Quantum symmetric states on free product C*-algebras

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independence	$\longleftrightarrow$	symmetries of tensor products

Introduction and Motivation

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$$\varphi(X_{i_1} \cdots X_{i_n}) = \varphi(X_{\pi(i_1)} \cdots X_{\pi(i_n)})$$

for all $n \in \mathbb{N}$, all $i_1, \ldots, i_n \in \mathbb{N}$, and all permutations π

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- $\varphi(X_1^2 X_4^5 X_3) = \varphi(X_6^2 X_1^5 X_9)$

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Theorem (De Finetti 1931, Hewitt & Savage 1955,...)

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  $\equiv$ exchangeable sequence of 0-1-valued random variables

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- **. . . dynamical co-symmetry:** there is a *-homomorphism $\alpha \colon L^\infty(\mathbb{A}) \to C(\mathbb{S}_k) \otimes L^\infty(\mathbb{A})$ such that

$$\mathrm{id} \otimes \varphi \circ \alpha(\bullet) = \varphi(\bullet)\mathbb{1} \qquad \text{(invariance)}$$

$$\alpha(X_i) = \sum_{j=1}^{k} e(\pi)_{ij} \otimes X_j \qquad \text{(coaction)}$$

for all permutation matrices $e(\pi)$ with $e(\pi)_{ij} = \delta_{\pi(i)j}$

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Exchangeability as a co-symmetry can be ‘quantized’:
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◇ Replace X_k's by **operators** $x_1, x_2, \ldots$

◇ Replace $C(\mathbb{S}_k)$ by **quantum permutation group** $A_s(k)$

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The **abelianization** of $A_s(k)$ is $C(\mathbb{S}_k)$, the continuous functions on the symmetric group $\mathbb{S}_k$

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$$\varphi(x_{i_1} \cdots x_{i_n}) 1\!\!1_{A_s(k)} = \sum_{j_1, \ldots, j_n = 1}^{k} e_{i_1 j_1} \cdots e_{i_n j_n} \; \varphi(x_{j_1} \cdots x_{j_n})$$

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Remark

$$\boxed{\text{quantum exchangeability}} \;\substack{\Longrightarrow \\ \not\Longleftarrow}\; \boxed{\text{exchangeability}}$$

A free analogue of de Finetti's theorem

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of noncommutative random variables $x_1, x_2, \ldots$ in a W*-probability space $(\mathcal{A}, \varphi)$:

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(a) the sequence is **quantum exchangeable**

(b) the sequence is identically distributed and **freely independent with amalgamation over** $\mathcal{T}$

Here $\mathcal{T}$ denotes the **tail von Neumann algebra**

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathrm{vN}(x_k | k \geq n)$$

Quantum symmetric states

Definition (Dykema & K & Williams)

Let A be a unital C*-algebra, $(\mathfrak{B}, \phi)$ a C*-probability space and $\lambda_i \colon \mathfrak{A} \to \mathfrak{B}$ unital *-homomorphisms ($i \in \mathbb{N}$). The sequence $(\lambda_i)_i$ is said to be **quantum exchangeable w.r.t.** ψ or the state ψ is said to be **quantum symmetric** w.r.t. $(\lambda_i)_i$ if

$$\psi(\lambda_{i_1}(a_1)\cdots\lambda_{i_n}(a_n))\mathbb{1}_{A_s(k)}$$

$$= \sum_{j_1,\dots,j_n=1}^{k} e_{i_1 j_1}\cdots e_{i_n j_n}\ \psi(\lambda_{j_1}(a_1)\cdots\lambda_{j_n}(a_n))\mathbb{1}_{A_s(k)}$$

for all $n \in \mathbb{N}$, all $i_1, \dots i_n \in \{1, \dots, k\}$, all $a_1, \dots a_n \in A$, all $k \times k$-matrices $(e_{ij})_{ij}$ satisfying defining relations for $A_s(k)$.

Standing notation — Goal

Throughout A is a unital C*-algebra and $\mathfrak{A} := *_1^\infty A$ denotes the universal, unital free product of infinitely many copies of A.

QSS(A) denotes the set of all quantum symmetric states on $\mathfrak{A}$.

TQSS(A) denotes the set of all tracial quantum symmetric states on $\mathfrak{A}$.

Proposition

For any unital C*-algebra A the sets QSS(A) and TQSS(A) are compact, convex subsets of the Banach space dual of $\mathfrak{A}$ in the weak*-topology.

Goal

Study and characterize QSS(A) and TQSS(A) as far as possible.

What is the free probability counterpart to the following 'classical' result?

