

Quantum symmetric states on free product C^* -algebras

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Joint work with
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Focus Program on Noncommutative Distributions in Free Probability

Workshop on Combinatorial and Random Matrix Aspects of
Noncommutative Distributions and Free Probability
Fields Institute, Toronto, July 2-6, 2013

Introduction and Motivation

Classical Probability

independence

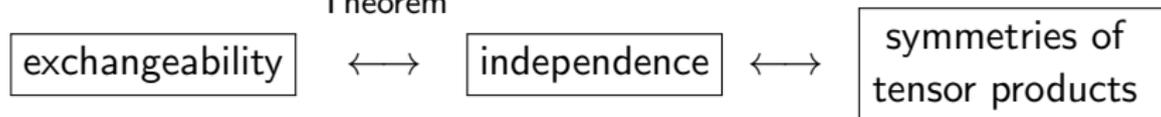


symmetries of
tensor products

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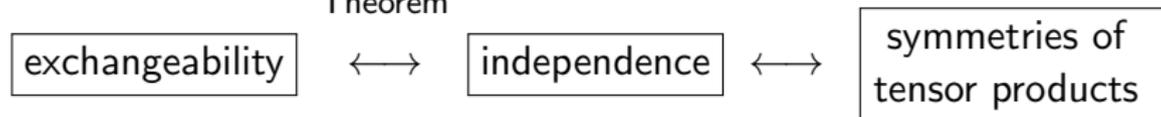
De Finetti's
Theorem



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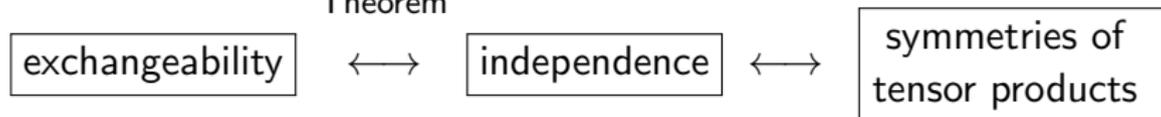


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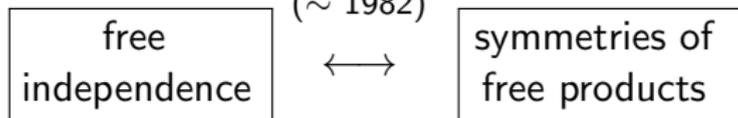
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Free Probability

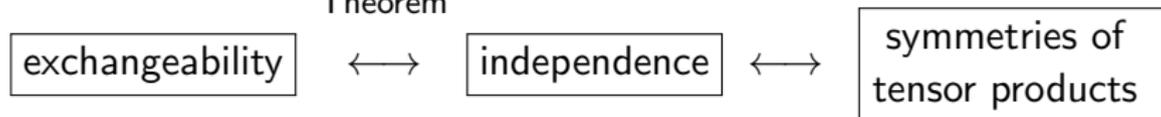
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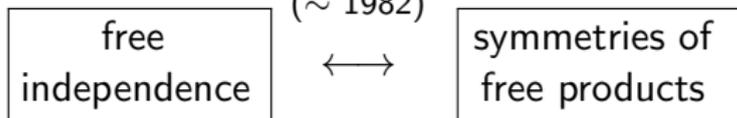
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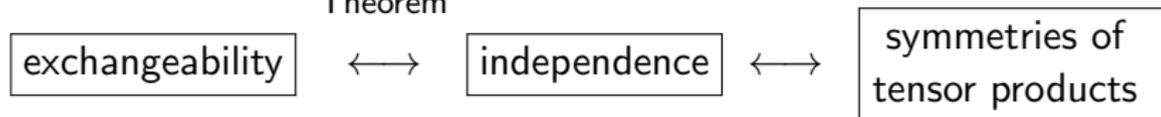


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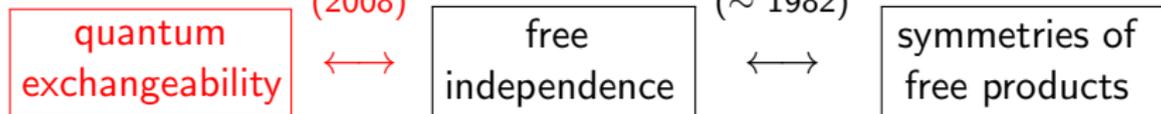


