^\alpha \Big\|. \end{aligned}$$

Then apply $\theta : u_{ij}^\alpha \otimes u_{kl}^\alpha \mapsto X_{j,k}^\alpha u_{il}^\alpha$ to get

$$\sum e_{ij} \otimes e_{kl} \otimes \sqrt{\frac{\lambda_j^\alpha}{\lambda_i^\alpha}} X_{ik}^\alpha u_{jl}^\alpha.$$

Then norm of this is $\leq \|\theta\|_{cb}$ so the aim is to bound $\|\theta\|_{cb}$ below.

Row/Column spaces

- Recall that C_n is the n-dim column Hilbert space, and R_n the row space.
- For an operator space $E \subseteq \mathcal{B}(H)$ we have

$$\Big\| \sum_{i=1}^{n} e_i \otimes x_i \Big\|_{C_n \otimes E} = \Big\| \sum x_i^* x_i \Big\|_{\mathcal{B}(H)}, \quad \Big\| \sum_{i=1}^{n} e_i \otimes x_i \Big\|_{R_n \otimes E} = \Big\| \sum x_i x_i^* \Big\|_{\mathcal{B}(H)}.$$

- Then $\mathbb{M}_n \cong C_n \otimes R_n$ via $e_{ij} \leftrightarrow e_i \otimes e_j$.
- All tensor products are minimal/spacial Operator Space ones.
- $C_n \otimes C_m = C_{n \times m}$ and $R_n \otimes R_m = R_{n \times m}$.

Apply this

$$\sum e_{ij} \otimes e_{kl} \otimes \sqrt{\frac{\lambda_j^\alpha}{\lambda_i^\alpha}} X_{ik}^\alpha u_{jl}^\alpha \rightsquigarrow \Big(\sum_{i,k} \frac{X_{ik}^\alpha}{\sqrt{\lambda_i^\alpha}} e_i \otimes e_k \Big) \otimes \Big(\sum_{j,l} e_j \otimes e_l \otimes \sqrt{\lambda_j^\alpha} u_{jl}^\alpha \Big)$$

$$\in M_n \otimes M_n \otimes L^\infty \cong C_n \otimes R_n \otimes C_n \otimes R_n \otimes L^\infty \rightsquigarrow (C_n \otimes C_n) \otimes (R_n \otimes R_n \otimes L^\infty).$$

- All minimal tensor products, so "shuffle" is a complete isometry.
- 1st part in C_{n^2} with norm

$$\Big(\sum_{i,k} \frac{|X_{ik}^\alpha|^2}{\lambda_i^\alpha} \Big)^{1/2}.$$

- 2nd part in $R_{n^2} \otimes L^\infty$ with norm (as u^α unitary)

$$\Big\| \sum_{j,l} \lambda_j^\alpha u_{jl}^\alpha (u_{jl}^\alpha)^* \Big\|^{1/2} = \Big\| \sum_j \lambda_j^\alpha 1 \Big\|^{1/2} = \Big(\sum_j \lambda_j^\alpha \Big)^{1/2} = \sqrt{m_\alpha}.$$

First bound

$$\|\theta\|_{cb} \geq \Big(\sum_{i,k} \frac{|X^\alpha_{ik}|^2}{\lambda^\alpha_i} \Big)^{1/2} \sqrt{m_\alpha} \geq \Big(\sum_{i} \frac{|X^\alpha_{ii}|^2}{\lambda^\alpha_i} \Big)^{1/2} \sqrt{m_\alpha}.$$

Now swap things around:

$$1 = \| u^\alpha \otimes (Q^\alpha)^{-1/2} (u^\alpha)^t (Q^\alpha)^{1/2} \| = \Big\| \sum e_{ij} \otimes e_{kl} \otimes u^\alpha_{ij} \otimes \sqrt{\frac{\lambda^\alpha_l}{\lambda^\alpha_k}} u^\alpha_{lk} \Big\|.$$

Applying θ we get

$$\sum \sum e_{ij} \otimes e_{kl} \otimes u^\alpha_{ik} X^\alpha_{jl} \sqrt{\frac{\lambda^\alpha_l}{\lambda^\alpha_k}} \rightsquigarrow \Big(\sum_{i,k} e_i \otimes e_k \otimes u^\alpha_{ik} \frac{1}{\sqrt{\lambda^\alpha_k}} \Big) \otimes \Big(\sum_{j,l} e_j \otimes e_l \otimes X^\alpha_{jl} \sqrt{\lambda^\alpha_l} \Big)$$

in $(C_n \otimes C_n \otimes L^\infty) \otimes (R_n \otimes R_n)$.

Second bound

Repeat the argument (and use intertwining relations again) to get:

$$\|\theta\|_{cb} \geq \Big(\sum_i |X_{ii}^\alpha|^2 \lambda_i^\alpha \Big)^{1/2} \sqrt{m_\alpha}.$$

Then

$$m_\alpha \sum_i |X_{ii}^\alpha|^2 \leq \Big(m_\alpha \sum_i \frac{|X_{ii}^\alpha|^2}{\lambda_i^\alpha} \Big)^{1/2} \Big(m_\alpha \sum_i |X_{ii}^\alpha|^2 \lambda_i^\alpha \Big)^{1/2} \leq \|\theta\|_{cb}^2$$

by Cauchy-Schwarz. Again by C.-S.

$$1 = \sum_{i=1}^{n_\alpha} X_{ii}^\alpha \leq \sqrt{n_\alpha} \Big(\sum_i |X_{ii}^\alpha|^2 \Big)^{1/2}$$

so conclude

$$\|\theta\|_{cb}^2 \geq \frac{m_\alpha}{n_\alpha}.$$

The trick

- If V is any finite-dimensional unitary corepresentation then can write V as a sum of irreducibles:
$$V = \sum_{i=1}^{m} u^{\alpha_i}.$$
- Then if $Q = \bigoplus Q^{\alpha_i}$ we have $V^t Q \overline{V} = Q$.
- Estimate from before gives:
$$\mathrm{Tr}(Q) = \sum_i \mathrm{Tr}(Q^{\alpha_i}) = \sum_i m_{\alpha_i} \leq \sum_i \|\theta\|_{cb}^2 n_{\alpha_i} = \|\theta\|_{cb}^2 \dim(V).$$
- Set $V = u^\alpha \textcircled{\top} u^\alpha \textcircled{\top} \cdots \textcircled{\top} u^\alpha$ say d times.
- Fact: Q for V is equal to $(Q^\alpha)^{\otimes d}$.
- So $m_\alpha^d = \mathrm{Tr}(Q^\alpha)^d \leq \|\theta\|_{cb}^2 n_\alpha^d$.
- $d \to \infty$ implies $m_\alpha \leq n_\alpha$ so $\mathbb{G}$ Kac.