Theorem (Størmer 1969)

Let A be a unital C*-algebra and $\bigotimes_1^\infty$ A the infinite minimal tensor product of A. Then the set of all symmetric states on $\bigotimes_1^\infty A$ is a Choquet simplex and an extreme symmetric state on $\bigotimes_1^\infty A$ is an infinite tensor product state of the form $\bigotimes_1^\infty \psi$, with $\psi \in \mathsf{S}(A)$.

Remark

So, for a given unital C*-algebra, one has a bijective correspondence between symmetric states on $\bigotimes_1^\infty A$ and probability measures on $\mathsf{S}(A)$.

Quantum symmetric states arising from freeness

Proposition

Let $\mathfrak{B}$ be a unital C*-algebra and $\mathfrak{D} \subseteq \mathfrak{B}$ be unital C*-subalgebra with conditional expectation $E \colon \mathfrak{B} \to \mathfrak{D}$ (i.e. a projection of norm one onto $\mathfrak{D}$). Suppose that

$$\pi_i \colon A \to \mathfrak{B} \qquad (i \in \mathbb{N})$$

are *-homorphisms such that $E \circ \pi_i$ is the same for all i and that $(\pi_i(A))_{i=1}^{\infty}$ is free with respect to E. Let $\pi = *_{i=1}^{\infty}\pi_i \colon \mathfrak{A} \to \mathfrak{B}$ be the resulting free product *-homomorphism. For a state ρ on $\mathfrak{D}$ consider the state $\psi = \rho \circ E \circ \pi$. Then ψ is **quantum symmetric**.

Remark

The proof uses Speicher's free $\mathfrak{D}$-valued cumulants and the defining properties of the projections e_{ij} from Wang's quantum permutation groups.

Tail algebras of symmetric states

Let ψ be a state on $\mathfrak{A} = *_1^\infty A$. Passing to the GNS representation $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ of $(\mathfrak{A}, \psi)$, put

$$\mathcal{M}_\psi = \pi_\psi(\mathfrak{A})'' \qquad \hat{\psi} := \langle \Omega_\psi, \bullet\, \Omega_\psi \rangle$$

The **tail algebra** of ψ is the von Neumann subalgebra

$$\mathcal{T}_\psi = \bigcap_{n=1}^{\infty} W^*\Big(\bigcup_{i=n}^{\infty} \pi_\psi(\lambda_i(A)) \Big) \quad \subset \mathcal{M}_\psi.$$

Proposition (Dykema & K & Williams)

Suppose ψ is symmetric. Then there exists a normal $\hat{\psi}$-preserving conditional expection E_ψ from $\mathcal{M}_\psi$ onto $\mathcal{T}_\psi$.

'Quantum symmetric noncommutative distributions' imply freeness with amalgamation

Theorem (Dykema & K & Williams)

Let ψ be a quantum symmetric state on $\mathfrak{A} = *_1^\infty A$ and put

$$\mathcal{B}_i := W^*\Big(\pi_\psi(\lambda_i(A)) \cup \mathcal{T}_\psi\Big).$$

Then $(\mathcal{B}_i)_{i=1}^\infty$ is free with respect to E_ψ.

Remark

- Our proof is modeled along the proof of K & Speicher (2009), but now starts in an C*-algebraic setting and does **not** assume the faithfulness of states.
- Curran's approach (2009) considers the more delicate situation of quantum exchangeability of finite sequences in a *-algebraic setting and obtains the result for infinite sequences as a limiting case, but also under the assumption of faithfulness of the state.

What tail algebras can appear?

In Størmer's setting of symmetric states on $\bigotimes_1^\infty A$, only abelian tail algebras can arise. But in our setting of quantum symmetric states one has:

Theorem (Dykema & K '2012)

Let $\mathcal{N}$ be a countable generated von Neumann algebra. Then there exists a unital C*-algebra A and a quantum symmetric state ψ on $\mathfrak{A} = *_1^\infty A$ such that $\mathcal{T}_\psi \simeq \mathcal{N}$.

Remark

Størmer's approach does **not** use tail algebras; the machinery of ergodic decomposition of states and Choquet theory is available.