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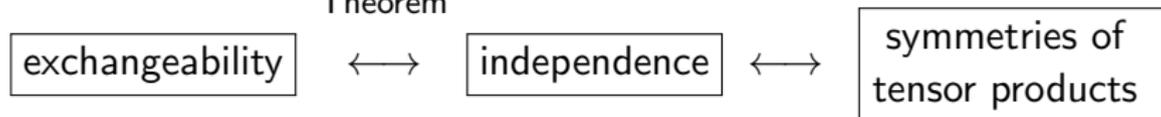


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↪ New direction of research in free probability

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$$\varphi(X_{i_1} \cdots X_{i_n}) = \varphi(X_{\pi(i_1)} \cdots X_{\pi(i_n)})$$

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$$\text{id} \otimes \varphi \circ \alpha(\bullet) = \varphi(\bullet) \mathbb{1} \quad (\text{invariance})$$

$$\alpha(X_j) = \sum_{i=1}^k e(\pi)_{ij} \otimes X_i \quad (\text{coaction})$$

for all permutation matrices $e(\pi)$ with $e(\pi)_{ij} = \delta_{\pi(i)j}$

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- ◇ Replace $C(\mathbb{S}_k)$ by **quantum permutation group** $A_s(k)$

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Definition and Theorem (Wang 1998)

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The **abelianization** of $A_s(k)$ is $C(\mathbb{S}_k)$, the continuous functions on the symmetric group \mathbb{S}_k

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Remark



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- (a) the sequence is **quantum exchangeable**
- (b) the sequence is identically distributed and **freely independent with amalgamation over \mathcal{T}**

Here \mathcal{T} denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \text{vN}(x_k | k \geq n)$$

Quantum symmetric states

Definition (Dykema & K & Williams)

Let A be a unital C^* -algebra, (\mathfrak{B}, ϕ) a C^* -probability space and $\lambda_i: \mathfrak{A} \rightarrow \mathfrak{B}$ unital $*$ -homomorphisms ($i \in \mathbb{N}$). The sequence $(\lambda_i)_i$ is said to be **quantum exchangeable w.r.t. ψ** or the state ψ is said to be **quantum symmetric** w.r.t. $(\lambda_i)_i$ if

$$\begin{aligned} & \psi(\lambda_{i_1}(a_1) \cdots \lambda_{i_n}(a_n)) \mathbb{1}_{A_s(k)} \\ = & \sum_{j_1, \dots, j_n=1}^k e_{i_1 j_1} \cdots e_{i_n j_n} \psi(\lambda_{j_1}(a_1) \cdots \lambda_{j_n}(a_n)) \mathbb{1}_{A_s(k)} \end{aligned}$$

for all $n \in \mathbb{N}$, all $i_1, \dots, i_n \in \{1, \dots, k\}$, all $a_1, \dots, a_n \in A$, all $k \times k$ -matrices $(e_{ij})_{ij}$ satisfying defining relations for $A_s(k)$.

Standing notation — Goal

Throughout A is a unital C^* -algebra and $\mathfrak{A} := *_1^\infty A$ denotes the universal, unital free product of infinitely many copies of A .

$\text{QSS}(A)$ denotes the set of all quantum symmetric states on \mathfrak{A} .

$\text{TQSS}(A)$ denotes the set of all tracial quantum symmetric states on \mathfrak{A} .

Proposition

For any unital C^* -algebra A the sets $\text{QSS}(A)$ and $\text{TQSS}(A)$ are compact, convex subsets of the Banach space dual of \mathfrak{A} in the weak*-topology.

Goal

Study and characterize $\text{QSS}(A)$ and $\text{TQSS}(A)$ as far as possible.

What is the free probability counterpart to the following 'classical' result?

Theorem (Størmer 1969)

Let A be a unital C^* -algebra and $\bigotimes_1^\infty A$ the infinite minimal tensor product of A . Then the set of all symmetric states on $\bigotimes_1^\infty A$ is a Choquet simplex and an extreme symmetric state on $\bigotimes_1^\infty A$ is an infinite tensor product state of the form $\bigotimes_1^\infty \psi$, with $\psi \in S(A)$.

Remark

So, for a given unital C^* -algebra, one has a bijective correspondence between symmetric states on $\bigotimes_1^\infty A$ and probability measures on $S(A)$.

