Operator space structure on Fourier and Fourier-Stieltjes algebras

Nico Spronk, U. Waterloo

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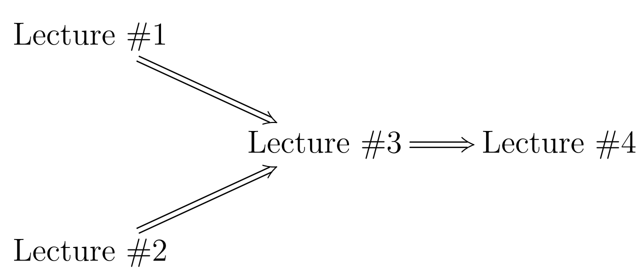
The following notes are meant to accompany lectures given March 27, 28, 31 and April 1, 2014, for the session "Banach and Operator Algebras over Groups", in the *Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras*, at the Fields Institute. The author is grateful to session organizer H. G. Dales, and to the general organizers of the thematic program, for the invitation to speak in this capacity. The author is very grateful to the audience for attending and asking valuable questions, and is further grateful to H. G. Dales for necessary copy editing.

Since these notes are accompanying a series of lectures, there will be only sketches of proofs, at best, possibly at the level of heuristics. The author has made an effort to thoroughly and accurately reference the sources of known facts. Any (inevitable) failure to do so is his fault.

My goal is for these notes to serve as an invitation to the study harmonic analysis, in particular on Fourier algebras, with the advent of operator spaces. The scope is being purposely and brutally limited in order to gain some depth and highlight some key results of about 10 years ago. The results being highlighted are chosen because they partially, though quite satisfactorily, answer quite classical-sounding questions with reasonably modern techniques.

Coarsely speaking, the logic flow is given thus.



Lecture #1: What are Fourier Stieltjes and Fourier algebras?

The Fourier-Stieltjes and Fourier algebras are objects meant to generalize Pontryagin duality from abelian groups. (Happily, these objects even generalize Tannaka-Krein duality for compact groups, but I will not touch on that explicitly.)

Measure algebras and group algebras. Let G be a locally compact group. We let $\mathrm{M}(G) \cong \mathcal{C}_0(G)^*$ denote the *measure algebra* whose algebric structure is realized by convolution: if $\mu, \nu \in \mathrm{M}(G)$, then

$$\int_G \varphi \, d(\mu * \nu) = \int_G \int_G \varphi(st) \, d\mu(s) \, d\nu(t)$$

for φ in $\mathcal{C}_0(G)$; hence $\mathrm{M}(G)$ is a Banach algebra We now let m denote the left Haar measure (which is unique up to scalar). Notice that $s \mapsto \delta_s$ (Dirac measure) defines an injective isomorphism form G into $\mathrm{M}(G)$ which is a homeomorphism when the weak* topology is used in the codomain.

We let the *group algebra*

$$\begin{aligned} \mathrm{L}^1(G) &= \{\mu \in \mathrm{M}(G) : \mu \ll m\} \lhd \mathrm{M}(G) && (\dagger) \\ &\cong \left\{ f : G \to \mathbb{C} \mid \int_G |f| \, dm < \infty \right\} / \sim_{a.e.} && (\ddagger) \end{aligned}$$

where the ideal property (†) holds by virtue of left-invariance on m, and the identification (‡) is thanks to the Radon-Nikodym Theorem. Observe that, qua functions (modulo a.e. equivalence), we have convolution

$$f * g(s) = \int_G f(t) g(t^{-1}s) \, dt$$

where $dt = dm(t)$. Furthermore we have a contactive approximate identity $\left(\frac{1}{m(U)} 1_U\right)_{U \in \mathcal{U}}$, where $\mathcal{U}$ denotes the relatively compact neighbourhoods of the identity e_G, partially ordered with respect to reverse inclusion.

Proposition. *Let $\mathcal{X}$ be a Banach space. Then there is a bijective correspondence between:*

(i) *weak operator continuous representations $\pi : G \to \mathcal{B}_{inv}(\mathcal{X})$ (bounded invertible operators) for which $\|\pi\|_\infty = \sup_{s \in G} \|\pi(s)\| < \infty$ [resp., the image is of isometries]; and*

(ii) *bounded [resp., contractive] homomorphisms* $\pi_1 : \mathrm{L}^1(G) \to \mathcal{B}(\mathcal{X})$ *such that* $\pi_1(\mathrm{L}^1(G))\mathcal{X}$ *is dense in* $\mathcal{X}$.

Idea. Given π as in (i), let $\pi_1(f) = \text{w.o.t.-}\int_G f(s)\pi(s)\,ds$. Thanks to approximation by compactly supported elements and a Krein-Smulian argument – see [Joh-Mem, §2] – this integral defines an element of $\mathcal{B}(\mathcal{X})$. Conversely, $\pi(s)\pi_1(f)\xi = \pi_1(\delta_s * f)\xi$ extends uniquely to a representation of G on $\mathcal{X}$. □

Moral:

G "=" $\mathrm{L}^1(G)$, with respect to actions on Banach spaces.

Let us finish this section by noting the unitary representations. We observe that $\mathrm{L}^1(G)$ is an involute algebra via $f^*(s) = \frac{1}{\Delta(s)}\overline{f(s^{-1})}$. We let Σ_G denote the class of weak operator continuous unitary representations of G. In particular let $\lambda : G \to \mathcal{U}(\mathrm{L}^2(G))$ denote the *left regular representation*: $\lambda(s)f(t) = f(s^{-1}t)$. Notice that $\lambda_1(f)\xi = f * \xi$. We then let for f in $\mathrm{L}^1(G)$

$$\|f\|_* = \sup\{\|\pi_1(f)\| : \pi \in \Sigma_G\}$$

which clearly defines a C*-seminorm: $\|f^* * f\|_* = \|f\|_*^2$. Observe that $\|\lambda_1(f)\| \leq \|f\|_* \leq \|f\|_1$, from which it follows that $\|\cdot\|_*$ is, in fact, a norm. Then let the *universal group C*-algebra* be given by the completion $\mathrm{C}^*(G) = \overline{\mathrm{L}^1(G)}^{\|\cdot\|_*}$.

Proposition. *There is a bijective correspondence between:*

(i) π *in* Σ_G*;*

(ii) *continuous* $*$*-representations* $\pi_1 : \mathrm{L}^1(G) \to \mathcal{B}(\mathcal{H}_\pi)$ *for which* $\pi_1(\mathrm{L}^1(G))\mathcal{H}_\pi$ *is dense in* $\mathcal{H}_\pi$*; and*

(iii) $*$*-representations* $\pi_* : \mathrm{C}^*(G) \to \mathcal{B}(\mathcal{H}_\pi)$ *for which* $\pi_*(\mathrm{C}^*(G))\mathcal{H}_\pi$ *is dense in* $\mathcal{H}_\pi$.

While the reader is encouraged to try this as an exercise, more detailed discussions can be found in [Dix-C*].

Pontryagin duality of abelian groups. Letting $\mathbb{T} = \mathcal{U}(\mathbb{C})$, Schur's lemma provides for a locally compact abelian G the identification

$$\widehat{G} = \{\pi \in \Sigma_G : \pi \text{ is irreducible}\}/\cong_u \; = \mathrm{Hom}_c(G, \mathbb{T})$$

where $\cong_u$ is the relation of unitary equivalence. Thus $\widehat{G}$ is a group via pointwise multiplication: $\sigma\tau(s) = \sigma(s)\tau(s)$; and further is a locally compact group when equipped with topology of uniform convergence on compacta. The latter unobvious fact should be checked in [HewRos-I, Fol-AbHarAn], as well as the following beautiful theorem.

Pontryagin's duality theorem. $\widehat{\widehat{G}} \cong G$ *isomorphically and homeomorphically; in fact,* $\widehat{\widehat{G}} = \{\hat{s} : s \in G\}$*.*

For non-abelian G, $\widehat{G} \not\cong \mathrm{Hom}_c(G, \mathbb{T})$. For example $\mathrm{Hom}_c(G, \mathbb{T}) = \{1\}$, the trivial character, if $G = S$ is itself a connected simple Lie group or $G = \mathbb{R}^n \rtimes S$, where S acts on $\mathbb{R}^n$ without non-trivial fixed points (say $S = \mathrm{SO}(n)$ or $\mathrm{SL}_n(\mathbb{R})$). Indeed, the Lie algebra of S admits no proper ideals, from which it can be deduced that any continuous finite-dimensional representation has either discrete kernel or is trivial. Hence any multiplicative character σ on $\mathbb{R}^n \rtimes S$, restricted to $\mathbb{R}^n$ must be constant on orbits of S, in particular on orbits of the maximal compact subgroup K of S ($K = \mathrm{SO}(n)$ in the case $S = \mathrm{SL}_n(\mathbb{R})$), which means that $\sigma(\mathbb{R}^n) = \{1\}$, as can be checked. In particular, this naive approach to Pontryagin duality is doomed to fail. Tannaka-Krein-duality in compact groups attempts to rectify this, but is very complicated; see, for example, [HewRos-II].

Hence we are motivated to see if the philosophy $\widehat{G}$" = "$\mathrm{L}^1(\widehat{G})$ may point us in the correct direction. We let $\hat{F} : \mathrm{L}^1(\widehat{G}) \to \mathcal{C}_0(G)$ denote the *Fourier transform*

$$\hat{F}(\hat{f})(s) = \int_{\widehat{G}} \hat{f}(\sigma)\sigma(s)\, d\sigma.$$

(That its range is within $\mathcal{C}_0(G)$ is the Riemann-Lebesgue lemma.) Again, see [HewRos-I, Fol-AbHarAn] for the following.

