Isotropic Entanglement

(Density of States of Quantum Spin Systems)

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The eigenvalue distribution: Motivation

Synonymous: (Energy) Spectrum; eigenvalue distribution, density of states, level densities etc.

- First step for all eigenvalue problems (e.g. quantum mechanics) of sums of matrices
- Physical: Partition function and therefore the thermodynamics of QMBS

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Goal: Given the geometry, local spin states, and type of local interaction, capture the spectrum of the H.

Complexity issues

Note: Generally the Spectrum of QMBS is hard to find *"exactly" (QMA-complete):*

- F.G.S.L. Brandao's Thesis (2008).
- B. Brown, S. T. Flammia, N. Schuch (2010), "Computational Difficulty of Computing the Density of States".

  Phys. Rev. Lett. 107, 040501 (2011)

Sums of non-commuting Hamiltonians

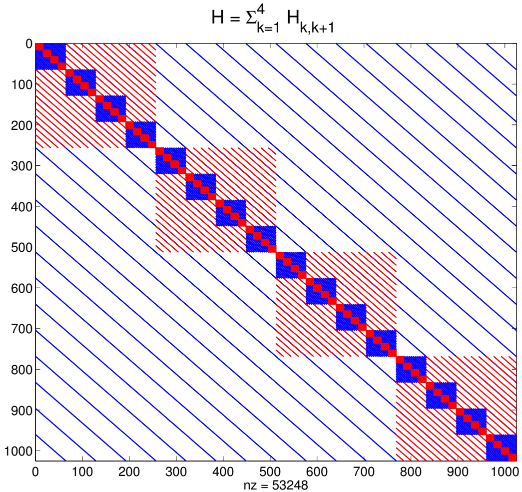
Figure: Qudit Chain

$$H_{k,k+1}: \quad d^2 \times d^2$$

- Generic local terms $\Longrightarrow$ Quantum Spin Glasses.

$$H = \sum_{k=1}^{N-1} \mathbb{I}_{d^{k-1}} \otimes H_{k,k+1} \otimes \mathbb{I}_{d^{N-k-1}}.$$

Non-zero elements of H



Why not just add eigenvalues?

$$\mathbb{I}_{d\times d} \otimes A_{d\times d} \quad + \quad B_{d\times d} \otimes \mathbb{I}_{d\times d}$$

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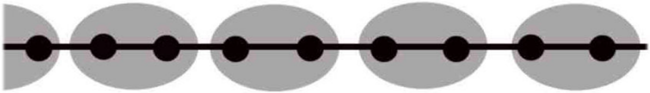
$$\mathbb{I}_{d\times d}\otimes A_{d\times d} \quad + \quad B_{d\times d}\otimes \mathbb{I}_{d\times d}$$

$$\mathbb{I}_{d\times d}\otimes A_{d^2\times d^2} \quad + \quad B_{d^2\times d^2}\otimes \mathbb{I}_{d\times d}$$

Interactions: $H = \sum_{k=1}^{N-1} \left(\mathbb{I} \otimes H_{k,k+1} \otimes \mathbb{I}\right) = H_{\rm odd} + H_{\rm even}$



Eigenvectors of odds: $Q_A = Q_1 \otimes Q_3 \otimes \cdots \otimes Q_{N-2} \otimes \mathbb{I}_{d\times d}$

Q_k : $d^2 \times d^2$ matrix of eigenvectors of $H_{k,k+1}$

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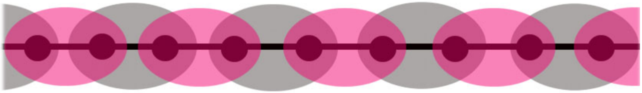


Eigenvectors of odds: $Q_A = Q_1 \otimes Q_3 \otimes \cdots \otimes Q_{N-2} \otimes \mathbb{I}_{d \times d}$

Eigenvectors of evens: $Q_B = \mathbb{I}_{d \times d} \otimes Q_2 \otimes Q_4 \otimes \cdots \otimes Q_{N-1}$

