From noncrossing partitions to ASM

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Fields Institute, Toronto

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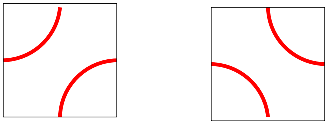
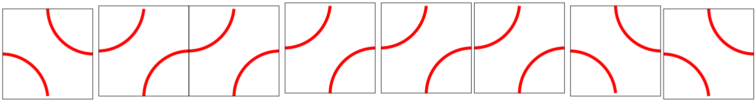
work with Hayat Cheballah

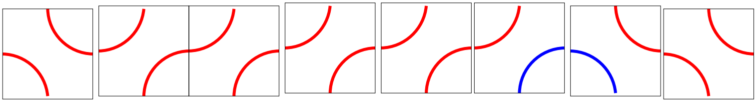
The $O(1)$ model

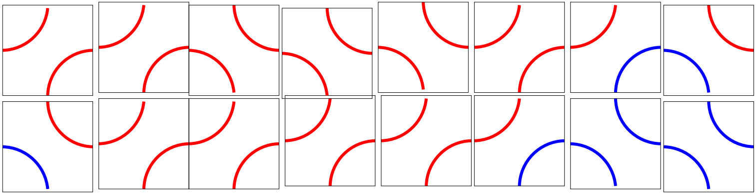
Take two types of tiles

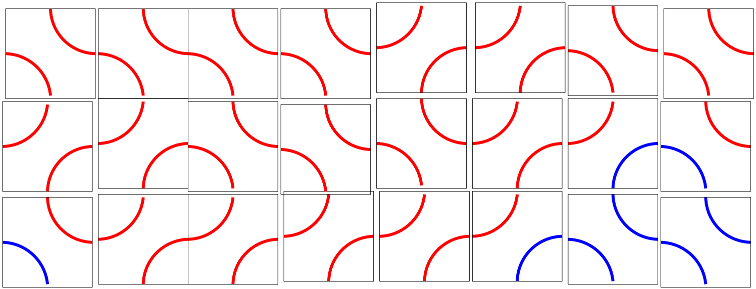