Plancherel's Theorem. $\hat{F}(\mathrm{L}^1 \cap \mathrm{L}^2(\widehat{G})) \subset \mathrm{L}^2(G)$ *and this map extends to a unitary* $U : \mathrm{L}^2(\widehat{G}) \to \mathrm{L}^2(G)$*.*

Notice that $U(\hat{s}\hat{\xi}) = \lambda(s)\xi$ where $\xi = U\hat{\xi}$. We then compute for $\hat{f}$ in $\mathrm{L}^1(\widehat{G})$, which we factor pointwise a.e. as a product of elements of $\mathrm{L}^2(\widehat{G})$, $\hat{f} = \hat{\xi}\bar{\hat{\eta}}$:

$$\hat{F}(\hat{f})(s) = \int_{\widehat{G}} \hat{\xi}(\sigma)\overline{\hat{\eta}(\sigma)}\sigma(s)\; ds = \int_G U(\hat{s}\hat{f})\overline{U\hat{\eta}}\, dm = \langle \lambda(s)\xi | \eta \rangle.$$

Hence $\mathrm{A}(G) := \hat{F}(\mathrm{L}^1(\widehat{G})) = \{\langle\lambda(\cdot)\xi|\eta\rangle : \xi, \eta \in \mathrm{L}^2(G)\}$. We also have that $\hat{F}(\hat{f} * \hat{g}) = \hat{F}(\hat{f})\hat{F}(\hat{g})$ (pointwise product), from which it follows that $\mathrm{A}(G)$ is a subalgebra of $\mathcal{C}_0(G)$, in fact a Banach algebra when equipped with the norm $\|\hat{F}(\hat{f})\|_{\mathrm{A}} = \|\hat{f}\|_1$.

Construction of Fourier-Stieltjes and Fourier algebras for general locally compact G. The last result above suggests that matrix coefficient functions of unitary representations may be useful. Let us advance this theme, albeit a tad circuitously. We begin by defining functions of *positive type* (or "positive definite" functions, as used more widely in the English language literature) by

$$\mathrm{P}(G) = \{u \in \mathcal{C}_b(G) : \forall s_1, \dots, s_n \in G, n \in \mathbb{N}, [u(s_j s_i)] \text{ is positive semidefinite}\}.$$

As an example, if $\pi \in \Sigma_G$, $\xi \in \mathcal{H}_\pi$, then $\langle\pi(\cdot)\xi|\xi\rangle$ is of positive type. Try, as an exercise, to see the converse, below; or look in [Dix-C*].

Gelfand-Naimark construction. *The following are equivalent:*

(i) $u \in \mathrm{P}(G)$*;*

(ii) $u = \langle\pi(\cdot)\xi|\xi\rangle$*, where* $\pi \in \Sigma_G$*,* $\xi \in \mathcal{H}_\pi$

(iii) $u \sim_{a.e.} u'$*,* $u' \in \mathrm{L}^\infty(G)$ *and* u' *defines a state on* $\mathrm{C}^*(G)$ *via the natural extentsion of the* L^1*-*L^∞ *duality.* $\langle\pi(\cdot)\xi|\xi\rangle$ for some π in Σ_G and ξ in $\mathcal{H}_\pi$.

We thus define the *Fourier-Stieltjes algebra* by

$$\boxed{\mathrm{B}(G) = \mathrm{span}\mathrm{P}(G).}$$

Proposition. (i) $\mathrm{B}(G) = \{\langle\pi(\cdot)\xi|\eta\rangle : \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$.

(ii) *We have that*

$$\mathrm{B}(G) \cong \left\{u \in \mathrm{L}^\infty(G) : \|u\|_{\mathrm{B}} = \sup\left\{\left|\int_G uf\,dm\right| : f \in \mathrm{L}^1(G), \|f\|_* \le 1\right\} < \infty\right\}$$

and $\mathrm{B}(G) \cong \mathrm{C}^*(G)^*$.

Observe that (i) is a direct application of the polarization identity $4\langle\xi|\eta\rangle = \sum_{k=0}^3 i^k\langle\xi + i^k\eta|\xi + i^k\eta\rangle$, and (ii) follows from the fact that states span the the dual of a C*-algebra. Furthermore, observe that $\mathrm{B}(G)$ is an algebra

$$\langle\pi(\cdot)\xi|\eta\rangle\langle\pi'(\cdot)\xi'|\eta'\rangle = \langle\pi \otimes \pi'(\cdot)\xi \otimes \xi'|\eta \otimes \eta'\rangle.$$

To see that $\mathrm{B}(G)$ is a Banach algebra with respect to $\|\cdot\|_{\mathrm{B}}$, we introduce the *universal von Neumman algebra* $\mathrm{W}^*(G) \cong \mathrm{C}^*(G)^{**}$, which is hence the dual of $\mathrm{B}(G)$. If we let for each u in $\mathrm{P}_1(G)$ (i.e. $\|u\|_{\mathrm{B}} = 1$), π_u denote the associated Gelfand-Naimark cyclic representation and $\varpi = \varpi_G = \bigoplus_{u \in \mathrm{P}_1(G)} \pi_u$ on $\mathcal{H}_\varpi = \ell^2\text{-}\bigoplus_{u \in \mathrm{P}_1(G)} \mathcal{H}_u$, then $\mathrm{W}^*(G) = \varpi(G)''$ (second commutant). If $\pi \in \Sigma_G$, we let $\mathrm{VN}_\pi = \pi(G)''$ and there is a unique $*$-homomorphism

$$\pi'' : \mathrm{W}^*(G) \twoheadrightarrow \mathrm{VN}_\pi \text{ such that } \pi''(\varpi(s)) = \pi(s) \text{ for } s \text{ in } G.$$

Hence $\varpi \otimes \varpi : G \to \mathcal{U}(\mathcal{H}_\varpi \otimes^2 \mathcal{H}_\varpi)$ begets $\Delta = (\varpi \otimes \varpi)'' : \mathrm{W}^*(G) \to \mathrm{W}^*(G) \bar{\otimes} \mathrm{W}^*(G)$ (von Neumann tensor product), which is a weak*-weak* continuous map satisfying $\Delta(\varpi(s)) = \varpi(s) \otimes \varpi(s)$. Let $\iota : \mathrm{B}(G) \otimes^\gamma \mathrm{B}(G) \to (\mathrm{W}^*(G) \bar{\otimes} \mathrm{W}^*(G))_*$ (projective tensor product to predual) be the natural contractive map. Then $\Delta_* \circ \iota : \mathrm{B}(G) \otimes^\gamma \mathrm{B}(G) \to \mathrm{B}(G)$ is contractive pointwise multiplication.

Returning briefly to abelian G, we observe the Jordan decomposition $\mathrm{M}(G) = \operatorname{span} \mathrm{M}(G)^+$ where $\mathrm{M}(G)^+ = \mathbb{R}^{\geq 0} \mathrm{Prob}(G)$.

Bochner's Theorem. *The Fourier-Stieltjes transform* $\widehat{FS}$ *satisfies* $\widehat{FS}(\mathrm{M}(\widehat{G})^+) = \mathrm{P}(G)$.

Hence for abelian G it follows that $\mathrm{B}(G) \cong \mathrm{M}(\widehat{G})$. The fact that $\mathrm{C}^*(G) \cong \mathcal{C}_0(\widehat{G})$ (Gelfand Theorem for commutative C*-algebras), in this case, shows that the above identification is isometric. We recall, from above, that $\mathrm{A}(G) = \hat{F}(\mathrm{L}^1(G)) = \{\langle \lambda(\cdot)\xi|\eta\rangle : \xi, \eta \in \mathrm{L}^2(G)\}$. This motivates us to the next step.

Let us return to a general locally compact G. If $\pi \in \Sigma_G$ we set

$$\mathrm{A}_\pi = \overline{\operatorname{span}}^{\|\cdot\|_{\mathrm{B}}} \{\langle \pi(\cdot)\xi|\eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}.$$

Theorem. [Ars] $\mathrm{A}_\pi^* \cong \mathrm{VN}_\pi \cong z_\pi \mathrm{W}^*(G)$, *where* z_π *is the* central cover *of* π.

Now we define the *Fourier algebra* [Eym, Stin59, Kre]:

$$\boxed{\mathrm{A}(G) = \mathrm{A}_\lambda.}$$

Outside of abelian or compact groups (where thanks to the Peter-Weyl theorem λ enjoys a natural quasi-equivalence to ϖ), it may be hard, a priori, to get excited by this definition. Allow me to try to generate some enthusiasm.

Theorem. $\mathrm{A}(G)$ *is an ideal in* $\mathrm{B}(G)$*, in particular a closed subalgebra.*

Idea 1. Fell's absorption principle: $\lambda \otimes \pi \cong_u \lambda^{(\dim \pi)}$, where $\dim \pi$ is the Hilbertian dimension of $\mathcal{H}_\pi$. More specifically, we obtain unitary intertwiners $\mathrm{L}^2(G) \otimes^2 \mathcal{H}_\pi \cong \mathrm{L}^2(G, \mathcal{H}_\pi) \cong \mathrm{L}^2(G)^{(\dim \pi)}$. Notice that on $\mathrm{L}^2(G, \mathcal{H}_\pi)$, $\lambda \otimes \pi$ takes the form $\xi \mapsto \pi(s)\xi(s^{-1}\cdot)$. Check that the unitary map $\xi \mapsto \pi(\cdot)^*\xi(\cdot)$ intertwines this representation with $\lambda(\cdot) \otimes \mathrm{id}_{\mathcal{H}} \cong \lambda^{(\dim \pi)}$.

Idea 2. [Eym] $\mathrm{A}(G) = \overline{\mathrm{B} \cap \mathcal{C}_c(G)}^{\|\cdot\|_{\mathrm{B}}}$. In [Ped-C*] there is a proof that $\mathrm{P} \cap \mathcal{C}_c(G) \subset \mathrm{A}(G)$, which helps establish the non-trivial inclusion, above. □

In light of the results for abelian groups, above, we philosophically think that

$$\mathrm{A}(G) = \text{“}\mathrm{L}^1(\widehat{G})\text{”}, \quad \mathrm{B}(G) = \text{“}\mathrm{M}(\widehat{G})\text{”}$$

even when the space $\widehat{G} = \{\pi \in \Sigma_G : \pi \text{ irreducible}\}/ \cong_u$ does not admit a structure as a Hausdorff topological space, hence the obnoxious "greengrocer's quotes".

