Introduction to Banach and Operator Algebras

Lecture 7

Zhong-Jin Ruan
University of Illinois at Urbana-Champaign

Winter School at Fields Institute
Thursday January 16, 2014

Let us first recall from Lecture 6 the following theorem.

Theorem: Let G be a discrete group. TFAE:

(1) G is amenable,

(2) There exists a net of unit vectors $\xi_\alpha \in \ell_2(G)$ (with finite support) such that $\|\lambda_s \xi_\alpha - \xi_\alpha\|_2 \to 0$ for all $s \in G$,

(3) There exists a net of (positive definite) contractive/bounded $\varphi_\alpha \in A(G)$ (with finite support) such that $\varphi_\alpha(s) \to 1$ for all $s \in G$.

(4) $A(G)$ has a contractive/bounded apprixImate identity,

(5) $C^*(G) = C^*_\lambda(G)$ or equivalently $B(G) = B_\lambda(G)$.

Theorem: For discrete group G, we can easily prove that TFAE:

(1) G is amenable,

(2) $C^*_\lambda(G)$ is nuclear,

(3) $C^*_\lambda(G)$ has the CPAP,

(4) $VN_\lambda(G)$ is semidiscrete.

How about non-amenable groups ?

What can we say about the free group $\mathbb{F}_2$ of 2-generators ?

How do we describe the correspondng property for their group C*-algebras and group von Neumann algebras ?

Completely Bounded/Herz-Schur Multipliers

A function $\varphi : G \to \mathbb{C}$ is a multiplier of $A(G)$ if the multiplication map

$$m_\varphi : f \in A(G) \to \varphi f \in A(G).$$

In this case, m_φ is automatically bounded on $A(G)$.

Since $A(G) = VN_\lambda(G)_*$, there is a natural operator space structure on $A(G)$. A multiplier φ is completely bounded (we also call it Herz-Schur multiplier) if $m_\varphi : A(G) \to A(G)$ is a cb map. In this case, we use the notion $\|\varphi\|_{cb} = \|m_\varphi\|_{cb}$.

Theorem: A function $\varphi : G \to \mathbb{C}$ is a cb multiplier with $\|m_\varphi\|_{cb} \leq 1$ if and only if there exist contractive maps $\alpha, \beta : G \to H$ for some Hilbert space H such that

$$\varphi(s^{-1}t) = \langle \alpha(t) \mid \beta(s) \rangle = \beta(s)^* \alpha(t).$$

We let $M_{cb}A(G)$ denote the space of all cb-multipliers of G.

Since every $\varphi \in B(G)$ is the coefficient of the universal representation of G. We can choose $\xi, \eta \in H_u$ such that

$$\varphi(s) = \langle u_s \xi | \eta \rangle \text{ and thus } \varphi(s^{-1}t) = \langle u_t \xi | u_s \eta \rangle$$

and $\|\varphi\|_{B(G)} = \|\xi\| \|\eta\|$. Therefore, we have

$$B(G) \subseteq M_{cb}A(G)$$

and

$$\|\varphi\|_{cb} \leq \|\varphi\|_{B(G)}.$$

In general, we have

$$A(G) \hookrightarrow B_\lambda(G) \hookrightarrow B(G) \subseteq M_{cb}A(G).$$

For any $\varphi \in A(G)$, we have

$$\|\varphi\|_{A(G)} = \|\varphi\|_{B_\lambda(G)} = \|\varphi\|_{B(G)} \geq \|\varphi\|_{cb}.$$

Theorem: A group G is amenable if and only if $B(G) = M_{cb}A(G)$.

So if G is non-amenable, then we have

$$\|\varphi\|_{cb} \leq \|\varphi\|_{A(G)}$$

for all $\varphi \in A(G)$.

Weakly Amenable Groups

A discrete group G is weakly amenable if there exists a net of finitely supported $\varphi_\alpha \in A(G)$ such that $\|\varphi_\alpha\|_{cb} \leq C < \infty$ and $\varphi_\alpha \to 1$ pointwisely.

Theorem: Let G be a discrete group. TFAE:

(1) G is weakly amenable (with $\|\varphi_\alpha\|_{cb} \leq C < \infty$),

(2) $C^*_\lambda(G)$ has the CBAP, i.e. there exists a net of finite rank cb maps $T_\alpha : C^*_\lambda(G) \to C^*_\lambda(G)$ such that $\|T_\alpha\|_{cb} \leq C$ and $\|T_\alpha(x) - x\| \to 0$ for all $x \in C^*_\lambda(G)$,

(3) $VN_\lambda(G)$ has the weak* CBAP, i.e. there exists a net of finite rank weak* continuous cb maps $T_\alpha : VN_\lambda(G) \to VN_\lambda(G)$ such that $\|T_\alpha\|_{cb} \leq C$ and $\langle T_\alpha(x) - x, \omega \rangle \to 0$ for all $x \in VN_\lambda(G)$ and $\omega \in VN_\lambda(G)_*$.