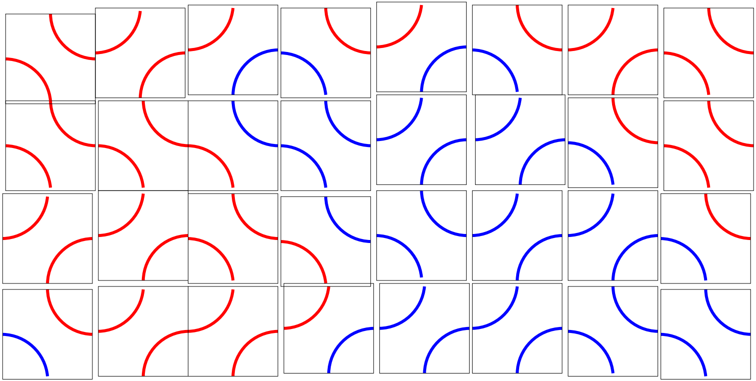
and tile a cylinder at random

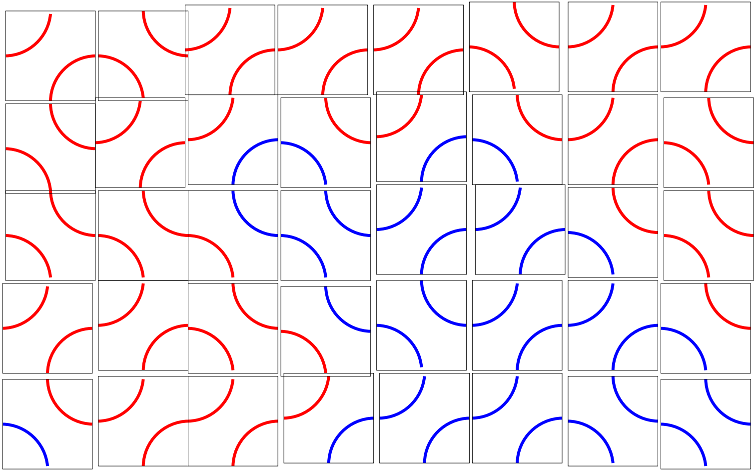


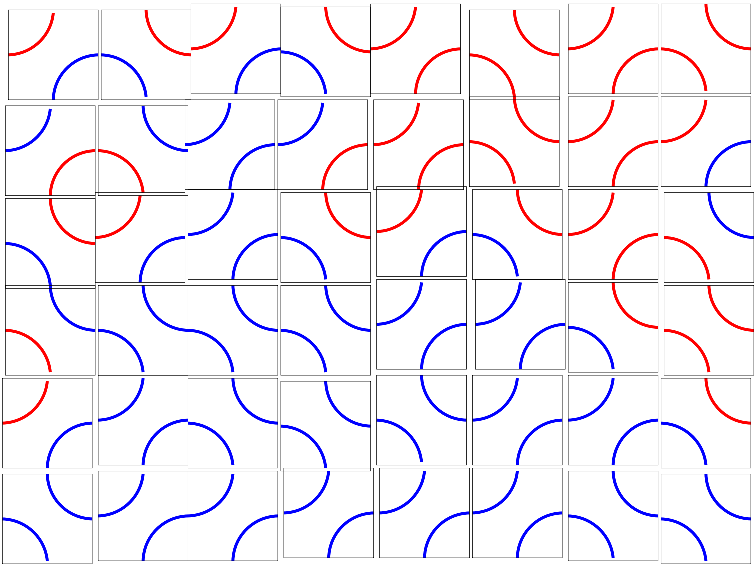










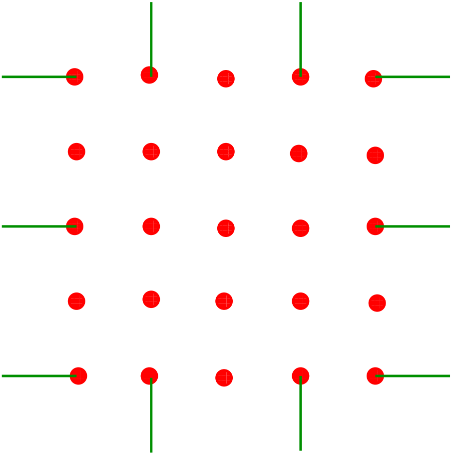


This gives a random noncrossing partition.

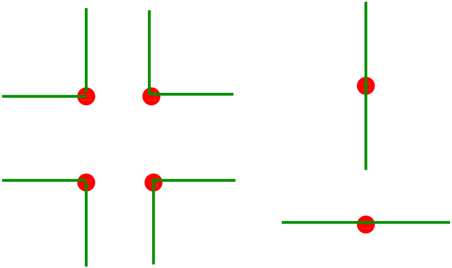
Problem: what is the distribution?

Fully packed loops

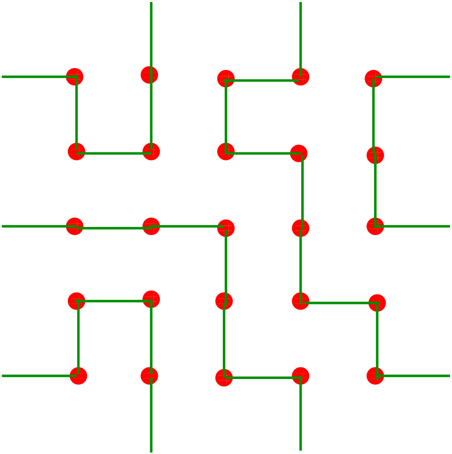
Consider a square grid



Colour the edges so that each each vertex has two incoming edges.