Theorem. (i)[Eym, Herz] *We have Gelfand spectrum* $\Gamma_{\mathrm{A}(G)} = \lambda(G) \cong G$.
(ii) [Eym, Haag75] $\mathrm{A}(G) = \{\langle \lambda(\cdot)\xi | \eta \rangle : \xi, \eta \in \mathrm{L}^2(G)\}$*, i.e. we do not require closed linear span.*

A consequence of Pontryagin's duality theorem for abelian groups tells us that $\widehat{G} \cong \widehat{H}$ isomorphically and homeomorphically only if $G \cong H$. The following should be regarded as generalizations of this fact for general locally compact G and H.

Theorem. (i)[Wen] $\mathrm{L}^1(G) \cong \mathrm{L}^1(H)$ *isometrically isomophically exactly when* $G \cong H$.
(ii) [Wal] $\mathrm{A}(G) \cong \mathrm{A}(H)$ *isometrically isomophically exactly when* $G \cong H$.

Group C*-algebras do not admit such a nice theorem as above. Consider the discrete direct sum groups $G_n = (\mathbb{Z}/n\mathbb{Z})^{\oplus \mathbb{N}}$ for $n = 2, 3, \ldots$. Then $\mathrm{C}^*(G_n) \cong \mathcal{C}(\widehat{G}_n)$, where $\widehat{G}_n \cong (\mathbb{Z}/n\mathbb{Z})^{\mathbb{N}}$ (direct product, product topology) is a Cantor set. For a non-abelian example, if G is either the dihedral group of order 8, or the quaternion group, then $\mathrm{C}^*(G) \cong \ell^\infty_4 \oplus_{\ell^\infty} \mathcal{B}(\ell^2_2)$.

We close by giving a list of conditions relating locally compact groups to their algebras. What is not referenced entails an exercise for the reader.

G	$\mathrm{L}^1(G)$	$\mathrm{A}(G)$
compact	Haar idempotent (HI)	unital
abelian	commutative	maximal operator space (Lec. 2)
amenable	amenable (Lec. 3)	bounded approximate identity [Lep] (Lec. 3)
compact normal subgroup	contractive δ_G-invariant projection [Wen, KawIto]	translation & conjugation invariant subalgebra [BekLauSch]
compact connected component of e_G	admits idempotent	admits idempotent
totally disconnected	bounded approximate identity consisting of idempotents ($*$)	span of idempotents dense
discrete	unital	Haar idempotent (HI')

(HI) There is norm 1 h in $\mathrm{L}^1 \cap \mathrm{Prob}(G)$ for which $h * f = \left(\int_G f\, dm\right) h$
(HI') There is norm 1 u in $\mathrm{A} \cap \mathrm{P}(G)$ for which $uv = v(e)u$.
Note that in the case of abelian G that $\hat{F}(\hat{f})(e) = \int_{\widehat{G}} \hat{f}(\sigma)\, d\sigma$.

The author is grateful to G. Willis for suggesting the characterization ($*$).

Thanks to [Glea], G is Lie if it admits no small normal subgroups, i.e. there exists neighbourhood U of e which contains no normal subgroup. Hence Lieness of G can be witnessed by $\mathrm{L}^1(G)$ in the absence of proper net of δ_G-invariant projections which tend strictly to δ_e; likewise by an absence of a dense family of proper translation and conjugation-invariant closed subalgebras of $\mathrm{A}(G)$.

Lecture #2: What are operator spaces?

The goal is to give a brief introduction to operator spaces, ultimately with specific emphasis on the preduals of von Neumann algebras. My main sources for this are the books [EffRuan-OpSp, Pis-OpSp].

Before beginning, let me summarize the desired objects. My goal is to suggest that defining and understanding the morphisms is the goal for understanding the objects.

objects	morphisms
complex vector spaces	linear maps
normed/Banach spaces	bounded linear maps or, contractive linear maps
operator spaces	completely bounded linear maps or, completely contractive linear maps

Note that in the category (BanSp,bounded), invertible morphisms are typically called isomorphisms, whilst in (BanSp,contractive), invertible morphisms are the isometric isomorphisms.

Definition of the objects. Let $\mathcal{V}$ be a complex vector space. We let $\mathrm{M}_n(\mathcal{V})$ denote the vector space of matrices with entries in $\mathcal{V}$; write $V = [v_{ij}]$ for an element of $\mathrm{M}_n(\mathcal{V})$.

1st definition. An *operator space structure* is a sequence $\left(\|\cdot\|_n : \mathrm{M}_n(\mathcal{V}) \to [0,\infty)\right)_{n=1}^{\infty}$ which satisfies

(D) $\left\| \begin{bmatrix} V & 0 \\ 0 & V' \end{bmatrix} \right\|_{n+m} = \max\{\|V\|_n, \|V'\|_m\}$, and

(M) $\|\alpha V \beta\| \leq \|\alpha\|_{\mathcal{B}(\ell^2_n)} \|V\|_n \|\beta\|_{\mathcal{B}(\ell^2_n)}$ where $\alpha, \beta \in \mathrm{M}_n \cong \mathcal{B}(\ell^2_n)$.

Notice that if $\mu, \nu \in \mathrm{U}(n)$ (unitary scalar), then $\|\mu V \nu\|_n = \|V\|_n$.

2nd definition. We consider norms $\|\cdot\|_n$, each on the tensor product space $\mathrm{M}_n \otimes \mathcal{V}$, for which

(D′) $(\mathrm{M}_n \otimes \mathcal{V}) \oplus_{\ell^\infty} (\mathrm{M}_m \otimes \mathcal{V}) \hookrightarrow \mathrm{M}_{n+m} \otimes \mathcal{V}$ isometrically, and

(M′) $\|\alpha \cdot V \cdot \beta\| \leq \|\alpha\|_{\mathcal{B}(\ell^2_n)} \|V\|_n \|\beta\|_{\mathcal{B}(\ell^2_n)}$ where $\alpha, \beta \in \mathrm{M}_n \cong \mathcal{B}(\ell^2_n)$

where, in (M'), $\alpha \cdot (\kappa \otimes v) \cdot \beta = \alpha\kappa\beta \otimes v$.

3rd definition. Let $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ denote the C*-algebra of compact operators. We consider a norm on the algebraic tensor product $\mathcal{K} \otimes \mathcal{V}$ such that

$$\text{(R)} \qquad \left\| \sum_{i=1}^{n} \alpha_i \cdot V_i \cdot \beta_i \right\| \leq \left\| \sum_{i=1}^{n} \alpha_i \alpha_i^* \right\|^{1/2} \max_{i=1,\dots,n} \|V_i\| \left\| \sum_{i=1}^{n} \beta_i^* \beta_i \right\|^{1/2}$$

where $\alpha \cdot V \cdot \beta$ is defined analogously as above.

It is an exercise to see that these three definitions are equivalent. Axioms (D) and (M) are due to Ruan; the formulation (R), named in honour of you-know-who, is given in [Pis-OpSp]. Let us consider some examples.

(O1) Let $\mathcal{V}$ be a normed space, and $J : \mathcal{V} \to \mathcal{B}(\mathcal{H})$ an isometry. Then let $\|[v_{ij}]\|_{J,n} = \|[Jv_{ij}]\|_{\mathcal{B}(\mathcal{H}^n)}$, i.e. we identify $\mathrm{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^n))$ in the usual way of multiplying columns of vectors by matrices.

(O2) Let $\mathcal{A}$ be a C*-algebra, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a faithful representation. Then the norms $(\|\cdot\|_{\pi,n})_{n=1}^{\infty}$ are the unique operator space norms for which each $*$-algebra $\mathrm{M}_n(\mathcal{A})$ is a C*-algebra. We shall deem this the "canonical" operator space structure on $\mathcal{A}$. In the context of the 2nd or 3rd definition, the uniqueness of C*-norms is exactly the fact that each M_n, or $\mathcal{K}$, is a nuclear C*-algebra.

(O3) Let $\mathcal{V}$ be a normed space and $\mathrm{M}_n \otimes^{\lambda} \mathcal{V}$ denote the injective tensor norm for Banach spaces. Then injectivity of this norm shows we get (D'), and (M') follows from the uniformity of this cross norm, i.e. the operator T on M_n, $T(\kappa) = \alpha\kappa\beta$, satisfies $\|T \otimes \mathrm{id}\|_{\mathcal{B}(\mathrm{M}_n)\otimes^{\lambda}\mathcal{V}} \leq \|T\|$.

(O3') Let $\mathcal{V}$ be a normed space. Consider the evaluation map isometry $E : \mathcal{V} \to \mathcal{C}(\mathrm{ball}(\mathcal{V}^*), \mathrm{weak}^*)$, $E(v)(f) = f(v)$. Letting $B = (\mathrm{ball}(\mathcal{V}^*), \mathrm{weak}^*)$, we observe that $\mathrm{M}_n(\mathcal{C}(B)) \cong \mathcal{C}(B, \mathrm{M}_n)$ as C*-algebras. It follows injectivity of injective tensor norm that $(\|\cdot\|_{E,n})_{n=1}^{\infty}$ gives the same operator space structure as (O3), above. We shall soon see that this should be called the "minimal" operator space structure on $\mathcal{V}$.

Definition of the morphisms. Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces, i.e. vector spaces equipped with fixed operator space structures. The amplification of a linear map $T : \mathcal{V} \to \mathcal{W}$ is given by $T^{(n)}[v_{ij}] = [Tv_{ij}]$. We shall say that T is

- *completely bounded (c.b.)* if $\|T\|_{cb} = \sup_n \left\|T^{(n)}\right\| < \infty$,
- *completely contractive (c.c.)* if $\|T\|_{cb} \leq 1$; and/or
- *completely isometric (c.i.)* if each $T^{(n)}$ is an isometry.

We shall have limited need for complete isomorphisms, so the short-form c.i. should cause no confusion ... to the author. The space $(\mathcal{CB}(\mathcal{V},\mathcal{W}), \|\cdot\|_{cb})$ of completely bounded maps is a normed space. It is complete if the normed space $(\mathcal{W} \cong \mathrm{M}_1(\mathcal{W}), \|\cdot\|_1)$ is complete.

The following shows that (O1), above, is actually the whole story about operator spaces, and is probably the reason we call them "operator spaces".