We let $\Lambda(G) = \inf\{C\}$ denote the Cowling-Haagerup constant. In general, we have $\Lambda(G) \geq 1$. We say that G has the CCAP if $\Lambda(G) = 1$.

Outline of Proof: (1) $\Rightarrow$ (2) and (3) If G is weakly amenable such that we have a net of finitely supported $\varphi_\alpha \in A(G)$ such that $\|\varphi_\alpha\|_{cb} \le C < \infty$ and $\varphi_\alpha \to 1$ poitwisely. Then for each α,

$$m_{\varphi_\alpha} : f \in A(G) \to \varphi_\alpha f \in A(G)$$

is a finite rank cb map on $A(G)$. Its adjoint map $T_\alpha = m^*_{\varphi_\alpha}$ is a weak* continuous finite rank cb map on the group von Neumann algebra $VN_\lambda(G)$ such that $\|T_\alpha\|_{cb} = \|m_{\varphi_\alpha}\|_{cb} \le C$ and

$$T_\alpha(\lambda_s) = \varphi_\alpha(s)\lambda_s.$$

It follows that the restriction of T_α to $C^*_\lambda(G)$ defines a net of finite rank cb maps on $C^*_\lambda(G)$.

Finally since $\varphi_\alpha(s) \to 1$ for every $s \in G$, we get

$$T_\alpha(\lambda_s) = \varphi_\alpha(s)\lambda_s \to \lambda_s$$

in the norm topology on $C^*_\lambda(G)$ (resp., in weak* topology on $VN_\lambda(G)$). This implies that $T_\alpha(x) \to x$ for all finite sum $x = \sum a_i \lambda_{s_i}$. Since $\{T_\alpha\}$ is uniformly bounded, this is also true for all $x \in C^*_\lambda(G)$) (resp., for all $x \in VN_\lambda(G)$).

(2) $\Rightarrow$ (1) Suppose that $\{T_\alpha\}$ is a net of finite rank maps on $C^*_\lambda(G)$ given in condition (2). We can prove that

$$\varphi_\alpha(s) = \langle \lambda_{s^{-1}} T_\alpha(\lambda_s)\delta_e | \delta_e \rangle = \langle T_\alpha(\lambda_s)\delta_e | \lambda_s \delta_e \rangle$$

is a net of bounded functions on G such that (i) each φ_α is contained in $A(G)$ and (ii) $\|\varphi_\alpha\|_{cb} \leq C$. The norm convergence $T_\alpha(\lambda_s) \to \lambda_s$ implies that

$$\varphi_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_e | \delta_s \rangle \to \langle \lambda_s \delta_e | \delta_s \rangle = 1$$

for all $s \in G$. This shows that G is weakly amenable with $\Lambda(G) \leq C$.

We can similarly prove (3) $\Rightarrow$ (1).

Proof of (i): It sufficies to consider that T_α is a rank one map, i.e. $T_\alpha(x) = f_\alpha(x) b_\alpha$ for some $f_\alpha \in B_\lambda(G)$ and $b_\alpha \in C^*_\lambda(G)$. In this case, we get

$$\varphi_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_e | \delta_s \rangle = f_\alpha(\lambda_s) \langle b\delta_e | \lambda_s \delta_e \rangle \in A(G).$$

Proof of (ii): Since $T_\alpha : C^*_\lambda(G) \to C^*_\lambda(G) \subseteq B(\ell_2(G))$ is completely bounded, we have the cb-representation

$$T_\alpha(x) = V^*\pi(x)W \text{ with } \|V\|\|W\| = \|T_\alpha\|_{cb}.$$

Then we obtain two bounded maps

$$\alpha(t) = \pi(t)W\lambda_{t^{-1}}\delta_e \text{ and } \beta(s) = \pi(s)V\lambda_{s^{-1}}\delta_e$$

such that

$$\begin{aligned} \langle \alpha(t) | \beta(s) \rangle &= \langle \pi(t)W\lambda_{t^{-1}}\delta_e | \pi(s)V\lambda_{s^{-1}}\delta_e \rangle = \langle V^*\pi(s^{-1})\pi(t)W\lambda_{t^{-1}}\delta_e | \lambda_{s^{-1}}\delta_e \rangle \\ &= \langle T_\alpha(\lambda_{s^{-1}t})\lambda_{t^{-1}}\delta_e | \lambda_{s^{-1}}\delta_e \rangle = \langle T_\alpha(\lambda_{s^{-1}t})\delta_e | \lambda_{s^{-1}t} \rangle = \varphi_\alpha(s^{-1}t) \end{aligned}$$