Q_k : $d^2 \times d^2$ matrix of eigenvectors of $H_{k,k+1}$

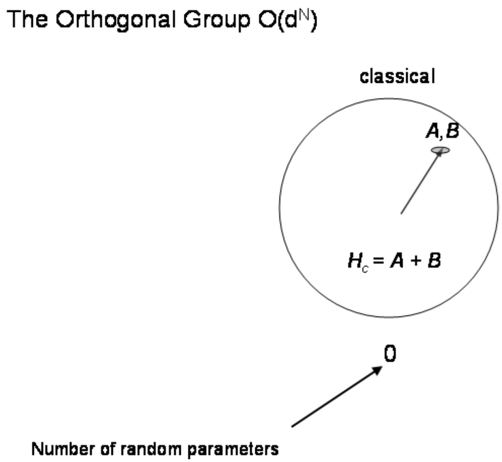
$H = H_{\text{odd}} + H_{\text{even}} = Q_A A Q_A^{-1} + Q_B B Q_B^{-1}$

Change bases such that H_{odd} is diagonal. Therefore,

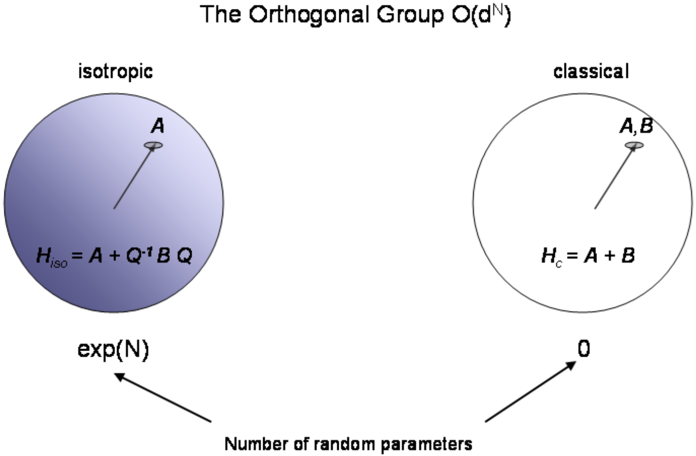
$$H = A + Q_q^{-1} B Q_q$$

$Q_q \equiv (Q_B)^{-1} Q_A \qquad \sim N \quad$ random parameters

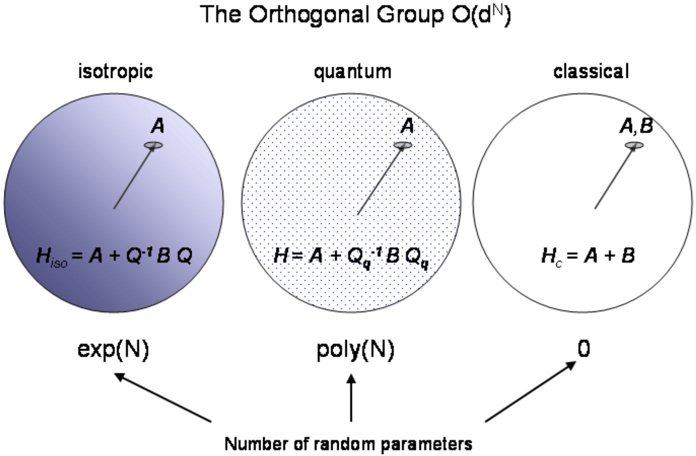
Classical sum



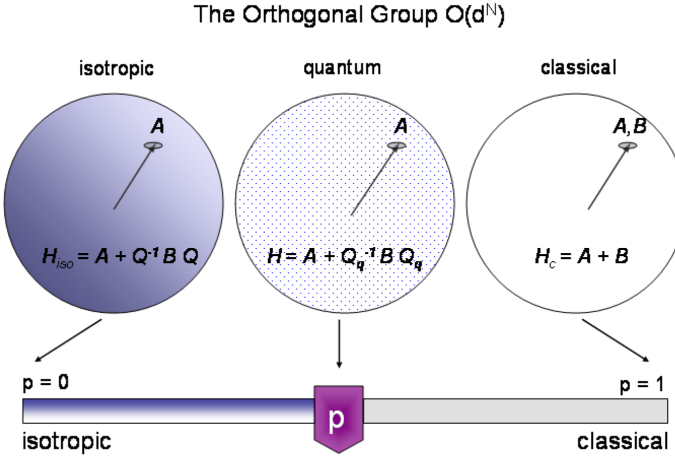
Isotropic (Free) sum



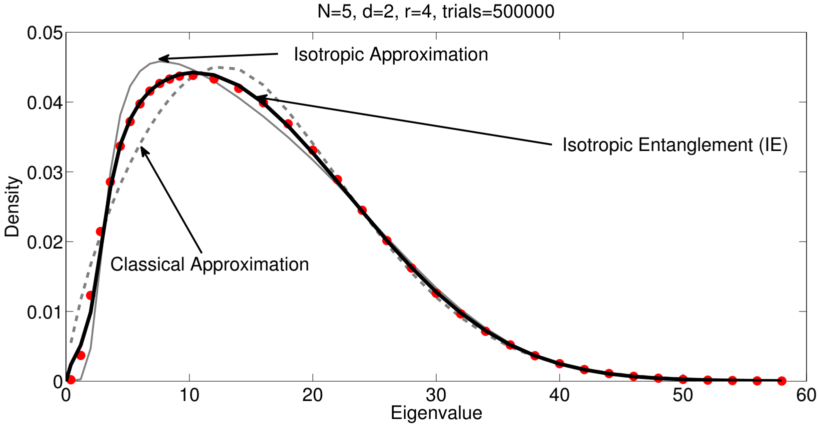
Isotropic, Quantum, and Classical



Quantum as a "sliding" sum of classical and iso



Local terms: Wishart matrices



The action starts at the fourth moment

Theorem

(The Matching Three Moments Theorem) *The first three moments of the quantum, iso and classical sums are equal.*

The Departure Theorem

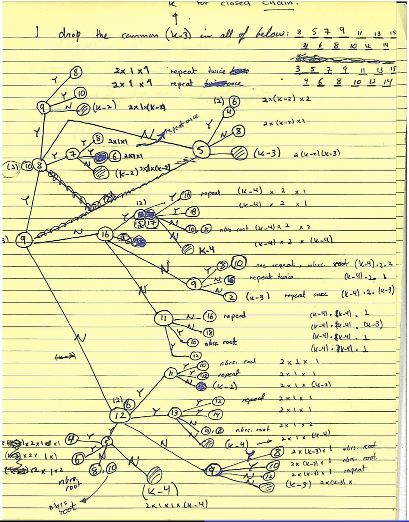
The Departure Theorem

$$m_4^{iso}=\frac{1}{dN}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3Q^{-1}BQ+4A^2Q^{-1}B^2Q+4AQ^{-1}B^3Q+\underline{\mathbf{2\left(AQ^{-1}BQ\right)^2}}+B^4\right]\right\}$$