Ruan's Theorem. *If $(\mathcal{V}, \|\cdot\|_{n=1}^{\infty})_{n=1}^{\infty}$ satisfies (D) and (M), then there is a complete isometry $J : \mathcal{V} \to \mathcal{B}(\mathcal{H})$.*

(O4) Given a normed space $\mathcal{V}$ define for V in $\mathrm{M}_n(\mathcal{V})$

$$\|V\|_{\max,n} = \sup\left\{\|J^{(n)}V\|_{\mathcal{B}(\mathcal{H}^n)} \mid J : \mathcal{V} \to \mathcal{B}(\mathcal{H}) \text{ is an isometry, } \mathcal{H} \text{ a Hilbert space}\right\}.$$

Convince yourself that $\|[v_{ij}]\|_{\max,n} \leq \sum_{i,j=1}^{n} \|v_{ij}\|$, so that this supremum is finite. We call this the *maximal operator space* structure on $\mathcal{V}$, and write $\max \mathcal{V} = (\mathcal{V}, (\|\cdot\|_{\max,n})_{n=1}^{\infty})$. Based on Ruan's theorem we have $\|V\|_n \leq \|V\|_{\max,n}$ for any operator space structure $(\|\cdot\|_n)_{n=1}^{\infty}$ on $\mathcal{V}$ for which $\|\cdot\|_1$ is the norm on $\mathcal{V}$. It follows that $\mathcal{B}(\mathcal{V},\mathcal{W}) = \mathcal{CB}(\max \mathcal{V}, \mathcal{W})$, isometrically, for any other operator space $\mathcal{W}$.

Proposition. $\mathcal{V}^* = \mathcal{BC}(\mathcal{V}, \mathbb{C})$, *isometrically.*

Proof. $\mathbb{C}$ admits only one operator space structure. If $f \in \mathcal{V}^*$, $V \in \mathrm{M}_n(\mathcal{V})$, $\xi, \eta \in \ell^2_n$ (vectors are column matrices) we have

$$\langle f^{(n)}(V)\xi|\eta\rangle = f(\xi^* V \eta)$$

from which it follows that $\|f^{(n)}\| \leq \|f\|$. □

Exercise. Show that if $\mathcal{W} \subseteq \mathcal{C}(B)$ (B compact Hausdorff space) then for any operator space $\mathcal{V}$, $\mathcal{B}(\mathcal{V},\mathcal{W}) = \mathcal{CB}(\mathcal{V},\mathcal{W})$, isometrically.

We call $\mathcal{W}$, with the operator space structure of (O3) or (O3'), the *minimal operator space* structure; we write $\min \mathcal{W}$ for this operator space. If $(\|\cdot\|_n)_{n=1}^{\infty}$ is any operator space on a normed space $\mathcal{V}$ for which $\|\cdot\|_1$ is the norm on $\mathcal{V}$, we have for V in $\mathrm{M}_n(\mathcal{V})$ that

$$\|V\|_{\min,n} \leq \|V\|_n \leq \|V\|_{\max,n}.$$

The following two results are extremely important and intimately intertwined as is shown in the proof of [Pis-SiPr]. These results go back to [Witt, Haag80], and are related to [Stin55, Arv]. See [Paul, Paul-CBOA] as well as [EffRuan-OpSp].

Structure theorem. *If $\mathcal{A}$ is a C^*-algebra and $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely bounded map, then there exist a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}')$ and linear maps $v, w : \mathcal{H} \to \mathcal{H}'$ such that*

$$T = v^* \pi(\cdot) w \text{ and } \|T\|_{cb} = \|v\| \, \|w\| \, .$$

Extension Theorem. *Given operator spaces $\mathcal{V}_0 \subset \mathcal{V}$ and T in $\mathcal{CB}(\mathcal{V}_0, \mathcal{B}(\mathcal{H}))$, there is $\tilde{T}$ in $\mathcal{CB}(\mathcal{V}, \mathcal{B}(\mathcal{H}))$ such that*

$$\tilde{T}|_{\mathcal{V}_0} = T \text{ and } \|\tilde{T}\|_{cb} = \|T\|_{cb} \, .$$

Notice that if $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$, then $\mathcal{W}$ is injective amongst operator spaces with c.c've maps if and only if there is a completely contractive projection $E : \mathcal{B}(\mathcal{H}) \to \mathcal{W}$. If $\mathcal{W}$ is a unital C*-algebra, any contractive projection E is automatically c.c've [Tom57, NakTakUme] — see also [Li-OA] (in fact a unital completely positive map).

Corollary. *Any $f \in \mathcal{A}^*$ is of the form $f = \langle \pi(\cdot)\xi | \eta \rangle$ with $\|\xi\| \, \|\eta\| = \|f\|$.*

Mapping spaces, quotient spaces. We observe that for any operator space, the c.i'c identifications $\mathrm{M}_n(\mathrm{M}_m(\mathcal{V})) \cong \mathrm{M}_{nm}(\mathcal{V})$ allow us to identify each $\mathrm{M}_m(\mathcal{V})$ as an operator space. Given two operator spaces we let

$$\mathrm{M}_n(\mathcal{CB}(\mathcal{V}, \mathcal{W})) \cong \mathcal{CB}(\mathcal{V}, \mathrm{M}_n(\mathcal{W})) : [T_{ij}] \mapsto (v \mapsto [T_{ij} v])$$

isometrically. Check that (D) and (M) are satisfied. In particular we see that $\mathcal{V}^*$ is an operator space:

$$\mathrm{M}_n(\mathcal{V}^*) \cong \mathcal{CB}(\mathcal{V}, \mathrm{M}_n).$$

See also [Ble], where a direct embedding into $\mathcal{B}(\mathcal{H})$ is exhibited. We also consider isometric identifications for closed $\mathcal{V}_0 \subset \mathcal{V}$:

$$\mathrm{M}_n(\mathcal{V}/\mathcal{V}_0) \cong \mathrm{M}_n(\mathcal{V})/\mathrm{M}_n(\mathcal{V}_0).$$

A map $T : \mathcal{V} \to \mathcal{W}$ is a *complete quotient (c.q.)* map if each $T^{(n)}$ is a quotient map; i.e. each $T^{(n)}(\text{ball}(\text{M}_n(\mathcal{V})))$ is contained in and is a dense subset of $\text{ball}(\text{M}_n(\mathcal{W}))$. Notice that if both $\mathcal{V} \cong \text{M}_1(\mathcal{V})$ and $\mathcal{W} \cong \text{M}_1(\mathcal{W})$ are complete, hence each higher matrix space is complete, then the open mapping theorem shows that each $T^{(n)}$ takes the open ball of the domain space onto that of the codomain. We observe the following relationships, whose proofs vary greatly in depth.

Proposition. **(i)** *The evaluation mapping $\mathcal{V} \hookrightarrow \mathcal{V}^{**}$ is a complete isometry.*

Let $\mathcal{V}, \mathcal{W}$ be operator space and $T : \mathcal{V} \to \mathcal{W}$ be linear.
(ii) *T is c.b. $\Leftrightarrow$ T^* is c.b.*
(iii) *T is a c.q. $\Leftrightarrow$ T^* is c.i'c.*
(iv) *T is c.i'c $\Leftrightarrow$ T^* is a c.q.*

Proposition. *We have completely isometric identifications for any normed space*

$$(\max \mathcal{V})^* = \min(\mathcal{V}^*) \text{ and } (\min \mathcal{V})^* = \max(\mathcal{V}^*).$$

Preduals of von Neumann algebras. Thanks to [Sak-C*W*], any von Neumann algebra $\mathcal{M}$ has a unique isometric predual $\mathcal{M}_*$. We assign $\mathcal{M}_*$ the operator space structure it gains as a subspace of $\mathcal{M}^*$. We observe that $\mathcal{M}$, being a von Neumann algebra itself, admits within $\mathcal{M}^{**}$ a unique central projection z for which $\mathcal{M} \cong z\mathcal{M}^{**}$. The adjoint of the inclusion $\mathcal{M}_* \hookrightarrow \mathcal{M}^*$ is the c.q. map $X \mapsto zX : \mathcal{M}^{**} \twoheadrightarrow z\mathcal{M}^{**} \cong \mathcal{M}$. Hence it follows from one of the recent propositions that $\mathcal{M}$, qua dual space of $\mathcal{M}_*$, is c.i. to $\mathcal{M}$, qua von Neumann algebra.

Let (X, μ) be any decomposable measure space (or σ-finite, if you prefer). Then since $\text{L}^\infty(X, \mu)$ is a commutative von Neumann algebra, it is a minimal operator space. Thus $\text{L}^1(X, \mu) \cong \text{L}^\infty(X, \mu)_*$ is a maximal operator space. We shall deem this a certain "spatial" commutivity of the operator space structure.

Theorem. (Grothendieck) $\text{L}^1(X, \mu) \otimes^\gamma \text{L}^1(Y, \nu) \cong \text{L}^1(X \times Y, \mu \times \nu)$, *isometrically.*

Theorem. (folklore, see [Los]) *For von Neumann algebras $\mathcal{M}, \mathcal{N}$, we have $\mathcal{M}_* \otimes^\gamma \mathcal{N}_* \cong (\mathcal{M}\bar{\otimes}\mathcal{N})_*$ isomorphically [resp., isometrically] $\Leftrightarrow$ at least one of $\mathcal{M}, \mathcal{N}$ is subhomogeneous [commutative].*

The rather sorry state of affairs for non-subhomogeneous preduals begs for a resolution. Operator space theory provides this elegantly.

Given two complete operator spaces there is a sequence of norms $\|\cdot\|_{\wedge,n} : \mathrm{M}_n(\mathcal{V} \otimes \mathcal{W}) \to [0, \infty)$ which gives the largest operator space structure for which $\|[v_{ij} \otimes w_{kl}]\|_{\wedge,nm} = \|[v_{ij}] \otimes [w_{kl}]\|_{\wedge,nm} = \|[v_{ij}]\|_n \|[w_{kl}]\|_m$, i.e. is a "matricial-cross" structure. The completion of $\mathcal{V} \otimes \mathcal{W}$ with respect to $\|\cdot\|_{\wedge,1}$ is denoted $\mathcal{V}\hat{\otimes}\mathcal{W}$ and called the *operator projective tensor product.* Let us observe its naturally properties.

