



The Category Cu .
Which maps are the
correct ones?

*-homomorphisms or
cpc order zero?

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- 2 The Bivariant Cuntz Semigroup

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The order in $W(A)$ is usually **not the algebraic** order.

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└ The Cuntz Semigroup

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Coward-Elliott-Ivanescu in 2008 defined $\mathcal{Cu}(A)$ for any C^* -algebra, which is a modified version of the Cuntz semigroup.
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Properties

- $\mathcal{Cu}(A)$ belongs to a category of semigroups called \mathcal{Cu} that admits inductive limits that are not algebraic.
- The assignment $A \mapsto \mathcal{Cu}(A)$ is sequentially continuous.

└ Motivation

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The category Cu

The category \mathcal{Cu}

Definition

Let a, b be elements in a partially ordered set S . Then, we will say that $a \ll b$ (**way-below**) if for any increasing sequence $\{y_n\}$ with supremum in S such that $b \leq \sup(y_n)$, there exists m such that $a \leq y_m$.

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Definition (\mathbf{Cu})

An object of \mathbf{Cu} is a partially ordered semigroup with zero element S such that:

- The order, in S , is compatible with the addition, i.e., if $x_i \leq y_i$, $i \in \{1, 2\}$ then $x_1 + x_2 \leq y_1 + y_2$,
- every increasing sequence in S has a supremum,
- for all $x \in S$ there exists a sequence $\{x_n\}$ such that $x = \sup(x_n)$ where $x_n \ll x_{n+1}$,
- the relation \ll and suprema are compatible with addition.

The maps of \mathbf{Cu} are those morphisms which preserve the order, the zero, the suprema of increasing sequences and the relation \ll .

Remark

In fact, $\langle (a - \varepsilon)_+ \rangle \ll \langle a \rangle$ in $\text{Cu}(A)$ for all $\varepsilon > 0$ and for all $a \in A_+$.

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Example

- Let X be a compact metric space. Then, if $\mathcal{O}(X)$ is the set of open sets in X ordered by inclusion, it follows that $\mathcal{O}(X) \in \mathbf{Cu}$. In this, we have that $U \ll V$ for $U, V \in \mathcal{O}(X)$, if there exists a compact subset $K \subseteq X$ such that $U \subseteq K \subseteq V$.

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- Let X be a finite-dimensional compact metric space, then $\text{Lsc}(X, \overline{\mathbb{N}}) \in \mathbf{Cu}$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

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Remark

Not all the maps between semigroups preserve \ll , usually maps between two semigroups just preserve $(+, \leq, \text{sup})$.

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Linear Maps

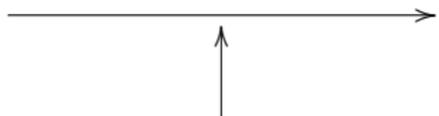
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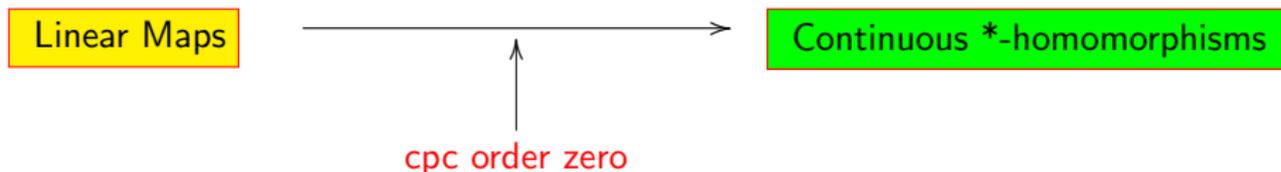
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cpc order zero

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Definition

- A map $\varphi : A \rightarrow B$ is positive if $\forall a \geq 0 \implies \varphi(a) \geq 0$, and it is completely positive (c.p.) if $\varphi^n : M_n(A) \rightarrow M_n(B)$ is positive.
- A c.p. map $\varphi : A \rightarrow B$ is order zero if for $a, b \in A^+$ such that $ab = 0 \implies \varphi(a)\varphi(b) = 0$.

Theorem (Winter-Zacharias '09)

Let A, B be C^* -algebras and $\varphi : A \rightarrow B$ a cpc order zero map and set $C = C^*(\varphi(A)) \subseteq B$. Then, there exists

- $h_\varphi \in \mathcal{M}(C) \cap C'$ a positive element
- a *-homomorphism $\pi_\varphi : A \rightarrow \mathcal{M}(C) \cap \{h\}'$

such that

$$\pi_\varphi(a)h_\varphi = \varphi(a) \quad \forall a \in A.$$

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Consequences

With the same notation:

- (Functional calculus on cpc_\perp) If $f \in C_0((0, 1])$, then $f(\varphi) : A \rightarrow B$ given by $f(\varphi)(a) = f(h_\varphi)\pi_\varphi(a)$ is a well-defined c.p. order zero map.

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via $\text{Cu}(\varphi)(\langle a \rangle) = \langle \varphi^k(a) \rangle$ if $a \in M_k(A)_+$. (They don't preserve \ll)

Proposition

Let A, B be C^* -algebras. Then every cpc order zero map $\varphi : A \rightarrow B$ naturally induces a map $\mathbf{Cu}(\varphi) : \mathbf{Cu}(A) \rightarrow \mathbf{Cu}(B)$ which preserves addition, order, the zero element and the suprema of increasing sequences, but, in general, not the way-below.

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(Possible answer) Study a bivariate version of Cuntz Semigroup (as done by KK -theory)

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Definition ($\text{Cu}(A, B)$)

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It follows that it is an abelian semigroup



Questions?