The six vertex configurations



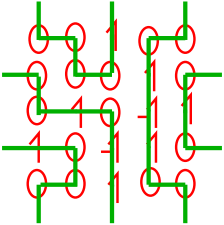
Such a configuration gives a noncrossing partition of the boundary edges.

Razumov-Stroganov (ex)-conjecture (2001)

Proved by Cantini and Sportiello (2010):

The two distributions on noncrossing pairings are the same.

From fully packed loops to ASM



One can associate to each FPL configuration an alternating sign matrix.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In each row and column the 1 and -1 alternate.

Enumeration of alternating sign matrices

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

$$A_n = 1 \qquad 2 \qquad 7 \qquad 42 \qquad 429 \qquad 7436 \qquad 218348$$

Conjectured by Mills, Robbins, Rumsey (1986).

Proved by D. Zeilberger (1995).

(see also G. Kuperberg, I. Fischer ...)

Gog Triangles

sum along columns and record the positions of the one's

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & \searrow & 2 & & 3 & & 4 & \nearrow & \\ & & & 2 & & 3 & & & \\ & & & & 3 & & & & \end{array}$$

Gog Triangles

sum along columns and record the positions of the one's

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & \searrow & 2 & & 3 & & 4 & \nearrow & \\ & & & 2 & & 3 & & & \\ & & & & 3 & & & & \end{array}$$

- Gog Triangles = Gelfand-Tsetlin triangles

Gog Triangles

sum along columns and record the positions of the one's

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ \searrow & & 2 & & 3 & & 4 & \nearrow & \\ & & & 2 & & 3 & & & \\ & & & & 3 & & & & \end{array}$$

- Gog Triangles = Gelfand-Tsetlin triangles
- strictly increasing in rows

Gog Triangles

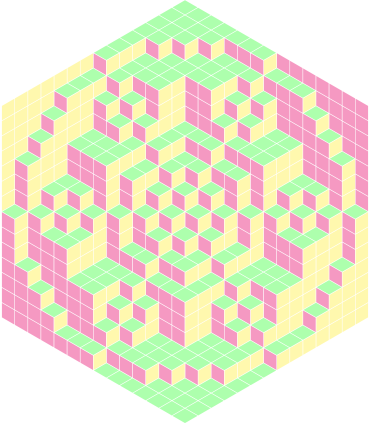
sum along columns and record the positions of the one's

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & \searrow & 2 & & 3 & & 4 & \nearrow & \\ & & & 2 & & 3 & & & \\ & & & & 3 & & & & \end{array}$$

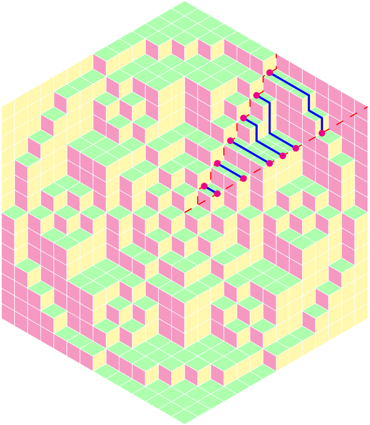
- Gog Triangles = Gelfand-Tsetlin triangles
- strictly increasing in rows
- upper row is $1, 2, 3, \ldots, n$.

Totally symmetric self-complementary plane partitions



Plane partitions, with hexagonal symmetry, and auto-complemented.