Proposition. (Duality) $(\mathcal{V}\hat{\otimes}\mathcal{W})^* \cong \mathcal{CB}(\mathcal{V}, \mathcal{W}^*)$, *c.i'lly.*
(Uniformity & projectivity) *If $T_j : \mathcal{V}_j \to \mathcal{W}_j$ $(j = 1, 2)$ are c.c's, then $T_1 \otimes T_2$ forms a complete contraction $\mathcal{V}_1\hat{\otimes}\mathcal{W}_1 \to \mathcal{V}_2\hat{\otimes}\mathcal{W}_2$. If each T_j is a complete quotient map, then so too is $T_1 \otimes T_2$.*

Finally we come to the elegant non-commutative Grothedieck formula of [EffRuan]. Though the commutative result admits an elementary proof, the present result actually requires the commutation theorem from Tomita-Takesaki theory. To my knowledge, no elementary proof is known.

Effros-Ruan tensor product formula. *There is a c.i'c identification*

$$\mathcal{M}_*\hat{\otimes}\mathcal{N}_* \cong (\mathcal{M}\bar{\otimes}\mathcal{N})_*.$$

From the perspective of non-commutative harmonic analysis, preduals of von Neumann algebras are the most basic spaces. Hence we consider the operator projective tensor product $\hat{\otimes}$ as more natural and assign it the "default" symbol – as opposed to $\otimes^\gamma$ for Banach space projective tensor product.

Notice that since $\mathcal{B}(\max \mathrm{L}^1(X, \mu), \mathcal{W}) = \mathcal{CB}(\max \mathrm{L}^1(X, \mu), \mathcal{W})$, isometrically it follows from duality that $\mathrm{L}^1(X, \mu) \otimes^\gamma \mathcal{W} = \max \mathrm{L}^1(X, \mu)\hat{\otimes}\mathcal{W}$, isometically.

Lecture #3: Amenability!

Our goal is to relate amenability of groups, to amenability of group algebras, to (operator) amenability of Fourier algebras. In particular, we wish to illustrate the necessity of the predual operator space structure to understanding the latter.

Amenable groups. A locally compact group G is *amenable* if it admits a *Følner net*: an increasing directed subset $\mathcal{F} \subset \mathrm{Borel}(G)$ for which

- $0 < m(F) < \infty$ for F in $\mathcal{F}$; and
- $\dfrac{m(sF \triangle F)}{m(F)} \xrightarrow{F\in\mathcal{F}} 0$ uniformly for s in compacta.

We prove the easy direction of the following; a general proof of the hard direction can be found in [Green-Mean, Pat-Amen], and an easier proof for discrete group can be found in [BroOza-C*App].

Theorem. *G is amenable $\Leftrightarrow$ there is a left-invariant mean M on* $\mathrm{L}^\infty(G)$.

Proof ($\Rightarrow$). M is any weak* cluster point of $\left(\frac{1}{m(F)}1_F\right)_{F\in\mathcal{F}}$ in $\mathrm{L}^\infty(G)^*$. □

Let us observe some examples and stability properties. Any unobvious fact should be checked in [Green-Mean, Pat-Amen], though they amount to exercises in the case of discrete groups.

(A1) Compact $\Rightarrow$ amenable: take $\mathcal{F} = \{G\}$.

(A2) $\mathbb{R}^d$ is amenable: $F_n = [-n, n]^d$; similar is true for $\mathbb{Z}^d$. We quickly obtain all compactly generated abelian groups.

(A3) $G = \bigcup_{i\in I} G_i$, increasing union, G_i open and amenable $\Rightarrow$ G amenable. From (A2) we obtain all l.c. abelian groups. [If you prefer means, apply the Markov-Kakutani fixed point theorem to translations by abelian G on the weak*-compact convex set of means on $\mathrm{L}^\infty(G)$.]

(A4) G amenable $\Rightarrow$ all closed $H \leq G$, all continuous quotient G/N amenable; $N \hookrightarrow G \twoheadrightarrow G/N$ with $N, G/N$ amenable $\Rightarrow$ G amenable.

Any group gained from (A1) and (A2), via operations of (A3) and (A4) is called *elementarily amenable.*

(N1) Any non-abelian free group F_n is not amenable. Draw the Cayley graph and convince yourself that for any finite $F \subset \mathrm{F}_n$ that there is $c > 0$ such that for $s \neq e$

$$c \leq \frac{|\partial F|}{2|F|} \leq \frac{|sF \triangle F|}{|F|}$$

where ∂F is the boundary of F.

(N2) Any non-compact semisimple connected Lie group is known to contain a discrete copy of F_2. Compare with (A4). Notice that semi-simple Lie groups contain dense copies of F_2 in abundance: the equation on the group in any finite 2-alphabet word generates a closed subset nowhere dense since we can find some free group; appeal to Baire category theorem. Why, for a compact such group, does this not contradict (A1) and (A4)?

Amenable Banach algebras. We observe that $\mathcal{A} \otimes^\gamma \mathcal{A}$ is a Banach algebra in the obvious manner. We let $\pi : \mathcal{A} \otimes^\gamma \mathcal{A} \to \mathcal{A}$ denote the product map. We interpret $a \otimes 1, 1 \otimes a$ as elements in the multiplier algebra, in the event that $\mathcal{A}$ is not unital.

We say that a Banach algebra $\mathcal{A}$ is *amenable* if there is a net $(m_\alpha)_\alpha \subset \mathcal{A} \otimes^\gamma \mathcal{A}$ such that

- $\sup_\alpha \|m_\alpha\|_\gamma < \infty$;
- $(\pi(m_\alpha))_\alpha$ is an approximate identity for $\mathcal{A}$; and
- $(a \otimes 1) \cdot m_\alpha - m_\alpha \cdot (1 \otimes a) \xrightarrow{\alpha} 0$ for a in $\mathcal{A}$.

The net $(m_\alpha)_\alpha$ is called a *bounded approximate diagonal (b.a.d.)* Standard references for what follows are [Dal-BanAlg, **?**]

Let us note some (non-)stability properties of amenability. Proofs of the first two are exercises; the latter follows from the cohomological characterization which follows shortly.

(B1) If $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range and $\mathcal{A}$ is amenable, then $\mathcal{B}$ is amenable.

(B2) If $\bigcup_{i \in I} \mathcal{A}_i$ is an increasing dense union of closed amenable subalgebras of $\mathcal{A}$, and each with a b.a.d. $(m_{i,\alpha(i)})_{\alpha(i)}$ for which $\sup_i \sup_{\alpha(i)} \left\|m_{i,\alpha(i)}\right\|_{\mathcal{A}_i \otimes^\gamma \mathcal{A}_i} < \infty$, then $\mathcal{A}$ is amenable.

(NB) It is <u>not</u> the case that a closed subalgebra of amenable $\mathcal{A}$ is amenable. Consider the disc algebra $\mathcal{A}(\mathbb{D})$ as a subalgebra of $\mathcal{C}(\mathbb{T})$. See the exercise below, to show that the latter is amenable. The derivation criterion we shall see shortly shows that $\mathcal{A}(\mathbb{D})$ is not amenable.

The following punctuates our Lecture #1 philosophy that G" = "$\mathrm{L}^1(G)$. We sketch the easy direction, only.

Theorem. [Joh-Mem, Run-Amen, Dal-BanAlg] *G is amenable $\Leftrightarrow$ $\mathrm{L}^1(G)$ is amenable.*

Sketch ($\Rightarrow$). If G is discrete, index over a Følner net

$$m_F = \frac{1}{|F|}\sum_{s\in F}\delta_x \otimes \delta_{x^{-1}}.$$

If G is not discrete, we work a bit harder. Thanks to [LosRin], by way of the fact that there is an inner-invariant mean of $\mathrm{L}^\infty(G)$, there exists a contactive *quasi-central approximate identity (q.c.a.i.)* for $\mathrm{L}^1(G)$: $(e_\alpha)_\alpha$, which means that

$$\|\delta_x * e_\alpha - e_\alpha * \delta_x\|_1 \xrightarrow{\alpha} 0 \text{ uniformly for } x \text{ in compacta.}$$

Now build a net with constituent elements

$$(e_\alpha \otimes e_\alpha) * \left[\frac{1}{m(F)}\int_F \delta_x \otimes \delta_{x^{-1}}\,dx\right]$$

where we build the index set in a manner such that we take the limit in F first, then in α; see [Kel-Top] and a use of this technique in [S]. $\square$

Exercise. Show that any commutative Banach algebra, generated by its invertible elements, is amenable.

A *(contractive) Banach $\mathcal{A}$-module* is a Banach space $\mathcal{X}$ which is a bimodule in the sense that there are a contractive homomorphism $\mathcal{A} \to \mathcal{B}(\mathcal{X}) : a \mapsto (x \mapsto a \cdot x)$ and a contractive anti-homomorphism $\mathcal{A} \to \mathcal{B}(\mathcal{X}) : a \mapsto (x \mapsto x \cdot a)$ with commuting ranges: $(a\cdot x)\cdot b = a\cdot(x\cdot b)$. The dual space $\mathcal{X}^*$ with adjoint actions $a\cdot f\cdot b(x) = f(b\cdot x\cdot a)$ is called a *(contractive) dual Banach $\mathcal{A}$-module.*

Within the next theorem lies Johnson's original formulation of the definition of amenability. The "averaging" nature of the b.a.d. $(m_\alpha)_\alpha$ is slightly apparent in the proof.