$$m_4^{q}=\frac{1}{dN}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3Q_q^{-1}BQ_q+4A^2Q_q^{-1}B^2Q_q+4AQ_q^{-1}B^3Q_q+\underline{\mathbf{2\left(AQ_q^{-1}BQ_q\right)^2}}+B^4\right]\right\}$$

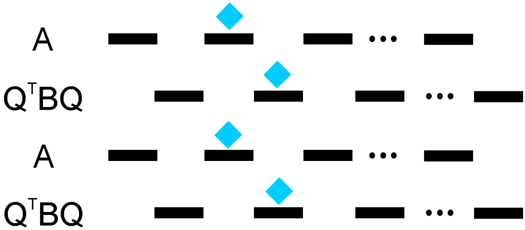
$$m_4^{c}=\frac{1}{dN}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3B+4A^2B^2+4AB^3+\underline{\mathbf{2(AB)^2}}+B^4\right]\right\}$$

Quantum agony



Resolving the agony

Lemma: Only these matter



$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(H^{(3)}\otimes\mathbb{I}_{d^{N-2}}\right)\left(\mathbb{I}\otimes H^{(4)}\otimes\mathbb{I}_{d^{N-3}}\right)\left(H^{(3)}\otimes\mathbb{I}_{d^{N-2}}\right)\left(\mathbb{I}\otimes H^{(4)}\otimes\mathbb{I}_{d^{N-3}}\right)\right]$$

$$=\frac{1}{d^3}\left\{\mathbb{E}\left(H^{(3)}_{i_3 i_4, j_3 j_4} H^{(3)}_{i_3 p_4, j_3 k_4}\right)\mathbb{E}\left(H^{(4)}_{j_4 i_5, k_4 k_5} H^{(4)}_{i_4 i_5, p_4 k_5}\right)\right\},$$