This shows that we have

$$\|\varphi_\alpha\|_{cb} \leq \|V\|\|W\| = \|T_\alpha\|_{cb} \leq C.$$

Properties About Cowling-Haagerup Constant

(1) Every amenable group is weakly amenable with $\Lambda(G) = 1$.

(2) Weak amenability is closed under subgroups, i.e. if $H \leq G$ is a subgroup, then $\Lambda(H) \leq \Lambda(G)$.

(3) Weak amenability is closed under the cartesian product, i.e. we have $\Lambda(G_1 \times G_2) = \Lambda(G_1) \cdot \Lambda(G_2)$.

(4) Weak amenability is not closed under group quotient or group semidirect product.

Length Function on the Free Group $\mathbb{F}_2$

Let $\mathbb{F}_2$ be the free group of 2-generators with generators u and v. Then $\mathbb{F}_2$ consists of all reduced words e (empty word), $u, v, u^{-1}, v^{-1}, uu, uv, uv^{-1}, v$

Given a reduced word $s = r_1 r_2 \cdots r_n$ (with $r_i = u, v, u^{-1}$ or v^{-1}), we use $|s| = n$ denote the length of s. This induces a metric

$$d(s, g) = |s^{-1}g|$$

on $\mathbb{F}_2$. It is known by Haagerup that there exists a map $f : \mathbb{F}_2 \to H_\Lambda$ such that $f(e) = 0$ and

$$d(s, g) = |s^{-1}g| = \|f(s) - f(g)\|^2.$$

Then the length function

$$(s, g) \in \mathbb{F}_2 \times \mathbb{F}_2 \to |s^{-1}g| = \|f(s) - f(g)\|^2$$

is a negative definite kernel, i.e. for all $s_1, \cdots s_n \in \mathbb{F}_2$ and $\alpha_1 \cdots \alpha_n \in \mathbb{C}$ with $\sum \alpha_i = 0$, we have

$$\sum |s_i^{-1} s_j| \alpha_i \bar{\alpha}_j = \sum \|f(s_i) - f(s_j)\|^2 \alpha_i \bar{\alpha}_j = -2\|\sum_i \alpha_i f(s_i)\|^2 \leq 0.$$

Positive Definite Functions associated with the Length Function

It follows from Schoenberg theorem that for each real number $t > 0$,

$$(s, g) \in \mathbb{F}_2 \times \mathbb{F}_2 \to e^{-t|s^{-1}g|}$$

is a positive definite kernel. Therefore,

$$\varphi_t : g \in \mathbb{F}_2 \to e^{-t|g|} \in [0, \infty)$$

is a positive definite function on $\mathbb{F}_2$.

Proposition: Let $t > 0$.

(1) Each φ_t is a positive definite function in $B(G)$ with $\varphi_t(e) = 1$.

(2) Each φ_t is contained in $c_0(G)$ since $\varphi_t(g) \to 0$ as $|g| \to \infty$,

(3) For each $g \in \mathbb{F}_2$, $\varphi_t(g) \to 1$ as $t \to 0$.

CCAP of $C^*_\lambda(\mathbb{F}_2)$

Theorem: $C^*_\lambda(\mathbb{F}_2)$ has the CCAP.

Outline of Proof: Let W_n denote the set of words with length n and let $E_n = \cup_{k=0}^n W_k$ be the set of all words with length $\leq n$. For $n \geq 1$, we have

$$|W_n| = 4 \times 3^{n-1} \text{ and } |E_n| = 1 + 4(\sum_{k=1}^{n} 3^{k-1}).$$

Then $\varphi_{n,t} = \varphi_t \chi_{E_n}$ is a net of functions on $\mathbb{F}_2$ with finite support and thus all contained in $A(\mathbb{F}_2)$.