Theorem. [Joh72, Run-Amen, Dal-BanAlg] *Let $\mathcal{BD}(\mathcal{A},\mathcal{X}^*)$ be the space of bounded derivations: $D(ab) = a\cdot D(b) + D(a)\cdot b$. Then $\mathcal{A}$ is amenable $\Leftrightarrow$ $\mathcal{BD}(\mathcal{A},\mathcal{X}^*) = \{a \mapsto a\cdot f = f\cdot a : f \in \mathcal{X}^*\}$.*

Sketch (⇒). We may suppose by density that $m_\alpha = \sum_{i=1}^{n_\alpha} a_{i,\alpha} \otimes b_{i,\alpha}$. Then $\left(\sum_{i=1}^{n_\alpha} a_{i,\alpha} \cdot D(b_{i,\alpha})\right)_\alpha$ can be shown to admit a weak* limit point f in $\mathcal{X}^*$. □

The condition that $\mathcal{BD}(\mathcal{A}, \mathcal{A}^*) = \{a \mapsto a \cdot f = f \cdot a : f \in \mathcal{A}^*\}$ is called *weak amenability*. For commutative $\mathcal{A}$ this notion was introduced in [BadCurDal] and means that $\mathcal{BD}(\mathcal{A}, \mathcal{A}^*) = \{0\}$; in fact $\mathcal{BD}(\mathcal{A}, \mathcal{S}) = \{0\}$ for any *symmetric* Banach $\mathcal{A}$-bimodule: $a \cdot s = s \cdot a$. [This is not to be confused with the notion of *weak amenability* for groups [deCaHaa]. To the authors knowledge there is no relationship between these notions.]

Theorem. [Joh91, DesGha] $\mathrm{L}^1(G)$ *is always weakly amenable.*

(Non-)Amenability of Fourier algebras. With the definition I have used, it is a tautology that amenability of an algebra entails that the algebra admits a bounded approximate identity (b.a.i.)

Leptin's theorem. [Lep] G *is amenable* ⇔ $\mathrm{B}(G)$ *admits a b.a.i.*

Sketch (⇒). Check that a Følner net satisfies

$$u_F(s) = \left\langle \lambda(s)\frac{1}{m(F)^{1/2}}1_F \middle| \frac{1}{m(F)^{1/2}}1_F \right\rangle = \frac{m(sF \cap F)}{m(F)} \xrightarrow{F \in \mathcal{F}} 1.$$

If $v \in \mathrm{P} \cap \mathrm{A}(G)$, $u_F v \xrightarrow{F\in\mathcal{F}} v$; see [GranLei]. Notice that the b.a.i. is really a contractive a.i., and, in fact, comprised of positve-type elements. □

The following result was considered surprising at the time of its publication.

Theorem. [Joh94] $\mathrm{A}(\mathrm{SO}(3))$ *is not weakly amenable.*

See [Ply, ForSamS] for extensions built on structure theory of compact groups. A different perspective is taken in [ChoGha], where the result is extended to such groups as the $ax+b$-group, $\mathrm{SL}_2(\mathbb{R})$ and its universal cover. It is widely suspected that $\mathrm{A}(G)$ is weakly amenable if and only the connected component of the identity G_e is abelian. See [ForRun], and other references in this paragraph.

Since amenability is a desirable property, it is regrettable that $\mathrm{A}(G)$ is infrequently amenable.

Theorem. [LauLoyWil, ForRun] $\mathrm{A}(G)$ *is amenable* ⇔ G *admits an abelian subgroup of finite index*$^{(a)}$.

(a) It seems best to refer to these as *virtually abelian*, which is consistent with group theory literature. In such literature *almost abelian* often refers to groups which are extensions of abelian groups by finite groups, though a Google search revealed as many uses as a synonym for virtually abelian. Interestingly "“almost abelian”" (scare-quotes within scare-quotes) is used casually by some authors to motivate intuition on finite nilpotent groups. I have never seen $\mathbb{R}^2 \rtimes \mathrm{S0}(2)$ referred to as "almost abelian", so this phrase seems almost never to admit the connotation of a compact extension of an abelian group.

Operator spaces to the rescue. Let us recall the happy fact that $\mathrm{A}(G)^* \cong \mathrm{VN}_\lambda =: \mathrm{VN}(G)$. Now we have a unitary equivalence

$$\mathrm{L}^2(G) \otimes^2 \mathrm{L}^2(H) \cong \mathrm{L}^2(G \times H) \text{ (Hilbertian tensor product)}$$

which intertwines $\lambda_G \times \lambda_H \cong \lambda_{G\times H}$, and hence gives us a spatial equivalence

$$\mathrm{VN}(G)\bar{\otimes}\mathrm{VN}(H) \cong \mathrm{VN}(G \times H).$$

Hence the Effros-Ruan tensor product formula gives us

$$\boxed{\mathrm{A}(G)\hat{\otimes}\mathrm{A}(H) \cong \mathrm{A}(G \times H), \text{ c.i'lly.}}$$

We note that [Los] has shown that $\mathrm{A}(G)\otimes^\gamma\mathrm{A}(H) \cong \mathrm{A}(G\times H)$, isomorphically, only when one of G or H admits an abelian subgroup of finite index. In fact, unless we know that one of $\mathrm{A}(G), \mathrm{A}(H)$ has the metric approximation property, it is not even clear to the author that $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(H)$ is semisimple [Tom60], hence injects into $\mathrm{A}(G \times H)$.

C.c've Banach algebras. A *completely contractive Banach algebra (c.c.B.a)* is an associative algebra equipped with an operator space structure with regards to which the product map extends to a complete contraction $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$.

By comments at the end of the last lecture for any Banach algebra $\mathcal{A}$, $\max \mathcal{A}$ is trivially a c.c.B.a. Of course $\mathrm{B}(G)$ and $\mathrm{A}(G)$ should be considered with their predual operator space structures.

Proposition. $\mathrm{B}(G)$*, hence* $\mathrm{A}(G)$*, is a c.c.B.a.*

Proof. Consider the chain of normal $*$-homomorphisms

$$\begin{gathered}\mathrm{W}^*(G) \to \mathrm{W}^*(G \times G) \to \mathrm{W}^*(G)\bar{\otimes}\mathrm{W}^*(G) \to \mathrm{W}^*(G)\bar{\otimes}\mathrm{VN}(G)\\ \varpi_G(s) \mapsto \varpi_{G\times G}(s,s) \mapsto \varpi_G(s) \otimes \varpi_G(s) \mapsto \lambda_G(s) \otimes \lambda_G(s)\end{gathered}$$

which admits c.c've preadjoint

$$\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) \hookrightarrow \mathrm{B}(G)\hat{\otimes}\mathrm{B}(G) \hookrightarrow \mathrm{B}(G \times G) \xrightarrow{R_{\mathrm{diag}}} \mathrm{B}(G).$$

The composition of these maps gives pointwise product. □

Operator amenability. A c.c.B.a. is *operator amenable* if it admits a b.a.d. $(m_\alpha)_\alpha$ which is bounded in $\mathcal{A}\hat{\otimes}\mathcal{A}$.

The contractive map $\mathcal{A} \otimes^\gamma \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ shows that amenability implies operator amenability.

For a C*-algebra $\mathcal{A}$ we have a facorization $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A} \otimes^h \mathcal{A} \to \mathcal{A}$, where $\otimes^h$ denotes the *Haagerup tensor product.* For lack of time and space I will not discuss this remarkable construction beyond this point.

The next result is famous and deep.

Theorem. [Con, Haag83, Ruan, EffKis] *Let $\mathcal{A}$ be a C*-algebra. Then the following are equivalent:*
(i) *$\mathcal{A}$ is nuclear;* **(ii)** *$\mathcal{A}$ is amenable;* *and* **(iii)** *$\mathcal{A}$ is operator amenable.*

However, for my present purposes, the following result, which motivates operator amenability, will be more useful.

Theorem. [Ruan] $\mathrm{A}(G)$ *is operator amenable* $\Leftrightarrow$ G *is amenable.*

Sketch. ($\Rightarrow$) See Leptin's theorem, above.

($\Leftarrow$) We follow [AriRunS, IlieS]. In [Stok04] a bounded q.c.a.i. $(e_\alpha)_\alpha$ for $\mathrm{L}^1(G)$ which is composed of compactly supported probabilities whose supports tend down to $\{e_G\}$. We set $\xi_\alpha = e_\alpha^{1/2}$ and then let

$$w_\alpha(s,t) = \langle\lambda(s)\rho(t)\xi_\alpha|\xi_\alpha\rangle \in \mathrm{P}(G \times G).$$

We have

$$w_\alpha(s,s) \xrightarrow{\alpha} 1 \text{ uniformly for } s \text{ in compacta.} \qquad (\heartsuit)$$

We also have that $(w_\alpha)_\alpha$ is eventually 0 on any compact set K such that $K \cap G_{\mathrm{diag}} = \varnothing$, where G_{diag} is the diagonal subgroup. The latter is a spectral subset for $\mathrm{A}(G \times G)$; see [Herz, TakTat]. In particular, approximating by compactly supported elements of $\mathrm{A}(G \times G)$ which are supported away from G_{diag}, we establish that

$$w_\alpha v \xrightarrow{\alpha} 0 \text{ for } v|_{G_{\mathrm{diag}}} \text{ in } \mathrm{A}(G \times G).$$

Hence it follows that for u in $\mathrm{A}(G)$ and any c.a.i. $(v_\beta)_\beta \subset \mathrm{P} \cap \mathrm{A}(G \times G)$, we have

$$\lim_\beta \lim_\alpha (u \otimes 1 - 1 \otimes u) w_\alpha v_\beta = 0. \qquad (\diamondsuit)$$

Combining ($\heartsuit$) and ($\diamondsuit$) in light of [GranLei], we can form the elements $w_\alpha v_\beta$ into a b.a.d. for $\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G)$; see [S]. $\square$

In pleasing analogy with [Joh91], the property of *operator weak amenability*, i.e. that completely bounded derivations to the dual space satisfy $\mathcal{CBD}(\mathcal{A}, \mathcal{A}^*)$ $= \{a \mapsto a \cdot f - f \cdot a : f \in \mathcal{A}^*\}$, is known for $\mathrm{A}(G)$.

Theorem. [S, Sam] $\mathrm{A}(G)$ *is always operator weakly amenable.*

Let us re-emphasize the fact that $\mathrm{L}^1(G)$, while often non-commutative qua Banach algebra, is "spatially" commutative: $\mathrm{L}^1(G)^* \cong \mathrm{L}^\infty(G)$. Meanwhile $\mathrm{A}(G)$ while being commutative qua Banach algebra is often "spatially" non-commutative: $\mathrm{A}(G)^* \cong \mathrm{VN}(G)$, the latter admits arbitrarily large matrix units if G is not virtually abelian.