Encoded by non-intersecting lattice paths



Using Lindström Gessel Viennot:

$$B_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

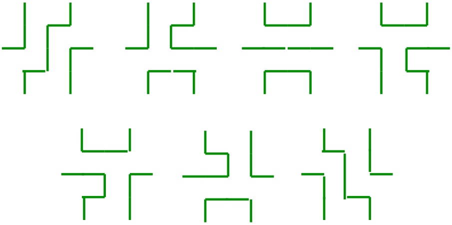
$B_n = 1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7436 \quad 218348$

$$B_n = A_n$$

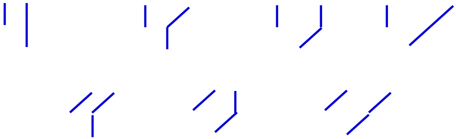
Bijective proof?

n=3

Fully packed loops



Nonintersecting paths



Magog Triangles

$$
\begin{array}{cccccccccc}
1 & & 1 & & 2 & & 2 & & 4 & \leq 5 \\
 & 1 & & 1 & & 2 & & 3 & & \leq 4 \\
 & \searrow & 1 & & 1 & & 3 & \nearrow & & \leq 3 \\
 & & & 1 & & 2 & & & & \leq 2 \\
 & & & & 1 & & & & & \leq 1
\end{array}
$$

Encode TSSCPPs or NIPs.

Problem: find an explicit bijection between Gog and Magog triangles.

$$
\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 2 & & 3 & & 5 & \\
& \searrow & 2 & & 3 & & 4 & \nearrow & \\
& & & 2 & & 3 & & & \\
& & & & 3 & & & &
\end{array}
$$

Gog

$$
\begin{array}{cccccccccc}
1 & & 1 & & 2 & & 2 & & 4 & \leq 5 \\
& 1 & & 1 & & 2 & & 3 & & \leq 4 \\
& \searrow & 1 & & 1 & & 3 & \nearrow & & \leq 3 \\
& & & 1 & & 2 & & & & \leq 2 \\
& & & & 1 & & & & & \leq 1
\end{array}
$$

Magog

1 2 3 1 2 3 1 2 3 1 2 3
1 2 1 2 1 3 1 3
1 2 1 2

1 2 3 1 2 3 1 2 3
1 3 2 3 2 3
3 2 3

Gog

1 1 1 1 1 2 1 1 2 1 1 3
1 1 1 1 1 2 1 1
1 1 1 1

1 1 3 1 2 2 1 2 3
1 2 1 2 1 2
1 1 1

Magog

Gog and Magog Trapezoids.

the k right SW-NE diagonals of a Gog (or Magog) triangle

		3		4		5
	2		3		5	
2		3		4		
	2		3			
		3				

		2		2		4
	1		2		3	
1		1		3		
	1		2			
		1				

Size $(5, 3)$.

Zeilberger (1995): for each (n, k) Gog and Magog trapezoids are equienumerated.

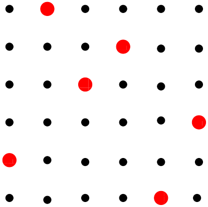
Approach to the bijection problem

Based on

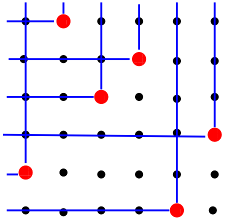
1 *inversions*.

2 *Schützenberger involution*.

