Connes-amenability of $B(G)$

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Amenable, locally compact groups

Definition

Let G be a locally compact group. A mean on $L^\infty(G)$ is a functional $M \in L^\infty(G)^*$ such that $\langle 1, M \rangle = \|M\| = 1$.

Definition (J. von Neumann 1929; M. M. Day, 1949)

G is amenable if there is a mean on $L^\infty(G)$ that is left invariant, i.e.,

$$\langle L_x\phi, M \rangle = \langle \phi, M \rangle \qquad (x \in G,\ \phi \in L^\infty(G)),$$

where

$$(L_x\phi)(y) := \phi(xy) \qquad (y \in G).$$

Some amenable and non-amenable groups

Examples

1. Compact groups are amenable: $M =$ Haar measure.
2. Abelian groups are amenable: use Markov–Kakutani to get M.
3. If G is amenable and $H < G$, then H is amenable.
4. If G is is amenable and $N \lhd G$, then G/N is amenable.
5. If G and $N \lhd G$ are such that N and G/N are amenable, then G is amenable.
6. $\mathbb{F}_2$, the free group in two generators, is not amenable.
7. If G contains $\mathbb{F}_2$ as a closed subgroup, then G is not amenable.

Banach $\mathfrak{A}$-bimodules and derivations

Definition

Let $\mathfrak{A}$ be a Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D : \mathfrak{A} \to E$ is called a derivation if

$$D(ab) := a \cdot Db + (Da) \cdot b \qquad (a, b \in \mathfrak{A}).$$

If there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}),$$

we call D an inner derivation.

Amenable Banach algebras

Remark

If E is a Banach $\mathfrak{A}$-bimodule, then so is E^*:

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E)$$

and

$$\langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E).$$

We call E^* a dual Banach $\mathfrak{A}$-bimodule.

Definition (B. E. Johnson, 1972)

$\mathfrak{A}$ is called amenable if, for every dual Banach $\mathfrak{A}$-bimodule E, every derivation $D : \mathfrak{A} \to E$, is inner.

Approximate and virtual diagonals, I

Definition (B. E. Johnson, 1972)

1. An approximate diagonal for $\mathfrak{A}$ is a bounded net $(\mathbf{d}_\alpha)_\alpha$ in the projective tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ such that

$$a \cdot \mathbf{d}_\alpha - \mathbf{d}_\alpha \cdot a \to 0 \qquad (a \in \mathfrak{A})$$

and

$$a\Delta\mathbf{d}_\alpha \to a \qquad (a \in \mathfrak{A})$$

with $\Delta : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ denoting multiplication.

2. A virtual diagonal for $\mathfrak{A}$ is an element $\mathbf{D} \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ such that

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \quad \text{and} \quad a \cdot \Delta^{**}\mathbf{D} = a \qquad (a \in \mathfrak{A}).$$

Approximate and virtual diagonals, II

Theorem (B. E. Johnson, 1972)

The following are equivalent for a Banach algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ has an approximate diagonal;*
2. *$\mathfrak{A}$ has a virtual diagonal;*
3. *$\mathfrak{A}$ is amenable.*

The meaning of amenability, I

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G:

1. *$L^1(G)$, the group algebra of G, is amenable;*
2. *G is amenable.*

Grand theme

Let $\mathcal{C}$ be a class of Banach algebras. Characterize the amenable members of $\mathcal{C}$!

The meaning of amenability, II

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ *is amenable;*
2. $\mathfrak{A}$ *is nuclear.*

Theorem (H. G. Dales, F. Ghahramani, & A. Ya. Helemskiĭ, 2002)

The following are equivalent for a locally compact group G:

1. $M(G)$*, the measure algebra of* G*, is amenable;*
2. G *is amenable and discrete.*

The meaning of amenability, III

Theorem (B. E. Forrest & VR, 2005)

The following are equivalent for a locally compact group G:

1. *$A(G)$, the Fourier algebra of G, is amenable;*
2. *G is almost abelian, i.e., has an abelian subgroup of finite index.*

Corollary

The following are equivalent for a locally compact group G:

1. *$B(G)$, the Fourier–Stieltjes algebra of G, is amenable;*
2. *G is almost abelian and compact.*

Dual Banach algebras

Definition

A dual Banach algebra is a pair $(\mathfrak{A}, \mathfrak{A}_*)$ of Banach spaces such that:

1. $\mathfrak{A} = (\mathfrak{A}_*)^*$;
2. $\mathfrak{A}$ is a Banach algebra, and multiplication in $\mathfrak{A}$ is separately $\sigma(\mathfrak{A}, \mathfrak{A}_*)$ continuous.