Quantum symmetric states arising from freeness

Proposition

Let \mathfrak{B} be a unital C^* -algebra and $\mathfrak{D} \subseteq \mathfrak{B}$ be unital C^* -subalgebra with conditional expectation $E: \mathfrak{B} \rightarrow \mathfrak{D}$ (i.e. a projection of norm one onto \mathfrak{D}). Suppose that

$$\pi_i: A \rightarrow \mathfrak{B} \quad (i \in \mathbb{N})$$

are $*$ -homomorphisms such that $E \circ \pi_i$ is the same for all i and that $(\pi_i(A))_{i=1}^{\infty}$ is free with respect to E . Let $\pi = *_{i=1}^{\infty} \pi_i: \mathfrak{A} \rightarrow \mathfrak{B}$ be the resulting free product $*$ -homomorphism. For a state ρ on \mathfrak{D} consider the state $\psi = \rho \circ E \circ \pi$. Then ψ is **quantum symmetric**.

Remark

The proof uses Speicher's free \mathfrak{D} -valued cumulants and the defining properties of the projections e_{ij} from Wang's quantum permutation groups.

Tail algebras of symmetric states

Let ψ be a state on $\mathfrak{A} = *_{1}^{\infty} A$. Passing to the GNS representation $(\mathcal{H}_{\psi}, \pi_{\psi}, \Omega_{\psi})$ of (\mathfrak{A}, ψ) , put

$$\mathcal{M}_{\psi} = \pi_{\psi}(\mathfrak{A})'' \quad \hat{\psi} := \langle \Omega_{\psi}, \bullet \Omega_{\psi} \rangle$$

The **tail algebra** of ψ is the von Neumann subalgebra

$$\mathcal{T}_{\psi} = \bigcap_{n=1}^{\infty} W^* \left(\bigcup_{i=n}^{\infty} \pi_{\psi}(\lambda_i(A)) \right) \subset \mathcal{M}_{\psi}.$$

Proposition (Dykema & K & Williams)

Suppose ψ is symmetric. Then there exists a normal $\hat{\psi}$ -preserving conditional expectation E_{ψ} from \mathcal{M}_{ψ} onto \mathcal{T}_{ψ} .

'Quantum symmetric noncommutative distributions' imply freeness with amalgamation

Theorem (Dykema & K & Williams)

Let ψ be a quantum symmetric state on $\mathfrak{A} = *_1^\infty A$ and put

$$\mathcal{B}_i := W^*\left(\pi_\psi(\lambda_i(A)) \cup \mathcal{T}_\psi\right).$$

Then $(\mathcal{B}_i)_{i=1}^\infty$ is free with respect to E_ψ .

Remark

- Our proof is modeled along the proof of K & Speicher (2009), but now starts in an C^* -algebraic setting and does **not** assume the faithfulness of states.
- Curran's approach (2009) considers the more delicate situation of quantum exchangeability of finite sequences in a $*$ -algebraic setting and obtains the result for infinite sequences as a limiting case. but also under the assumption of faithfulness of the state.

What tail algebras can appear?

In Størmer's setting of symmetric states on $\bigotimes_1^\infty A$, only abelian tail algebras can arise. But in our setting of quantum symmetric states one has:

Theorem (Dykema & K '2012)

Let \mathcal{N} be a countable generated von Neumann algebra. Then there exists a unital C*-algebra A and a quantum symmetric state ψ on $\mathfrak{A} = *_1^\infty A$ such that $\mathcal{T}_\psi \simeq \mathcal{N}$.

Remark

Størmer's approach does **not** use tail algebras; the machinery of ergodic decomposition of states and Choquet theory is available.

Description of quantum symmetric states $\text{QSS}(A)$

For a unital C^* -algebra A , let $\mathcal{V}(A)$ be the set (of all equivalence classes) of quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ such that

- (i) \mathcal{B} is a von Neumann algebra,
- (ii) \mathcal{D} is a unital von Neumann subalgebra of \mathcal{B} ,
- (iii) $E: \mathcal{B} \rightarrow \mathcal{D}$ is a normal conditional expectation onto \mathcal{D} ,
- (iv) $\sigma: A \rightarrow \mathcal{B}$ is a unital $*$ -homomorphism,
- (v) ρ is a normal state on \mathcal{D} ,
- (vi) the GNS representation of $\rho \circ E$ is a faithful represent. of \mathcal{B} ;
- (vii) $\mathcal{B} = W^*(\sigma(A) \cup \mathcal{D})$,
- (viii) \mathcal{D} is the smallest unital von Neumann subalgebra of \mathcal{B} that satisfies

$$E(d_0 \sigma(a_1) d_1 \cdots \sigma(a_n) d_n) \in \mathcal{D}$$

whenever $n \in \mathbb{N}$, $d_0, \dots, d_n \in \mathcal{D}$ and $a_1, \dots, a_n \in A$.