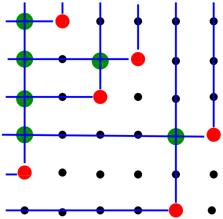
Inversions of permutation



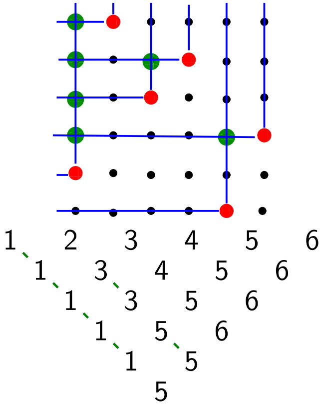
Inversions of a permutation



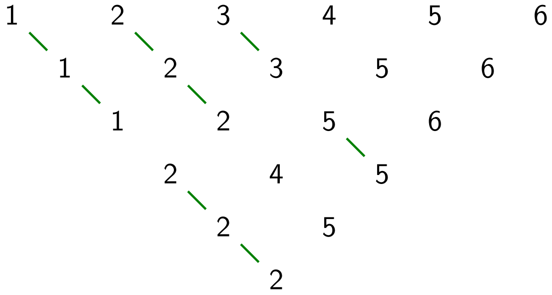
Inversions of a permutation



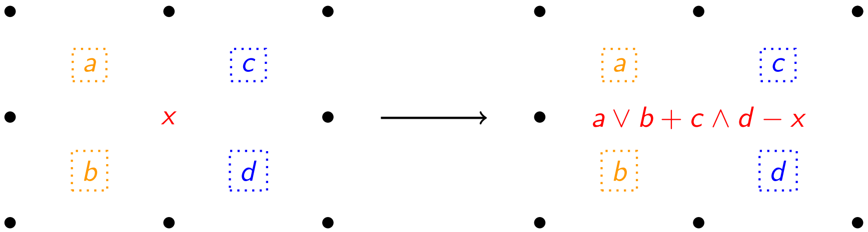
Inversions of a permutation



Inversions of a Gog triangle



Schützenberger involution



Apply to row i

$$
\begin{array}{ccccccccc}
1 & & 1 & & 2 & & 2 & & 4 \\
& 1 & & 1 & & 2 & & 3 & \\
& & 1 & & 1 & & 3 & & \\
\text{row } 2 \quad \rightarrow & & & 1 & & 2 & & & \\
& & & & 1 & & & &
\end{array}
$$

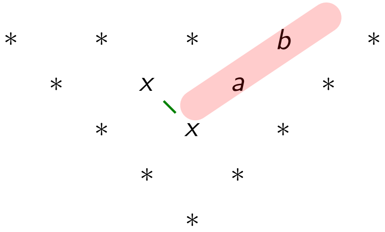
gives s_i

Schützenberger involution is

$$S = s_1 s_2 s_3 \dots s_{n-1} s_1 s_2 \dots s_{n-2} \dots s_1 s_2 s_1$$

Idea for bijection:

- Step 1: for each inversion, substract 1 from NE coefficients



Idea for bijection:

- Step 1: for each inversion, substract 1 from NE coefficients

$$\begin{array}{ccccccccc} * & & * & & * & & b-1 & & * \\ & * & & x & & a-1 & & * & \\ & & * & & x & & * & & \\ & & & * & & * & & & \\ & & & & * & & & & \end{array}$$

Idea for bijection:

- Step 1: for each inversion, substract 1 from NE coefficients

$$\begin{array}{ccccccccc}
* & & * & & * & & b-1 & & * \\
 & * & & x & & a-1 & & * & \\
 & & * & & x & & * & & \\
 & & & * & & * & & & \\
 & & & & * & & & &
\end{array}$$

- Step 2: apply Schützenberger

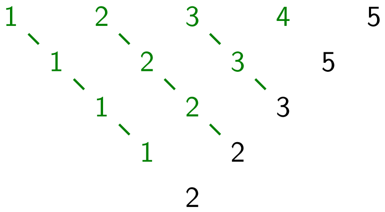
Works for $(n, 1)$ trapezoids

$$\begin{matrix} & & & & 5 \\ & & & 5 & \\ & & 3 & & \\ & 2 & & & \\ 2 & & & & \end{matrix}$$

Works for $(n, 1)$ trapezoids

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 2 & & 3 & & 5 & \\
& & 1 & & 2 & & 3 & & \\
& & & 1 & & 2 & & & \\
& & & & 2 & & & &
\end{array}$$

Works for $(n, 1)$ trapezoids



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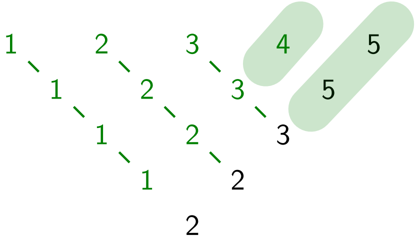