Examples

1. Every von Neumann algebra;
2. $(M(G), \mathcal{C}_0(G))$ for every locally compact group G;
3. $(M(S), \mathcal{C}(S))$ for every compact, semitopological semigroup S;
4. $(B(G), C^*(G))$ for every locally compact group G.

Normality and Connes-amenability

Definition (R. Kadison, BEJ, & J. Ringrose, 1972)

Let $\mathfrak{M}$ be a von Neumann algebra, and let E be a dual Banach $\mathfrak{M}$-bimodule. Then E is called normal if the module actions

$$\mathfrak{M} \times E \to E, \quad (a, x) \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are separately weak*-weak* continuous. If E is normal, we call a derivation $D : \mathfrak{M} \to E$ normal if it is weak*-weak* continuous.

Definition (A. Connes, 1976; A. Ya. Helemskiĭ, 1991)

A von Neumann algebra $\mathfrak{M}$ is Connes-amenable if, for every normal Banach $\mathfrak{M}$-bimodule E, every normal derivation $D : \mathfrak{M} \to E$ is inner.

Injectivity, semidiscreteness, and hyperfiniteness

Definition

A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$ is called

1. **injective** if there is a norm one projection $\mathcal{E} : \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}'$ (this property is independent of the representation of $\mathfrak{M}$ on $\mathfrak{H}$);
2. **semidiscrete** if there is a net $(S_\lambda)_\lambda$ of unital, weak*-weak* continuous, completely positive finite rank maps such that
$$S_\lambda a \overset{\text{weak}^*}{\longrightarrow} a \qquad (a \in \mathfrak{M});$$
3. **hyperfinite** if there is a directed family $(\mathfrak{M}_\lambda)_\lambda$ of finite-dimensional *-subalgebras of $\mathfrak{M}$ such that $\bigcup_\lambda \mathfrak{M}_\lambda$ is weak* dense in $\mathfrak{M}$.

Connes-amenability, and injectivity, etc.

Theorem (A. Connes, et al.)

The following are equivalent:

1. $\mathfrak{M}$ *is Connes-amenable;*
2. $\mathfrak{M}$ *is injective;*
3. $\mathfrak{M}$ *is semidiscrete;*
4. $\mathfrak{M}$ *is hyperfinite.*

The notions of normality and Connes-amenability make sense for every dual Banach algebra...

Normal, virtual diagonals, I

Notation

For a dual Banach algebra $\mathfrak{A}$, let $\mathcal{B}_\sigma^2(\mathfrak{A}, \mathbb{C})$ denote the separately weak* continuous bilinear functionals on $\mathfrak{A}$.

Observations

1. $\mathcal{B}_\sigma^2(\mathfrak{A}, \mathbb{C})$ is a closed submodule of $(\mathfrak{A} \hat{\otimes} \mathfrak{A})^*$.
2. $\Delta^* \mathfrak{A}_* \subset \mathcal{B}_\sigma^2(\mathfrak{A}, \mathbb{C})$, so that $\Delta^{**} : (\mathfrak{A} \hat{\otimes} \mathfrak{A})^{**} \to \mathfrak{A}^{**}$ drops to a bimodule homomorphism $\Delta_\sigma : \mathcal{B}_\sigma^2(\mathfrak{A}, \mathbb{C})^* \to \mathfrak{A}$.