It is known by Haagerup that for each $t > 0$, $\|\varphi_{n,t}\|_{cb} \to \|\varphi_t\|_{cb} = 1$. Then $\psi_{t,n} = \varphi_{t,n}/\|\varphi_{t,n}\|_{cb}$ is a net of functions with finite support such that $\|\psi_{t,n}\|_{cb} \leq 1$ and $\psi_{t,n}(g) \to 1$ for all $g \in \mathbb{F}_2$. This shows that $C^*_\lambda(\mathbb{F}_2)$ is weakly amenable with $\Lambda(\mathbb{F}_2) = 1$.

Corollary: For any $2 \leq n \leq \infty$, $C^*_\lambda(\mathbb{F}_n)$ has the CCAP.

Proof: Since $\mathbb{F}_n$ is a subgroup of $\mathbb{F}_2$, we have $\Lambda(\mathbb{F}_n) = \Lambda(\mathbb{F}_2) = 1$.

More Examples

- If G_1 and G_2 are weakly amenable with $\Lambda(G_1) = \Lambda(G_2) = 1$, then the free product $G_1 \star G_2$ is weakly amenable such that $\Lambda(G_1 \star G_2) = 1$.

  It follows that $\mathbb{F}_2 = \mathbb{Z} \star \mathbb{Z}$ and $\mathbb{Z}_2 \star \mathbb{Z}_3$ are weakly amenable with Cowling-Haagerup constant 1.

- $\Lambda(SL(2, \mathbb{Z})) = 1$.

- Any lattice Γ of $Sp(1, n)$ is weakly amenable with Cowling-Haagerup constant equal $2n - 1$.

- $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ and $SL(3, \mathbb{Z})$ are not weakly amenable.

Haagerup Property for Groups

Definition: A group G has the Haagerup property (or a-T-menable in Gromov's sense) if there exists a sequence of positive definite functions $\varphi_n : G \to \mathbb{C}$ such that
1) each φ_n is contained in $C_0(G)$,
2) $\varphi_n(s) \to 1$ for every $s \in G$.

Remark: Since $0 < \varphi_n(e) \to 1$, we can assume that $\varphi_n(e) = 1$ in the definition.

As we have seen from the above discussion, the free group C*-algebra $C^*_\lambda(\mathbb{F}_2)$ has the Haagerup property. In this case,

$$\varphi_t(g) = \mathrm{e}^{-t|g|} \quad !t > 0$$

is a net of positive definite functions on $\mathbb{F}_n$ satisfying the above conditions 1) and 2).

Groups with the Haagerup Property

- Amenable groups
- Free groups, $SL(2,\mathbb{Z})$,
- subgroups, cartesian product, free product, increasing unions, ...

Groups without the Haagerup Property

- $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ $SL(3,\mathbb{Z})$, $Sp(n,1)$, or any group with property (T)

A group has the property (T) if any sequence of (normalized) positive definite functions, converging uniformly on compact sets, must converge uniformly on G

Von Neumann Algebra Haagerup Property

Definition: A von Neumann algebra M with a normal faithful trace τ has the Haagerup property if there exists a net of unital normal cp maps Φ_i on M such that
0) $\tau \circ \Phi_i \leq \tau$
1) each Φ_i extends to a compact operator on $L_2(M,\tau)$
2) $\|\Phi_i(x) - x\|_2 \to 0$ for every $x \in M$ (resp. for every $x \in L_2(M,\tau)$).

Theorem [Choda 1983]: A discrete group has the Haagerup property if and only if its group von Neumann algebra $L(G)$ with the canonical trace τ has the von Neumann algebra Haagerup property.

Definition A unital C*-algebra A with a faithful trace (or state) τ has the Haagerup property if there exists a net of unital cp maps Φ_i on A such that

0) $\tau \circ \Phi_i \leq \tau$

1) each Φ_i extends to a compact operator on $L_2(A, \tau)$

2) $\|\Phi_i(x) - x\|_2 \to 0$ for every $x \in A$ (resp. for every $x \in L_2(A, \tau)$).

Theorem [Dong 2010]: A discrete group has the Haagerup property if and only if its reduced group C*-algebra $C^*_\lambda(G)$ with the canonical trace τ has the C*-algebra Haagerup property.

References

(1) M. Bożejko and G. Fendler, Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. Boll. Un. Mat. Ital. A (6), 3 1984, 297-302.

(2) C*-algebras and Finite-Dimensional Approximations, N. Brown and N. Ozawa

(3) Groups with the Haagerup Property, P-A Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette

(4) U. Haagerup, An example of nonnuclear C*-algebra which has the metric approximation property. Invent. Math. 50(1979), 279293.

(5) Introduction to Operator Spaces, G. Pisier