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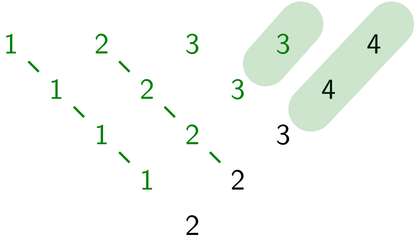

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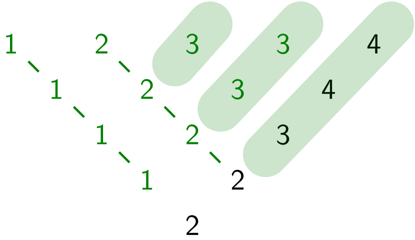

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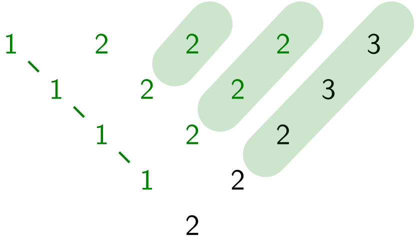
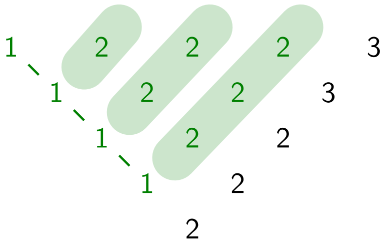
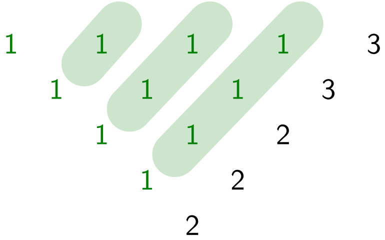


Works for $(n, 1)$ trapezoids



Works for $(n, 1)$ trapezoids



Works for $(n, 1)$ trapezoids

1		1		1		1		3
	1		1		1		3	
		1		1		2		
			1		2			
				2				

Works for $(n, 1)$ trapezoids

$$
\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 3 \\
& 1 & & 1 & & 1 & & 2 & \\
& & 1 & & 1 & & 2 & & \\
& & & 1 & & 2 & & & \\
& & & & 1 & & & &
\end{array}
$$

Can be modified for $(n, 2)$ trapezoids.

Hayat Cheballah, P.B. Gog and Magog Triangles, and the Schützenberger Involution Séminaire Lotharingien de Combinatoire, [B66d] (2012)

GOGAm triangles

Images by Schützenberger of Magog triangles.

Left trapezoids

The k NW-SE left diagonals

```
1     2     3
   1     2     3
      2     3     4     Gog
         2     3
            3

1     1     2
   1     1     2
      1     1     3     GOGAm
         1     3
            2
```

Conjecture:

Gog and GOGAm (n, k) left trapezoids are equienumerated

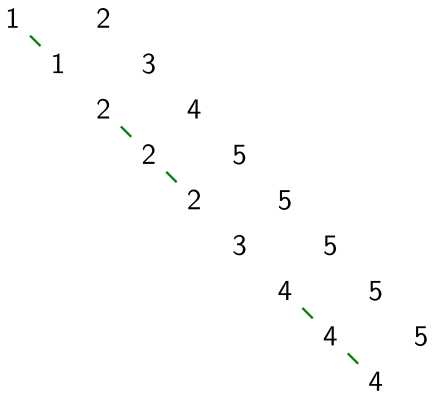
Bijective proof for $k = 1, 2$.