Description of quantum symmetric states QSS(A)

For a unital C*-algebra A, let $\mathcal{V}(A)$ be the set (of all equivalence classes) of quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ such that

(i) $\mathcal{B}$ is a von Neumann algebra,

(ii) $\mathcal{D}$ is a unital von Neumann subalgebra of $\mathcal{B}$,

(iii) $E\colon \mathcal{B} \to \mathcal{D}$ is a normal conditional expectation onto $\mathcal{D}$,

(iv) $\sigma\colon A \to \mathcal{B}$ is a unital *-homomorphism,

(v) ρ is a normal state on $\mathcal{D}$,

(vi) the GNS representation of $\rho \circ E$ is a faithful represent. of $\mathcal{B}$;

(vii) $\mathcal{B} = W^*\Big(\sigma(A) \cup \mathcal{D}\Big)$,

(viii) $\mathcal{D}$ is the smallest unital von Neumann subalgebra of $\mathcal{B}$ that satisfies

$$E(d_0\sigma(a_1)d_1 \cdots \sigma(a_n)d_n) \in \mathcal{D}$$

whenever $n \in \mathbb{N}$, $d_0, \ldots, d_n \in \mathcal{D}$ and $a_1, \ldots a_n \in A$.

Theorem (Dykema & K & Williams)

There is a bijection $\mathcal{V}(A) \to \mathrm{QSS}(A)$ that assigns to $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ the quantum symmetric state $\psi = \psi_W$ as follows. Let

$$(\mathcal{M}, \widetilde{E}) = (*_{\mathcal{D}})_{i=1}^{\infty}(\mathcal{B}, E)$$

be the amalgamated free product of von Neumann algebras, let

$$\pi_i \colon A \xrightarrow{\sigma} \mathcal{B} \overset{i\text{-th comp}}{\hookrightarrow} \mathcal{M}, \qquad \pi := *_{i=1}^{\infty}\pi_i \colon \mathfrak{A} \to \mathcal{M}$$

free product *-homomorphism

and set $\psi = \rho \circ \widetilde{E} \circ \pi$. Under this correspondence, the following identifications of objects and resulting constructions can be made:

from GNS construction	$\mathcal{T}_\psi$	$\mathcal{M}_\psi$	π_ψ	$\hat{\psi}$	E_ψ
from quintuple W	$\mathcal{D}$	$\mathcal{M}$	π	$\rho \circ \widetilde{E}$	$\widetilde{E}$

N-pure states on von Neumann algebras

A state ψ on a C*-algebra is said to be **pure** if whenever ρ is a state on this C*-algebra with $t\rho \leq \psi$ for some $0 < t < 1$, it follows $\rho = \psi$.

Proposition

Let ρ be a normal state on the von Neumann algebra $\mathcal{D}$. TFAE:

(i) the support projection of ρ is a minimal projection in $\mathcal{D}$,

(ii) ρ is pure.

To emphasize this support property, a normal pure state ρ on $\mathcal{D}$ is called an **n-pure state**.

Remark

Von Neumann algebras without discrete type *I* parts possess **no** n-pure states, but such von Neumann algebras **may** appear as tail algebra for a quantum symmetric state.

Extreme quantum symmetric states

Theorem (Dykema & K & Williams)

Let $\psi \in \mathrm{QSS}(A)$ and let $W = (\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ be the quintuple corresponding to ψ (under the bijection as indicated in the previous theorem).

$$\psi \in \partial_e \mathrm{QSS}(A) \quad \Longleftrightarrow \quad \rho \text{ is an n-pure state on } \mathcal{D}.$$

Remark

- Though having an n-pure state is a restriction on the tail algebra and forces it to have a discrete type I part, the tail algebra can still be quite complicated.
- For various examples of quantum symmetric states with 'exotic tail algebras' see our preprint.

Central quantum symmetric states

Recall that $\mathfrak{A} = *_1^\infty A$.

Notation

$\mathsf{ZQSS}(A) := \{\psi \in \mathsf{QSS}(A) \mid \mathcal{T}_\psi \subset \mathcal{Z}(\pi_\psi(\mathfrak{A})'')\}$
$\mathsf{ZTQSS}(A) := \mathsf{TQSS}(A) \cap \mathsf{ZQSS}(A)$

Theorem (Dykema & K & Williams)

ZQSS(A) and ZTQSS(A) are compact, convex subsets of QSS(A) and both are Choquet simplices, whose extreme points are the free product states and free product tracial states, respectively:

$$\begin{aligned}\partial_e(\mathsf{ZQSS}(A)) &= \{*_1^\infty \phi \mid \phi \in \mathsf{S}(A)\}\\ \partial_e(\mathsf{ZTQSS}(A)) &= \{*_1^\infty \tau \mid \tau \in \mathsf{TS}(A)\}\end{aligned}$$

Remark

ZQSS(A) is that part of QSS(A) which is in analogy to Størmer's result on symmetric states on the minimal tensor product $\otimes_1^\infty A$.

A final question . . .

It is known that $\mathrm{TS}(A)$, if non-empty, forms a Choquet simplex.

Question

Is $\mathrm{TQSS}(A)$ a Choquet simplex whenever $\mathrm{TS}(A)$ is non-empty?

Thank you for your attention!