Summary Theorem. *The following are equivalent:*

(i) G *is amenable;*

(ii) $\mathrm{L}^1(G)$ *is amenable;*

(ii') $\mathrm{L}^1(G)$ *is operator amenable;*

(iii) $\mathrm{A}(G)$ *is operater amenable.*

Furthermore $\mathrm{L}^1(G)$ *is always (operator) weakly amenable and* $\mathrm{A}(G)$ *is always operator weakly amenable.*

The author self-indulgently recommends the survey article [S-Surv] for more on this theme.

Lecture #4: Applications of operator amenability to two problems in harmonic analysis.

Application #1: Homomorphisms of Fourier algebras. In a delightfully whimsical bout of harmonic analysis chauvinism, a colleague of mine characterised the main result of [Coh] as "Cohen's last great theorem". Therein, for abelian groups G and H, the structure of all bounded homomorphisms from $\mathrm{A}(G) \cong \mathrm{L}^1(\widehat{G})$ to $\mathrm{B}(H) \cong \mathrm{M}(\widehat{H})$ were characterized. This result was extended to the case that G is virtually abelian, in [Host], and to the case that G is discrete in [Ilie]. The latter, implicitly rediscovered (and used) the fact that $\mathrm{A}(G)$ is operator biprojective in this case; see [Wood, Ari].

A complete characterisation for the analogous problem on $\mathrm{L}^1(G)$ for non-commutative G remains unknown. See [Gre, Stok12] for more on this. Let me hint that this problem will be completely understood if we can understand both the structure of the idempotents and the structure of the groups supported by the said idempotents. For discrete groups this entails no less than a resolution of Kaplansky's idempotent conjecture in $\mathbb{C}[G]$.

My goal now is to look at a procedure from [IlieS] to indicate how operator amenability gives an appealing partial resolution of the problem of understanding bounded homomorphisms between $\mathrm{A}(G)$ and $\mathrm{B}(H)$. Notice that we recover, and hence generalize, the results of [Coh, Host], mentioned above.

Piecewise affine maps. A *coset* of H is any $C \subseteq H$ for which

$$r, s, t \in C \quad \Rightarrow \quad rs^{-1}t \in C.$$

Notice that $C^{-1}C, CC^{-1}$ are groups and $C = sC^{-1}C = CC^{-1}s$ for any s in C. We let $\Omega(H)$ denote the *coset ring*, the Boolean algebra generated by cosets G. A map $\alpha : C \subset H \to G$ is called *affine* provided that C is a coset and $\alpha(rs^{-1}t) = \alpha(r)\alpha(s)^{-1}\alpha(t)$ for r, s, t in C. A map $\alpha : Y \subseteq H \to G$ is called *piecewise affine (p.a.)* if

- Y admits a partition $Y_1, \ldots, Y_n \subset \Omega(H)$,
- $\exists$ cosets $C_j \supseteq Y_j$ and affine maps $\alpha_j : C_j \to G$ such that

$$\alpha|_{Y_j} = \alpha_j|_{Y_j},\ j = 1, \ldots, n.$$

This family of maps contains homomorphisms from subgroups and well as translations. Verify (i), below, as an exercise.

Proposition. *Let* $\alpha : Y \subseteq H \to G$. *Then:*

(i) α *is affine* $\Leftrightarrow$ $\mathrm{Graph}(\alpha) \subset H \times G$ *is a coset; and*

(ii) α *is p.a.* $\Leftrightarrow$ $\mathrm{Graph}(\alpha) \in \Omega(H \times G)$.

The main result of [Host] is not a homomorphism theorem but an idempotent theorem. Let $\Omega_o(H)$ be the Boolean algebra generated by the open cosets.

Host's Theorem. $\{u \in \mathrm{B}(H) : u = u^2\} = \{1_Y : Y \in \Omega_o(H)\}$.

Now suppose that we have a bounded homomorphism $\Phi : \mathrm{A}(G) \to \mathrm{B}(H)$. Note that Gelfand spectrum $H \cong \varpi_H(H) \subseteq \Gamma_{\mathrm{B}(H)}$ (the so-called Wiener-Pitt phenomenon, see [Kat-HarAn], for example, tells us that $\varpi_{\mathbb{Z}}(\mathbb{Z})$ is not even dense in $\Gamma_{\mathrm{B}(\mathbb{Z})} \cong \Gamma_{\mathrm{M}(\mathbb{T})}$), while $\Gamma_{\mathrm{A}(G)} = \lambda_G(G) \cong G$. Hence $\Phi^*(\varpi_H(H)) \subseteq \lambda_G(G) \cup \{0\}$. Let $Y = \{s \in H : \Phi(\varpi_H(s)) \neq 0\}$. We thus induce a map $\alpha : Y \subset H \to G$. We have for u in $\mathrm{A}(G)$ that

$$\Phi u(s) = \begin{cases} u(\alpha(s)) & \text{if } s \in Y \\ 0 & \text{if } s \notin Y \end{cases}, \quad \text{i.e. } \Phi u = 1_Y u \circ \alpha. \tag{$\spadesuit$}$$

We wish to understand α, and it is here that we shall

assume that Φ *is completely bounded and that* G *is amenable.*

The value of c.b'ness of Φ can be observed in the following analogue of a proposition in [deCaHaa], with practically the same proof.

Proposition. *The following are equivalent:*

(i) Φ *is c.b.*

(ii) $\Phi \otimes \mathrm{id} : \mathrm{A}(G)\hat{\otimes}\mathrm{A}(S) \to \mathrm{B}(H)\hat{\otimes}\mathrm{A}(S)$ *is bounded for any l.c. group* S

(iii) $\Phi \otimes \mathrm{id} : \mathrm{A}(G)\hat{\otimes}\mathrm{A}(\mathrm{SU}(2)) \to \mathrm{B}(H)\hat{\otimes}\mathrm{A}(\mathrm{SU}(2))$ *is bounded.*

We now consider the following sequence of c.q. $*$-homomorphisms

$$\mathrm{W}^*(H_d \times G_d) \twoheadrightarrow \mathrm{W}^*(H \times G) \twoheadrightarrow \mathrm{W}^*(H)\bar{\otimes}\mathrm{VN}(G)$$

where F_d is the discretised version of F, $F = G, H$. We thus consider the composition of maps

$$\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) \xrightarrow{\Phi\otimes\mathrm{id}} \mathrm{B}(H)\hat{\otimes}\mathrm{A}(G) \hookrightarrow \mathrm{B}(H_d \times G_d)$$

which is bounded. Notice that $(\Phi\otimes \mathrm{id})w(s,t)=1_Y(s)w(\alpha(s),t)$. Now we invoke the assumption that G is amenable, hence $\mathrm{A}(G)$ is operator amenable and has a b.a.d. $(m_\iota)_\iota$. We observe that in the pointwise topology in $\mathrm{B}(H_d\times G_d)$ — which equals the weak* topology on bounded sets — that

$$(\Phi\otimes \mathrm{id})m_\iota(s,t)=1_Y(s)m_\iota(\alpha(s),t)\xrightarrow{\iota}\left.\begin{cases}1 & \text{if } s\in Y, t=\alpha(s)\\ 0 & \text{otherwise}\end{cases}\right\}=1_{\mathrm{Graph}(\alpha)}.$$

Hence $1_{\mathrm{Graph}(\alpha)}\in \mathrm{B}(H_d\times G_d)$. Then Host's theorem tells us that $\mathrm{Graph}(\alpha)\in\Omega(H\times G)$ (all sets are open in discrete topology) and hence α is p.a. Using regularity of $\mathrm{A}(G)$ we may determine that α is continuous, and some more effort then reveals that Y partitions with components in $\Omega_o(H)$.

Establishing that a continuous p.a. map on such open Y gives rise to a c.b. homomorphism form $\mathrm{A}(G)$ to $\mathrm{B}(H)$ is tedious, but easier. We summarise.

Theorem. [IlieS] *If G is amenable, then there is a bijection between c.b. homomorphisms $\Phi:\mathrm{A}(G)\to\mathrm{B}(H)$ and continuous p.a. maps $\alpha:Y\subset H\to G$, as given by (♠). Furthermore, $\Phi(\mathrm{A}(G))\subseteq\mathrm{A}(H)\Leftrightarrow\alpha$ is* proper, *i.e. $\alpha^{-1}(K)$ is compact whenever K is compact.*

Let us consider the necessity of our assumptions.

(i) Let F_n be a non-commutative free group and E an infinite free set in F_n. Then $u\mapsto 1_Eu:\mathrm{A}(\mathrm{F}_n)\to\mathrm{A}(\mathrm{F}_n)$ is a c.b. homomorphism but $1_E\notin\mathrm{B}(\mathrm{A}(\mathrm{F}_n))$, [Lei, BozFen]. The associated $\alpha:E\to\mathrm{F}_n$ is not p.a. as $E\notin\Omega(\mathrm{F}_n)$.

(ii) The map $u\mapsto\check{u}:\mathrm{A}(G)\to\mathrm{A}(G)$ ($\check{u}(s)=u(s^{-1})$) is contractive, but c.b. $\Leftrightarrow G$ is virtually abelian; [ForRun]. It follows that the anti-diagonal $\{(s^{-1},s):s\in G\}\in\Omega(G\times G)\Leftrightarrow G$ is virtually abelian. It seems that it should be possible to verify this by group theory techniques alone, but no such proof is known to the author.

An *anti-affine* map is a map $\alpha:C\subset H\to G$ from a coset which satisfies $\alpha(rs^{-1}t)=\alpha(t)\alpha(s)^{-1}\alpha(r)$, i.e. and affine map composed with inversion. Techniques, much different form ours, were used to show the following.

Theorem. [Pham] *There is a bijection between contractive homomorphisms $\Phi:\mathrm{A}(G)\to\mathrm{B}(H)$ and continuous affine or anti-affine maps $\alpha:Y\subset H\to G$, as given by (♠).*

I leave it to the reader to formulate a reasonable conjecture about the structure of bounded homomorphisms $\Phi : \mathrm{A}(G) \to \mathrm{B}(H)$, at least when G is amenable. I would be delighted to learn, and learn of, any new results in the direction.