Theorem (Dykema & K & Williams)

There is a bijection $\mathcal{V}(A) \rightarrow \text{QSS}(A)$ that assigns to $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ the quantum symmetric state $\psi = \psi_W$ as follows. Let

$$(\mathcal{M}, \tilde{E}) = (*_{\mathcal{D}})_{i=1}^{\infty}(\mathcal{B}, E)$$

be the amalgamated free product of von Neumann algebras, let

$$\pi_i: A \xrightarrow{\sigma} \mathcal{B} \xrightarrow{i\text{-th comp}} \mathcal{M}, \quad \pi := *_{i=1}^{\infty} \pi_i: \mathfrak{A} \rightarrow \mathcal{M}$$

free product *-homomorphism

and set $\psi = \rho \circ \tilde{E} \circ \pi$. Under this correspondence, the following identifications of objects and resulting constructions can be made:

from GNS construction	\mathcal{T}_{ψ}	\mathcal{M}_{ψ}	π_{ψ}	$\hat{\psi}$	E_{ψ}
from quintuple W	\mathcal{D}	\mathcal{M}	π	$\rho \circ \tilde{E}$	\tilde{E}

N-pure states on von Neumann algebras

A state ψ on a C^* -algebra is said to be **pure** if whenever ρ is a state on this C^* -algebra with $t\rho \leq \psi$ for some $0 < t < 1$, it follows $\rho = \psi$.

Proposition

Let ρ be a normal state on the von Neumann algebra \mathcal{D} . TFAE:

- (i) the support projection of ρ is a minimal projection in \mathcal{D} ,
- (ii) ρ is pure.

To emphasize this support property, a normal pure state ρ on \mathcal{D} is called an **n-pure state**.

Remark

Von Neumann algebras without discrete type I parts possess **no** n-pure states, but such von Neumann algebras **may** appear as tail algebra for a quantum symmetric state.

Extreme quantum symmetric states

Theorem (Dykema & K & Williams)

Let $\psi \in \text{QSS}(A)$ and let $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ be the quintuple corresponding to ψ (under the bijection as indicated in the previous theorem).

$$\psi \in \partial_e \text{QSS}(A) \iff \rho \text{ is an } n\text{-pure state on } \mathcal{D}.$$

Remark

- Though having an n -pure state is a restriction on the tail algebra and forces it to have a discrete type I part, the tail algebra can still be quite complicated.
- For various examples of quantum symmetric states with 'exotic tail algebras' see our preprint.

Central quantum symmetric states

Recall that $\mathfrak{A} = *_1^\infty A$.

Notation

$ZQSS(A) := \{\psi \in QSS(A) \mid \mathcal{T}_\psi \subset \mathcal{Z}(\pi_\psi(\mathfrak{A})'')\}$

$ZTQSS(A) := TQSS(A) \cap ZQSS(A)$

Theorem (Dykema & K & Williams)

$ZQSS(A)$ and $ZTQSS(A)$ are compact, convex subsets of $QSS(A)$ and both are Choquet simplices, whose extreme points are the free product states and free product tracial states, respectively:

$$\begin{aligned}\partial_e(ZQSS(A)) &= \{*_1^\infty \phi \mid \phi \in S(A)\} \\ \partial_e(ZTQSS(A)) &= \{*_1^\infty \tau \mid \tau \in TS(A)\}\end{aligned}$$

Remark

$ZQSS(A)$ is that part of $QSS(A)$ which is in analogy to Størmer's result on symmetric states on the minimal tensor product $\otimes_1^\infty A$.

A final question . . .

It is known that $TS(A)$, if non-empty, forms a Choquet simplex.

Question

Is $TQSS(A)$ a Choquet simplex whenever $TS(A)$ is non-empty?

Thank you for your attention!