Hayat Cheballah, P.B. Gog , Magog and Schützenberger II: left trapezoids
FPSAC 2013

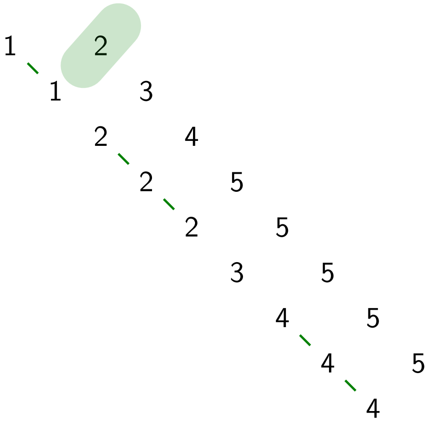
The $(n, 2)$ Gog→GOGAm bijection.

1		2							
	1		3						
		2		4					
			2		5				
				2		5			
					3		5		
						4		5	
							4		5
								4	

The $(n, 2)$ Gog→GOGAm bijection.



The $(n,2)$ Gog→GOGAm bijection.



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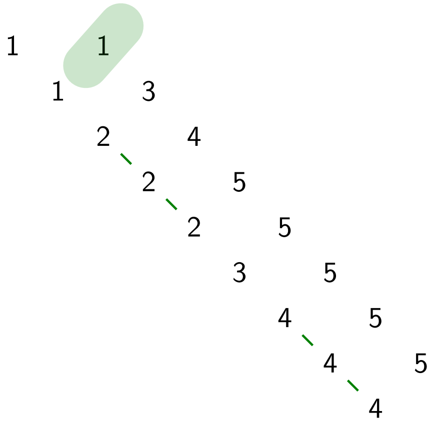

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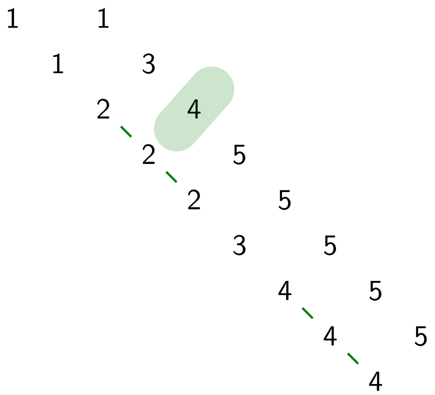

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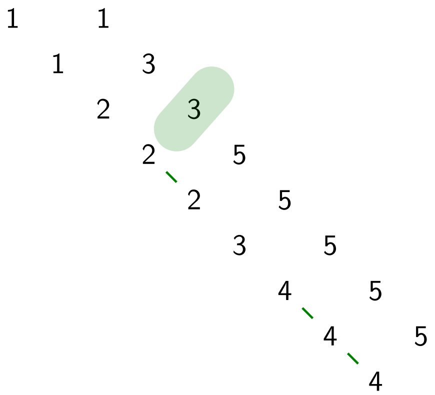

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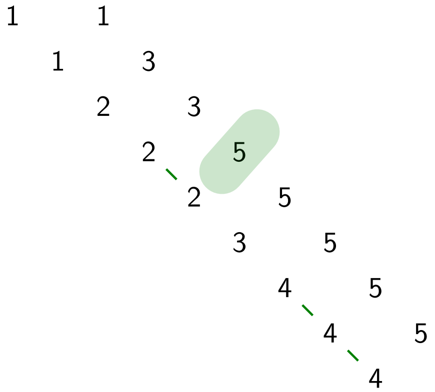