Application #2: Ideals with bounded approximate identities in Fourier algebras. The problem of considering, for an abelian G, those ideals in $\mathrm{A}(G) \cong \mathrm{L}^1(\widehat{G})$ admitting a b.a.i. goes back over 40 years. See [Ros, Gil, Sch, LiuRooWan]. Advances were made for small invariant neighbourhood groups in [For].

Let us first consider where to look. If $\mathcal{J} \lhd \mathrm{A}(G)$ is such an ideal with b.a.i. $(u_\alpha)_\alpha$, then in $\mathrm{B}(G_d) \supseteq \mathrm{A}(G)$, the regularity of G can be used to show that weak*-$\lim_\alpha u_\alpha = 1_E$ where $E = \mathrm{hull}(\mathcal{J}) = \{s \in G : u(s) = 0\ \forall u \in \mathcal{J}\}$. But Host's theorem tells us $E \in \Omega(G)$ and further that E is closed. Based on [Gil, Sch], it was shown in [For] that

$$E = \bigcup_{i=1}^{n} \left(C_i \setminus \bigcup_{j=1}^{n_i} K_{ij} \right)$$

where each C_i is a closed coset and each K_{ij} is a relatively open coset in C_i, hence closed in C_i, whence closed in G. We may allow $n_i = 0$ and understand that empty unions represent the empty set. At this point we may conclude that $\mathcal{J} \subseteq \mathrm{k}(E)$; if we knew that E were spectral we would be able to conclude that $\mathcal{J} = \mathrm{k}(E) := \{u \in \mathrm{A}(G) : u|_E = 0\}$.

Closed subgroups, hence closed cosets, are spectral [Herz, TakTat]; note that translations are isometric automorphisms of $\mathrm{A}(G)$. However, the union problem for spectral sets is unsolved, even for abelian G. But, if each closed coset C allowed that $\mathrm{k}(C)$ admitted a bounded approximate identity, then each such C would be a Ditkin set (i.e. we approximate each element of the b.a.i. by elements from $\mathrm{j}(C)$). Since Ditkin sets satisfy that unions of such are Ditkin, it seems as if it is possible to conclude that such E is spectral. Indeed it is possible, but takes some effort: see [KanForLauS]. A good introduction to sets of spectral synthesis is in [Kan-CBA].

My main goal is to summarise how we show that $\mathrm{k}(H)$ admits a b.a.i. for every closed subgroup of an amenable group G. The role of a b.a.i. for $\mathrm{A}(G)$ will be indispensable beyond a certain point, so Leptin's theorem tells us that we might as well assume from the beginning that G is amenable.

If, for example, G is discrete and non-amenable and $F \subset G$ is finite, then $\mathrm{k}(F)$ will not admit a b.a.d.

Observe that the statement of the result below has no reliance on the operator space structure of $\mathrm{A}(G)$. The fact that they are so useful in the proof speaks to the naturality of using this extra structure.

Theorem. [KanForLauS] *If G is amenable and $H \leq G$ is closed, then* $\mathrm{k}(H)$ *admits a b.a.i.*

Outline of proof. **(1)** We will first see that the annihilator $\mathrm{k}(H)^\perp$ is completely complemented in $\mathrm{VN}(G)$.

We first observe the restriction theorem [Herz, McM, Ars]: $R_H : \mathrm{A}(G) \to \mathrm{A}(H)$ is surjective. Its adjoint $R_H^* : \mathrm{VN}(H) \to \mathrm{VN}(G)$ satisfies $R_H^*\lambda_H(s) = \lambda_G(s)$ which can be seen to extend to an injective $*$-homomorphims $\mathrm{VN}(H) \to \mathrm{VN}_G(H) := \overline{\mathrm{span}}^{w*}\lambda_G(H) \subset \mathrm{VN}(G)$, hence a c.i. Furthermore, it is then basic to observe that $\mathrm{k}(H) = \ker R_H$ thus satisfies that $\mathrm{k}(H)^\perp = \mathrm{VN}_G(H)$.

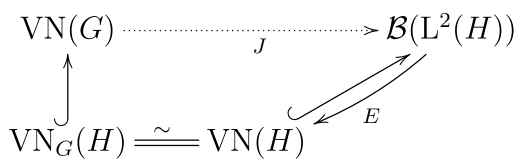
Since H is amenable (closed subgroup of G), there is a left-invariant mean M on $\mathrm{L}^\infty(H)$. We use this to see that $\mathrm{VN}(H)$ is injective as an operator space:

$$E : \mathcal{B}(\mathrm{L}^2(H)) \to \mathrm{VN}(H), \quad \langle E(a)\xi|\eta\rangle = M(s \mapsto \langle \rho(s)a\rho(s^{-1})\xi|\eta\rangle)$$

where $\rho : H \to \mathcal{U}(\mathrm{L}^2(H))$ is the right regular representation. It is well-known, but non-trivial, that $\mathrm{VN}(H) = \mathrm{VN}_\rho''$; see [Dix-vN]. It follows that $\mathrm{ran}E = \mathrm{VN}(H)$ and $E^2 = E$. The complete boundedness is automatic, but is easily checked manually:

$$\left\langle E^{(n)}[a_{ij}] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \middle| \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \right\rangle = M\left(s \mapsto \left\langle [a_{ij}] \begin{bmatrix} \rho(s^{-1})\xi_1 \\ \vdots \\ \rho(s^{-1})\xi_n \end{bmatrix} \middle| \begin{bmatrix} \rho(s^{-1})\eta_1 \\ \vdots \\ \rho(s^{-1})\eta_n \end{bmatrix} \right\rangle \right).$$

The extension theorem provides us a c.i. J making the diagram, below, commute.



Let $P = E \circ J : \mathrm{VN}(G) \to \mathrm{VN}_G(H) = \mathrm{k}(H)^\perp$.

(2) We now use a b.a.d. $(m_\alpha)_\alpha$, provided by the operator amenability of $\mathrm{A}(G)$, to "average P to invariance". This device is from [CurLoy] but was first discovered by Helemskii. See [Hel-Hom, Run-Amen].

By density we may suppose each $m_\alpha = \sum_{i=1}^{n_\alpha} u_i^\alpha \otimes v_i^\alpha$. Let $P_\alpha(x) = \sum_{i=1}^{n_\alpha} u_i^\alpha \cdot P(v_i^\alpha \cdot x)$. The fact that P is c.b. allows $(P_\alpha)_\alpha$ to be bounded in $\mathcal{CB}(\mathrm{VN}(G), \mathrm{k}(H)^\perp) \cong (\mathrm{VN}(G)\hat{\otimes}(\mathrm{A}(G)/\mathrm{k}(H))^*$, hence bounded in $\mathcal{B}(\mathrm{VN}(G), \mathrm{k}(H)^\perp) \cong (\mathrm{VN}(G) \otimes^\gamma (\mathrm{A}(G)/\mathrm{k}(H))^*$. [If we only knew that P were bounded, we would have no means to arrive at this conclusion, since $(m_\alpha)_\alpha$ is generally <u>not</u> bounded in $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$.] Hence we may obtain a weak* cluster point $\tilde{P}$ and we have

$$\tilde{P}(u \cdot x) = u \cdot \tilde{P}(x).$$

[If we could construct a net $(u_\alpha)_\alpha \subset \mathrm{P}(G)$ for which $u_\alpha \xrightarrow{\alpha} 1_H$ pointwise, this would allow a different construction to obtain $\tilde{P}$. This is trivially available for H open, and easy to find if $H \lhd G$. Indeed, we would take $P_\alpha = S_{u_\alpha}$, $S_u x = u \cdot x$, and extract a cluster point.

J. Crann kindly informs me that a method in [CraNeu] allows one to generally and directly to obtain $\tilde{P}$ without taking a limit, assuming that H is amenable but without assuming G is amenable! The proof is very nice, but requires introduction of some fundamental unitaries, and circumvents the use of operator space techniques. This is adding to evidence that these techniques are valuable, even in the setting of cocommutative quantum groups, of the type the author adores.]

(3) We continue with devices from [CurLoy, Hel-Hom] to use the invariant projection $\tilde{P}$ to obtain a b.a.i. for $\mathrm{k}(H)$. From here on we need to know only that $\mathrm{A}(G)$ has a b.a.i. and that our invariant projection is bounded.

We have a split short exact sequence of $\mathrm{A}(G)$-module maps:

$$\mathrm{k}(H)^\perp \underset{\tilde{P}}{\overset{\hookrightarrow}{\leftarrow}} \mathrm{VN}(G) \underset{Q}{\twoheadrightarrow} \mathrm{VN}(G)/\mathrm{k}(H)^\perp \cong \mathrm{k}(H)^*.$$

Hence we may obtain an $\mathrm{A}(G)$-linear embedding $i : \mathrm{k}(H)^* \cong \ker \tilde{P} \subset \mathrm{VN}(G)$ for which $Q \circ i = \mathrm{id}$, and hence $i^* : \mathrm{A}(G)^{**} \cong \mathrm{VN}(G)^* \to \mathrm{k}(H)^{**}$ is a surjective $\mathrm{A}(G)$-linear map. For any b.a.i. $(u_\beta)_\beta$ for $\mathrm{A}(G)$, any cluster point U of $(i^*(u_\beta))_\beta$ can be shown to satisfy $u \cdot U = U$ (second adjoint action of u in $\mathrm{k}(H)$ on $\mathrm{k}(H)^{**}$) so U is called a mixed identity. It is shown in [BonDun-CNA] that this implies that $\mathrm{k}(H)$ itself admits a b.a.d. □

We note in passing that the property that $\mathcal{J} \lhd \mathcal{A}$ ($\mathcal{A}$ [c.c've] Banach algebra) admitting a b.a.i. $(u_\beta)_\beta$ implies that $\mathcal{J}^\perp$ is complemented in $\mathcal{A}$; we call this *weakly complemented*. Indeed let P be any weak*-cluster point of of S_{u_β} where $S_u f = u \cdot f$. Hence we have for [op.] amenable $\mathcal{A}$ that $\mathcal{J}$ admits a b.a.i. $\Leftrightarrow$ $\mathcal{J}$ is weakly [c.ly] complemented.

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