Quantum as a convex combination of *classical* and *iso*

- Use fourth moments to form a hybrid theory

$$\gamma_2^q = p\gamma_2^c + (1-p)\,\gamma_2^{iso}$$

$\gamma_2^{(\bullet)}$ is found from the fourth moments

The Slider Theorem

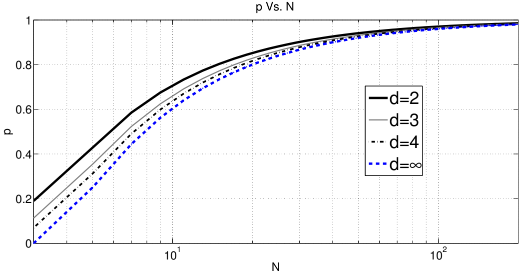
Theorem

(The Slider Theorem) *The quantum kurtosis lies in between the classical and the iso kurtoses,* $\gamma_2^{iso} \leq \gamma_2^{q} \leq \gamma_2^{c}$. *Therefore there exists a* $0 \leq p \leq 1$ *such that* $\gamma_2^{q} = p\gamma_2^{c} + (1-p)\,\gamma_2^{iso}$. *Further,* $\lim_{N\to\infty} p = 1$.

Universality of p

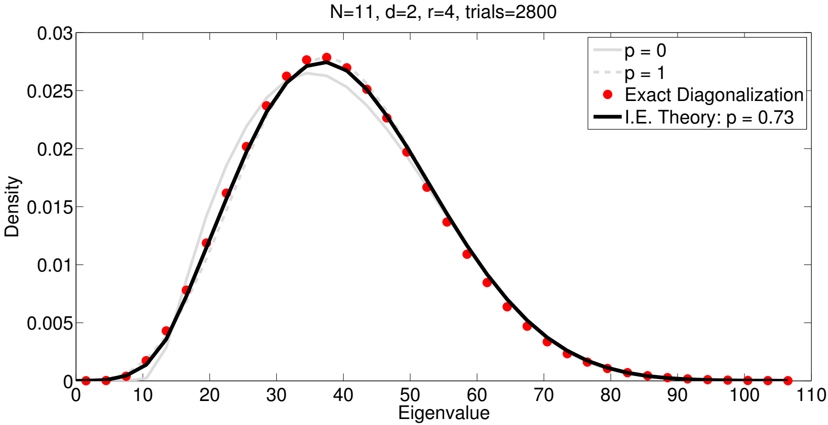
Corollary

(Universality) $p \mapsto p(N,d,\beta)$, *namely, it is independent of the distribution of the local terms.*

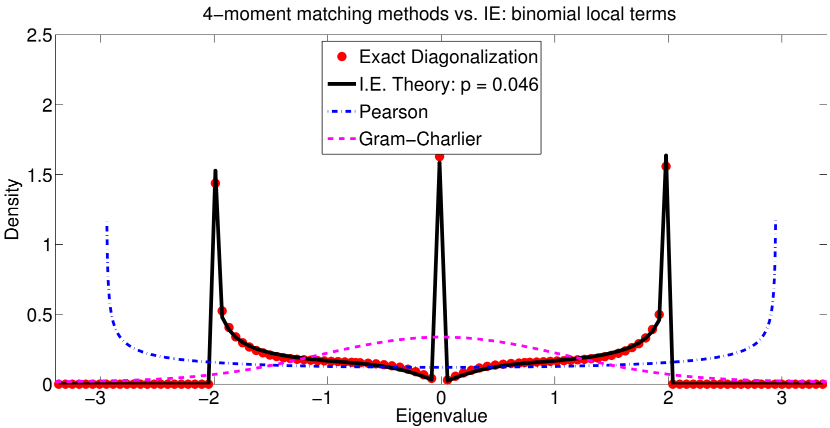


Here $\beta = 1$. Therefore, p only depends on eigenvectors!