Normal, virtual diagonals, II

Definition (E. G. Effros, 1988; for von Neumann algebras)

Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathbf{D} \in \mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})^*$ is called a normal, virtual diagonal for $\mathfrak{A}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{A})$$

and

$$a \Delta_\sigma \mathbf{D} = a \qquad (a \in \mathfrak{A}).$$

Proposition

Suppose that $\mathfrak{A}$ has a normal, virtual diagonal. Then $\mathfrak{A}$ is Connes-amenable.

Normal, virtual diagonals and Connes-amenability

Question

Is the converse true?

Theorem (E. G. Effros, 1988)

A von Neumann algebra $\mathfrak{M}$ is Connes-amenable if and only if $\mathfrak{M}$ has a normal virtual diagonal.

Theorem (VR, 2003)

The following are equivalent for a locally compact group G:

1. *G is amenable;*
2. *$M(G)$ is Connes-amenable;*
3. *$M(G)$ has a normal virtual diagonal.*

Weakly almost periodic functions

Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called weakly almost periodic if $\{L_x f : x \in G\}$ is relatively weakly compact in $\mathcal{C}_b(G)$. We set

$$\mathcal{WAP}(G) := \{f \in \mathcal{C}_b(G) : f \text{ is weakly almost periodic}\}.$$

Remark

$\mathcal{WAP}(G)$ is a commutative C^*-algebra. Its character space $G_{\mathcal{WAP}}$ is a compact, semitopological semigroup containing G as a dense subsemigroup. This turns $\mathcal{WAP}(G)^* \cong M(G_{\mathcal{WAP}})$ into a dual Banach algebra.

Connes-amenability without a normal, virtual diagonal

Proposition

The following are equivalent:

1. G *is amenable;*
2. $\mathcal{WAP}(G)^*$ *is Connes-amenable.*

Theorem (VR, 2006 & 2013)

Suppose that G *is a* [SIN]*-group. Then the following are equivalent:*

1. $\mathcal{WAP}(G)^*$ *has a normal virtual diagonal;*
2. G *is compact.*

$\mathcal{C}_\sigma^w$-elements, I

Definition

Let $\mathfrak{A}$ be a dual Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. We call $x \in E$ a $\mathcal{C}_\sigma^w$-element if the maps

$$\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are weak*-weakly continuous.

Notation

$$\mathcal{C}_\sigma^w(E) := \{x \in E : x \text{ is a } \mathcal{C}_\sigma^w\text{-element}\}.$$

$\mathcal{C}_\sigma^w$-elements, II

Observations

1. $\mathcal{C}_\sigma^w(E)$ is a closed submodule of E.
2. $\mathcal{C}_\sigma^w(E)^*$ is normal.
3. E^* is normal if and only if $E = \mathcal{C}_\sigma^w(E)$.
4. If $\theta \colon E \to F$ is a bounded, $\mathfrak{A}$-bimodule homomorphism, then $\theta(\mathcal{C}_\sigma^w(E)) \subset \mathcal{C}_\sigma^w(F)$.
5. As $\mathfrak{A}_* \subset \mathcal{C}_\sigma^w(\mathfrak{A}^*)$, we have $\Delta^*\mathfrak{A}_* \subset \mathcal{C}_\sigma^w((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)$, and so $\Delta^{**} : (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**} \to \mathfrak{A}^{**}$ drops to a bimodule homomorphism $\Delta_\sigma^w \colon \mathcal{C}_\sigma^w((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^* \to \mathfrak{A}$.

$\mathcal{C}_\sigma^w$-diagonals and Connes-amenability

Definition (VR, 2004)

Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathbf{D} \in \mathcal{C}_\sigma^w((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^*$ is called a $\mathcal{C}_\sigma^w$-diagonal for $\mathfrak{A}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{A})$$

and

$$a\Delta_\sigma^w \mathbf{D} = a \qquad (a \in \mathfrak{A}).$$

Theorem (VR, 2004)

For a dual Banach algebra $\mathfrak{A}$, the following are equivalent:

1. *$\mathfrak{A}$ is Connes-amenable;*
2. *$\mathfrak{A}$ has a $\mathcal{C}_\sigma^w$-diagonal.*

From $C^*(G \times G)$ into $\mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)$...

