Maximal left ideals of operators acting on a Banach space

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Joint work with Garth Dales (Lancaster),
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- *finitely generated:* there exist $n \in \mathbb{N}$ and $b_1, \dots, b_n \in \mathscr{L}$ such that

$$\mathscr{L} = \{a_1 b_1 + \cdots + a_n b_n : a_1, \dots, a_n \in \mathscr{A}\}.$$

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Question I. Is this conjecture true for $\mathscr{A} = \mathscr{B}(E)$, the Banach algebra of all bounded, linear operators acting on a Banach space E?

A partial answer to Question I

Theorem (DKKKL). *Let E be a separable Banach space with a countable, unconditional Schauder decomposition. Then $\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ maximal left ideals, but only $\mathfrak{c}$ finitely-generated, maximal left ideals, where $\mathfrak{c} = 2^{\aleph_0}$.*

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Terminology. A *countable, unconditional Schauder decomposition* of a Banach space E is a sequence $(E_n)_{n\in\mathbb{N}}$ of non-zero, closed subspaces of E such that, for each $x \in E$, there is a unique sequence (x_n) with $x_n \in E_n$ $(n \in \mathbb{N})$ such that

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Remark. The above theorem can be extended to non-separable Banach spaces.

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In each of the following cases, the Banach space E is separable and has a countable, unconditional Schauder decomposition:

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Perspective. Gowers's Dichotomy Theorem: *an infinite-dimensional Banach space is either hereditarily indecomposable (in the sense that none of its closed subspaces can be decomposed into the direct sum of two closed, infinite-dimensional subspaces), or it contains a subspace which has an unconditional Schauder basis.*

A refinement of the question

Observation. *Let E be a Banach space. For each $x \in E \setminus \{0\}$,*

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Question II — the infinite-dimensional case

Let E be an infinite-dimensional Banach space. Then

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Corollary. A positive answer to Question II implies a positive answer to Q. I:

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Question III. Is $\mathscr{F}(E)$ ever contained in a finitely-generated, maximal left ideal of $\mathscr{B}(E)$?

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An equivalent, more algebraic, definition is that T is inessential if and only if $T + \mathscr{K}(E)$ belongs to the Jacobson radical of the Calkin algebra $\mathscr{B}(E)/\mathscr{K}(E)$.

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Corollary. Questions II and III are equivalent, in the following sense:

Every finitely-generated, maximal left ideal of $\mathscr{B}(E)$ is fixed if and only if no finitely-generated, maximal left ideal of $\mathscr{B}(E)$ contains $\mathscr{F}(E)$.

Positive answers to Question II

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The Dichotomy Theorem for Hilbert spaces follows from these facts because each pure state λ on $\mathscr{B}(H)$ is either a vector state, or $\mathscr{K}(H) \subseteq \ker \lambda$, in which case $\mathscr{K}(H) \subseteq \mathscr{N}_\lambda$.

A negative answer to Question II: Argyros–Haydon's Banach space

Theorem (Argyros–Haydon 2011). *There is a Banach space X_{AH} which has the following three properties:*

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More precisely, $\mathscr{K}_1$ is generated as a left ideal by the operator

$$L = \begin{pmatrix} 0 & 0 \\ VU^*\kappa & W \end{pmatrix},$$

where $\kappa\colon X_{\mathrm{AH}} \to X_{\mathrm{AH}}^{**}$ is the canonical embedding, while $U\colon \ell_1 \to X_{\mathrm{AH}}^*$, $V\colon \ell_1^* = \ell_\infty \to \ell_\infty(2\mathbb{N}-1)$ and $W\colon \ell_\infty \to \ell_\infty(2\mathbb{N})$ are isomorphisms.

How about Question I?

Recall: $E = X_{\mathsf{AH}} \oplus \ell_\infty$.

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In particular, the answer to Question I is positive for E.

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Question: Is there a *separable* Banach space E such that $\mathscr{B}(E)$ contains a finitely-generated, non-fixed, maximal left ideal?

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Theorem (Kania–L). *Argyros and Haydon's Banach space X_{AH} contains a closed, infinite-dimensional subspace Y of infinite codimension such that:*

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A separable example *(continued)*

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where $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2} \in \mathbb{C}$ and the operators $K_{1,1}, K_{1,2}, K_{2,1}, K_{2,2}$ are compact.

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Theorem (Kania–L).

(i) *There are exactly two non-fixed, maximal left ideals of* $\mathscr{B}(E)$, *namely*

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(iii) *$\mathscr{M}_2$ is not finitely generated as a left ideal.*

Open problems

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Open problems and references

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Key references

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