Conclusions, Outlook, Questions

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Contents:

Topics in 10th and last lecture:

Conclusions of classification results and questions.

1. Structure of algebras with ideal-system preserving zero-homotopy.

2. Constructions of examples of algebras with given second countable locally compact sober T_0 spaces (not necessarily Hausdorff).

3. Minimal requirement for a weak version of a universal coefficent theorem for ideal-equivariant classification.

4. Optional: Equivariant versions for actions of compact groups (up to 2-cocycle equivalence) ?.

Consider the case where $X \cong \mathrm{Prim}(B)$ and X acts non-degenerate, lower semi-continuous and monotone upper semi-continuos (i.e., "way below" continuous) on A. Then the observations applied under the – from now on overall – assumptions that A and B are *non-zero, separable, stable and strongly p.i.* – here equal to strong $\mathcal{O}_\infty$-absorption ! – lead to the following:

Theorem (1)

The elements of $\mathrm{KK}_X(A, B)$ *can be represented by* X*-action compatible mono-morphisms* $h\colon A \to B$.

h *and* $h \oplus h_0$ *are in the same class.*

It is in the same KK_X*-class as* $h' : A \to B$ *if and only if* $h \oplus h_0$ *and* $h' \oplus h_0$ *are unitarily homotopic.*

h *and* $h \oplus h_0$ *are unitarily homotopic, if and only if the action* $J \mapsto h^{-1}(J) \subseteq \Psi_X(J)$ *coincides always with* Ψ_X *of* X *on* A.

The last remark excludes e.g. $h = 0$ which is in $\mathrm{KK}_X(A, B)$ the same as $h_0\colon A \to B$ that defines the action of X on A – via $\mathrm{Prim}(B) \cong X$ – and "extends" to a non-degenerate monomorphism from $A \otimes \mathcal{O}_2$ into B: $[h_0] = [0]$.

Corollary (2)

If the action of X on A comes from an homeomorphism of X with $\mathrm{Prim}(A)$, then $h\colon A \to B$ is a KK_X-equivalence, if and only if, $h \oplus h_0$ is unitarily homotopic to an X-equivariant isomorphism φ from A onto B.

Application to X-equivariant zero homotopies:

Here comes a case where this difference between 0 and h_0 plays an important role:

Suppose now that B has an ideal-system equivariant zero-homotopy, i.e., there exists $\varphi\colon B \to \mathrm{C}([0,1], B)$ given by a point-norm continuous path $\varphi_t \in \mathrm{Hom}_X(B, B)$ with $\varphi_1 = \mathrm{id}_B$ and $\varphi_0 = 0$.

If we now add to this homotopy our "big zero" $h_0\colon B \to B$, then $[h_0] = [h_0 \oplus \mathrm{id}_B] = [\mathrm{id}_B]$ in $\mathrm{KK}_X(B, B)$, homotopy invariance of $\mathrm{KK}_X(B, B)$. This implies that id_B is unitarily homotopic to h_0, and then that $\mathcal{M}(B)$ contains a central sequence of copies of $\mathcal{O}_2$. It implies $B \cong B \otimes \mathcal{O}_2$.

The absorption of $\mathcal{O}_2$ and the X-compatible zero-homotopy allow to construct asymptotic embeddings of B into an inductive limit of certain AH-algebras coming from 1-dim simplicial complexes and also asymptotic embeddings of those AH-algebras into B. A controlled approximation procedure then gives that B is an AH-algebra of the below described type.

Our (M. Rørdam and me) approximation itself is not X-compatible and is not compatible with the given X-compatible zero homotopy.

Theorem (3)(GAFA, **15**, 2005)

If A is a separable, nuclear, strongly purely infinite C-algebra that is homotopic to zero in an ideal-system preserving way, then A is the inductive limit of C*-algebras of the form* $\mathrm{C}_0(\Gamma, v) \otimes M_k$*, where* Γ *is a graph (and* $\mathrm{C}_0(\Gamma, v)$ *is the algebra of continuous functions on* Γ *that vanish at a distinguished point* $v \in \Gamma$*).*

It would be interesting if one can find an AH-approximation that is compatible with the ideal structure and is invariant under – some variant of – the zero homotopy.

Corollary (4)

If B is any separable, nuclear C-algebra, then $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is isomorphic to a crossed product $D \rtimes_\alpha \mathbb{Z}$, where D is an inductive limit of algebras $\mathrm{C}_0(\Gamma, v) \otimes M_k$ (and D is $\mathcal{O}_2$-absorbing and homotopic to zero in an ideal-system preserving way).*

An other consequence is the following observation:

Corollary (5)

If B is separable and amenable then $B \otimes \mathcal{O}_2$ *contains a regular abelian C*-subalgebra.*

In particular, $\mathcal{I}(B)$ *is isomorphic a sub-lattice* Ω *of the open sets* $\mathbb{O}(P)$ *of some locally compact Polish space* P *that is closed under suprema and infima.*

Thus B has Abelian factorization.