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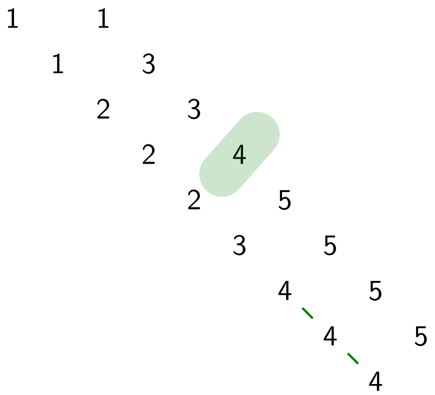
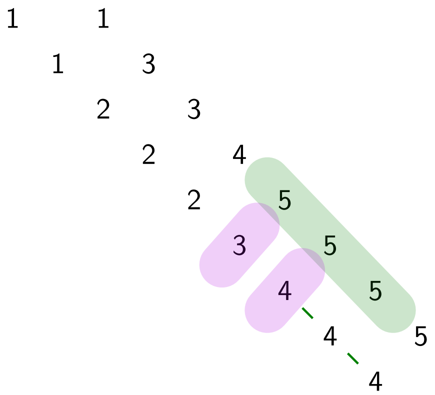
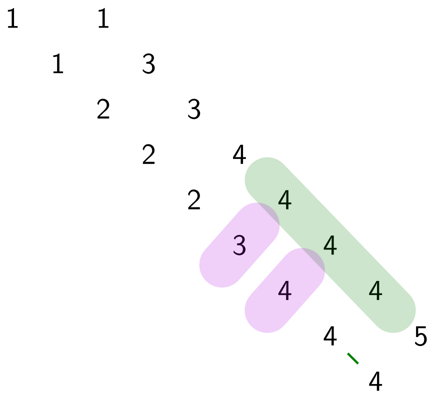


The $(n, 2)$ Gog→GOGAm bijection.



The $(n, 2)$ Gog→GOGAm bijection.



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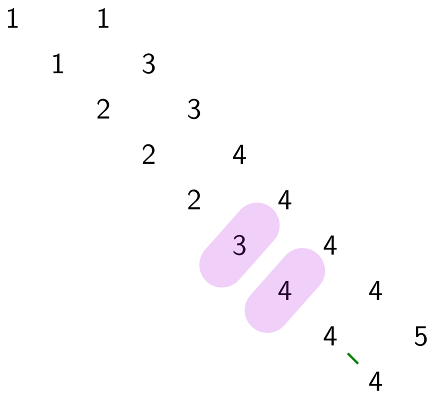

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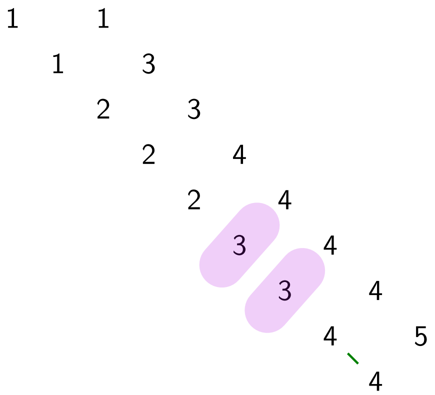
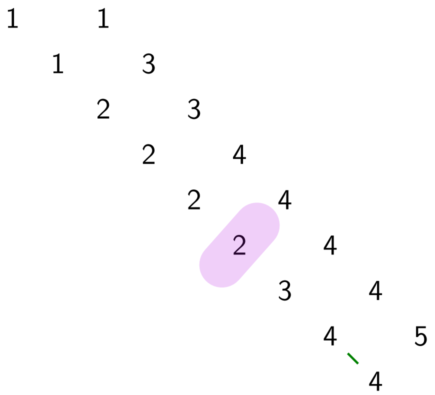


The $(n, 2)$ Gog→GOGAm bijection.

1		1							
	1		3						
		2		3					
			2		4				
				2		4			
					3		4		
						3		4	
							4		5
								4	

The $(n, 2)$ Gog→GOGAm bijection.



The $(n, 2)$ Gog→GOGAm bijection.

1		1							
	1		3						
		2		3					
			2		4				
				2		4			
					2		4		
						3		4	
							4		5
								4	

The $(n, 2)$ Gog→GOGAm bijection.

1		1							
	1		3						
		2		3					
			2		4				
				2		4			
					2		4		
						3		4	
							4		4
								4	

The $(n, 2)$ Gog→GOGAm bijection.

1		1							
	1		3						
		2		3					
			2		4				
				2		4			
					2		4		
						3		4	
							4		4
								4	