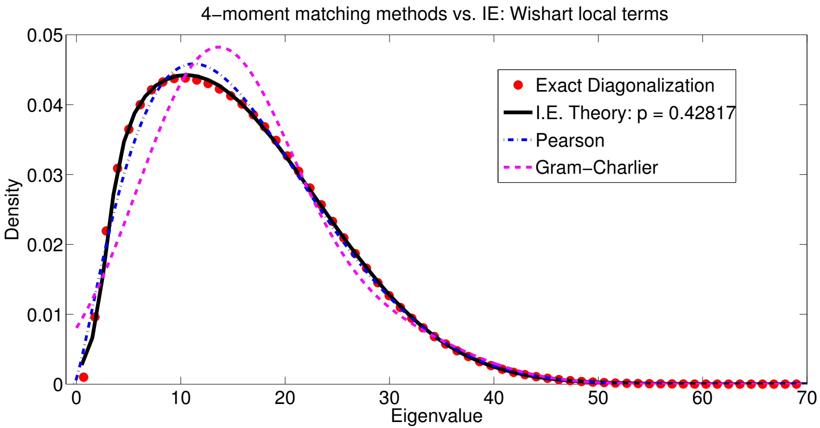
Local terms: Wishart matrices



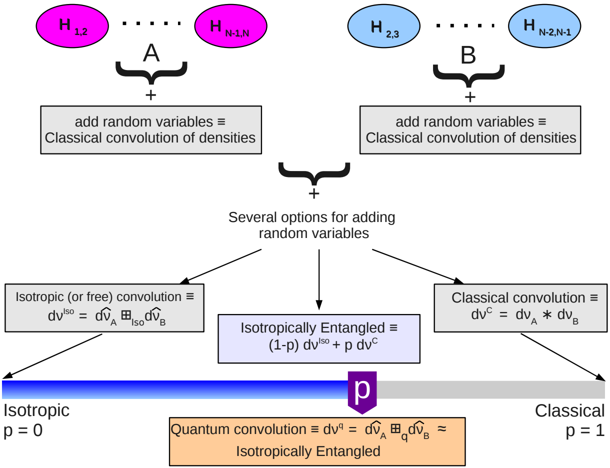
Suppose you have the first four moments



Suppose you have the first four moments



Summary: Method of Isotropic Entanglement



Others got excited about it too!

Error analysis of free probability approximations to the density of states of disordered systems

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Theoretical studies of localization, anomalous diffusion and ergodicity breaking require solving the electronic structure of disordered systems. We use free probability to approximate the ensemble-averaged density of states without exact diagonalization. We present an error analysis that quantifies the accuracy using a generalized moment expansion, allowing us to distinguish between different approximations. We identify an approximation that is accurate to the eighth moment across all noise strengths, and contrast this with the perturbation theory and isotropic entanglement theory.

Phys. Rev. Lett. 109, 036403 (2012)

Question for you: Why do we do so well?

$$m_5=\frac{1}{m}\mathbb{E}\mathrm{Tr}\Big(A^5+5A^4Q_\bullet^TBQ_\bullet+5A^3Q_\bullet^TB^2Q_\bullet+5A^2Q_\bullet^TB^3Q_\bullet+\underline{\mathbf{5A}\big(\mathbf{AQ_\bullet^TBQ_\bullet}\big)^2}+ \underline{\mathbf{5}\big(\mathbf{AQ_\bullet^TBQ_\bullet}\big)^2\mathbf{Q_\bullet^TBQ_\bullet}}+5AQ_\bullet^TB^4Q_\bullet+B^5\Big) \quad (1)$$

Question for you: Why do we do so well?

$$\begin{aligned}
\mathbb{E}\mathrm{Tr}\left\{\ldots Q^{-1}B^{\geq 1}QA^{\geq 1}Q^{-1}B^{\geq 1}Q\ldots\right\} &\leq \\
\mathbb{E}\mathrm{Tr}\left\{\ldots Q_q^{-1}B^{\geq 1}Q_qA^{\geq 1}Q_q^{-1}B^{\geq 1}Q_q\ldots\right\} &\leq \\
\mathbb{E}\mathrm{Tr}\left\{\ldots B^{\geq 1}A^{\geq 1}B^{\geq 1}\ldots\right\} &.
\end{aligned}$$

e.g. $\mathbb{E}\mathrm{Tr}\left\{\left(AQ^{-1}BQ\right)^k\right\} \leq \mathbb{E}\mathrm{Tr}\left\{\left(AQ_q^{-1}BQ_q\right)^k\right\} \leq \mathbb{E}\mathrm{Tr}\left\{(AB)^k\right\}$

$$K > 2$$

Lastly...

Thank*Q*