Examples.

Since we have no ideal-equivariant UCT available, one can attempt to construct test-examples. This will be supported by the below given theorem:

Let Ω a sub-lattice of the lattice of open subsets of an l.c. Polish space P that is closed under supremum and infimum, then this action defines a canonical lower semi-continuous action of $\mathbb{O}(P)$ on P,

and Ω is naturally isomorphic to the lattice of open subsets of the locally compact space X of its prime elements.

We get from the action of X on $\mathrm{C}_0(P)$ a self-action of P on a Hilbert bi-module $\mathcal{H}$ over $\mathrm{C}_0(P, \mathbb{K})$ in general position.

Theorem

(6) Under above assumptions is the Toeplitz-Pimsner algebra and the Cuntz-Pimsner algebra is the same C-algebra, is strongly purely infinite and its ideal lattice is isomorphic to Ω.*
The Pimsner construction of a homotopy equivalence between $\mathrm{C}_0(P, \mathbb{K})$ and $\mathcal{T}(\mathcal{H})$ is X-equivariant.

One can use that this homotopy produces also KK_Y-equivalences for coarser topologies, e.g. $\mathrm{KK}_X(A, B) \to \mathrm{KK}(A, B)$ to distinguish easier KK_X classes.

In general there are only some very special cases where there is a chance to derive invariants for the verification of KK_X-equivalence practical. See further below.

Example

Consider $P_0 := (0,1]$, $X_0 := (0,1]_{\mathrm{lsc}} := (0,1]$, with topology $\mathbb{O}(X_0) := \Omega := \{\emptyset, (t,1]\,;\ t \in [0,1)\} \subset \mathbb{O}(P)$.

Alternatively:

$P_1 :=$ probability measures on $[0,1]$ *except* the character δ_0 with topology "wage" ($\cong \sigma(\mathrm{C}[0,1]^*, \mathrm{C}[0,1])$- topology on the states of $\mathrm{C}[0,1]$).

Define a map $\gamma\colon \mu \in P_1 \to (0,1]$ by $\gamma(\mu) = \max(\text{support of } \mu)$. Then $\gamma\colon P_1 \to X_0$ is continuous and open.

The corresponding Cuntz-Pimsner algebras are the same and coincide with the example studied by Mortensen and Rødram, that we call A_0.

It has by our descriptions a ideal-system preserving zero homotopy and $t \mapsto t^2$ is implemented by an automorphism of A_0 that is properly outer because each power $t \to t^{2n}$ moves all ideals (notice here that $\{ 1\}$ is neither open nor closed).

The crossed product by this action is $A_0 \rtimes \mathbb{Z} \cong \mathcal{O}_2 \otimes \mathbb{K}$.

If B is an amenable C^*-algebra, then $B \otimes A_0$ has an ideal-system preserving zero-homotopy.

Thus, $B \otimes A_0$ has the structure considered in the last Theorem.

The natural inclusion $B \otimes A_0 \subset B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ implies that a regular Abelian C^*-subalgebra of $B \otimes A_0$ is also regular in
$B \otimes \mathcal{O}_2 \otimes \mathbb{K} \subset B \otimes \mathcal{O}_2$.

Crossed product with the $\mathbb{Z}$-action defined by $t \to t^2$ leads to a proof of the Corollary.

Except the idea of Rødram for the T_0 space $\{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$ (for 1-step extensions of UCT-class p.i. stable separable C^*-algebras).

It can be described by transformations between of 6-term sequences.

Some progress exists also for 2-step extension $\{0, 1, 2\}$ with topology $\{\emptyset, \{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The case of linear ordered $\mathbb{O}(X)$ is difficult enough.

Next we use for an action $\Psi\colon \mathbb{O}(Y) \to \mathcal{I}(A)$ of X on A the notation $A|Z := \Psi(U)/\Psi(U) \cap \Psi(U \setminus Z)$ if Z is a closed subset of an open subset $U \subset X$. We say that X acts on A *continuously* if Ψ is a lattice *monomorphism* and is both upper semi-continuous and lower semicontinuous. (not only monotone)

Conjecture: Let X a second countable locally quasi-compact point-complete T_0 space.

Conjecture: TFAE:

(i) If A is nuclear and separable, $\mathrm{Prim}(A) \cong X$, and $A/J \cong (A/J) \otimes \mathcal{O}_2$ for every *primitive* ideal, then $A \cong A \otimes \mathcal{O}_2$.

(ii) If X acts on separable nuclear C^*-algebras A and B continuously, and if $\psi\colon A \to B$ is an X-equivariant *-monomorphism such that, for each $x \in X$, the induced morphism $A|\overline{\{x\}} \to B|\overline{\{x\}}$ defines a $\mathrm{KK}(\overline{\{x\}}; A|\overline{\{x\}}, B|\overline{\{x\}})$ equivalence, *then* ψ *defines a* $\mathrm{KK}(X; A, B)$ *equivalence.*