Lemma

Let $\mathfrak{A}$ be a dual Banach algebra. Then the canonical map from $\mathfrak{A}_*\check{\otimes}\mathfrak{A}_*$ into $(\mathfrak{A}\hat{\otimes}\mathfrak{A})^*$ is an isometric $\mathfrak{A}$-bimodule homomorphism with range in $\mathcal{C}_\sigma^w((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)$.

Corollary

Let G be a locally compact group. Then there is a canonical contractive $B(G)$-bimodule homomorphism from $C^*(G \times G)$ into $\mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)$.

...and from $\mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)^*$ into $B(G_d \times G_d)$

Observation

Let $\theta\colon C^*(G \times G) \to \mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)$ be the canonical $B(G)$-bimodule homomorphism.

- There is a canonical $B(G)$-bimodule homomorphism $\pi\colon C^*(G_d \times G_d) \to W^*(G \times G)$.
- Thus, $(\pi \circ \theta^{**})^*\colon (\mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*))^{***} \to B(G_d \times G_d)$ is a $B(G)$-bimodule homomorphism.

Let $\kappa\colon \mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)^* \to (\mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*))^{***}$ be the canonical embedding, and set $\Theta := (\pi \circ \theta^{**})^* \circ \kappa$.

- Then $\Theta\colon \mathcal{C}_\sigma^w(B(G)\hat{\otimes}B(G))^*)^* \to B(G_d \times G_d)$ is a $B(G)$ bimodule homomorphism.

$B(G)$ with a $\mathcal{C}_\sigma^w$-diagonal, I

Proposition

Let G be a locally compact group such that $B(G)$ is Connes-amenable, and let $\mathbf{D} \in \mathcal{C}_\sigma^w((B(G)\hat{\otimes}B(G))^*)^*$ be a $\mathcal{C}_\sigma^w$-diagonal for $B(G)$. Then $\Theta(\mathbf{D}) \in B(G_d \times G_d)$ is the indicator function of the diagonal of $G \times G$, i.e., of

$$\{(x,x) : x \in G\}.$$

$B(G)$ with a $\mathcal{C}_\sigma^w$-diagonal, II

Theorem (VR & F. Uygul, 2013)

The following are equivalent for a locally compact group G:

1. *$B(G)$ is Connes-amenable;*
2. *$B(G)$ has a $\mathcal{C}_\sigma^w$-diagonal;*
3. *$B(G)$ has a normal, virtual diagonal;*
4. *G is almost abelian.*

$B(G)$ with a $\mathcal{C}_\sigma^w$-diagonal, III

Proof.

We shall prove (ii) $\Longrightarrow$ (iv).
For $f \in B(G)$, define $\check{f} \in B(G)$ by

$$\check{f}(x) := f(x^{-1}).$$

Let

$$^\vee : B(G) \to B(G), \quad f \mapsto \check{f}.$$

Easy:

$$(\mathrm{id} \otimes {}^\vee)^* : (B(G)\hat{\otimes}B(G))^* \to (B(G)\hat{\otimes}B(G))^*$$

maps $\mathcal{C}_\sigma^w((B(G)\hat{\otimes}B(G))^*)$ into itself.

$B(G)$ with a $\mathcal{C}_\sigma^w$-diagonal, IV

Proof (continued).

Let $\mathbf{D} \in \mathcal{C}_\sigma^w((B(G)\hat{\otimes}B(G))^*)^*$ be a $\mathcal{C}_\sigma^w$-diagonal for $B(G)$, and set

$$\chi := \theta((\mathrm{id} \otimes \; ^\vee)^{**}(\mathbf{D})) \in B(G_d \times G_d).$$

Then χ is the indicator function of the anti-diagonal of $G \times G$, i.e.,

$$\{(x, x^{-1}) : x \in G\}.$$

This means that $^\vee : B(G) \to B(G)$ is completely bounded, which is possible only if $C^*(G)$ is subhomogeneous, i.e., G is almost abelian. $\square$