Cauchy-Stieltjes kernel families

Wlodek Bryc

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NEF versus CSK families

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- Cauchy-Stieltjes kernel families (CSK):

$$P_\theta(dx) = \frac{1}{L(\theta)} \frac{1}{1 - \theta x} \mu(dx)$$

  μ is a probability measure with support bounded from above.
  The "generic choice" for Θ is $\Theta = (0, \theta_+)$.

A specific example of CSK

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- "Canonical" parametrization: $p = \frac{1}{2-\theta}$
- $Q_p := P_{2-\frac{1}{p}} = (1-p)\delta_0 + p\delta_1, \;\; p \in (0,1)$

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- $Q_p := P_{2-\frac{1}{p}} = (1-p)\delta_0 + p\delta_1, \;\; p \in (0,1)$
- Bernoulli family parameterized by probability of success p.
- $p = \int x Q_p(dx)$ (parametrization by the mean)

Parametrization by the mean

$$m(\theta) = \int x P_\theta(dx) = \begin{cases} \frac{L'(\theta)}{L(\theta)} & \text{NEF} \\ \\ \frac{L(\theta)-1}{\theta L(\theta)} & \text{CSK} \end{cases}$$

- For non-degenerate measure μ, function $\theta \mapsto m(\theta)$ is strictly increasing and has inverse $\theta = \psi(m)$.

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- $\theta \mapsto m(\theta)$ maps $(0, \theta_+)$ onto (m_0, m_+), "the domain of means".
- Parameterizations by the mean:

$$\mathcal{K}(\mu) = \{Q_m(dx) : m \in (m_0, m_+)\}$$

where $Q_m(dx) = P_{\psi(m)}(dx)$

Variance function

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- Variance function $V(m)$ (together with $m_0 = m(0) \in \mathbb{R}$, the mean of μ) determines measure μ uniquely (hence determines CSK uniquely).

Example: a CSK with quadratic variance function

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- The variance function is $V(m) = m(1-m)$
- The generating measure $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ is determined uniquely once we specify its mean $m_0 = 1/2$.
  That is, there is no other μ that would have mean $1/2$ and generate CSK with variance function $V(m)$ that would equal to $m(1-m)$ for all $m \in (1/2-\delta, 1/2+\delta)$

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Morris class. Meixner laws

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- Letac-Mora (1990): cubic $V(m)$
- Various other classes Kokonendji, Letac, ...

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2. *μ is the Marchenko-Pastur (free Poisson) type law iff $V(m) = 1 + am$ with $a \neq 0$*
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4. *μ is the free binomial type law (Kesten law, McKay law) iff $V(m) = 1 + am + bm^2$ with $-1 \leq b < 0$*

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Reproductive properties of NEF and CSK

Theorem (NEF: Jörgensen (1997))

If μ is a probability measure in NEF with variance function $V(m)$, then for $r \in \mathbb{N}$ the r-fold convolution $\mu_r := \mu^{*r}$, is in NEF with variance function $rV(m/r)$.

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If a probability measure μ generates CSK with variance function $V_\mu(m)$, then the free additive convolution power $\mu_r := \mu^{\boxplus r}$ generates the CKS family with variance function $rV_\mu(m/r)$.

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- The domains of means behave differently.
- The ranges of admissible $r \geq 1$ are different.

End now

Pseudo-Variance function for CSK

- The variance

$$V(m) = \frac{1}{L(\psi(m))} \int \frac{(x-m)^2}{1-\psi(m)x} \mu(dx)$$

is undefined if $m_0 = \int x\mu(dx) = -\infty$. (This issue does not arise for NEF)

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$$\mathbb{V}(m) = m\left(\frac{1}{\psi(m)} - m\right) \tag{1}$$

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  where $\psi(\cdot)$ is the inverse of $\theta \mapsto m(\theta) = \int x P_\theta(dx)$ on $(0, \theta_+)$.
- Expression (1) defines a "pseudo-variance" function $\mathbb{V}(m)$ that is well defined for all non-degenerate probability measures μ with support bounded from above.

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Theorem (WB-Hassairi (2011))

Let $\mathbb{V}_\mu$ be a pseudo-variance function of the CSK family generated by a probability measure μ with support bounded from above and mean $-\infty \leq m_0 < \infty$. Then for $m > rm_0$ close enough to rm_0,

$$\mathbb{V}_{\mu^{\boxplus r}}(m) = r\mathbb{V}_\mu(m/r). \tag{2}$$

Example: CKS family with cubic pseudo-variance function

Measure μ generating CSK with $\mathbb{V}(m) = m^3$ has density

$$f(x) = \frac{\sqrt{-1-4x}}{2\pi x^2} 1_{(-\infty,-1/4)}(x) \tag{3}$$

From reproductive property it follows that μ is $1/2$-stable with respect to $\boxplus$, a fact already noted before: [Bercovici and Pata, 1999, page 1054], [Pérez-Abreu and Sakuma, 2008]

$$\left\{ Q_m(dx) = \frac{m^2\sqrt{-1-4x}}{2\pi(m^2+m-x)x^2} 1_{(-\infty,-1/4)}(x)dx : m \in (-\infty, m_+) \right\}$$

What is m_+?

End now

25 min?

Domain of means: $\{Q_m : m \in (m_0, m_+)\}$

For $\mathbb{V}(m) = m^3$ the domain of means is $(-\infty, m_+)$, where:

1. $\theta \mapsto m(\theta)$ is increasing, so $m_+ = \lim_{\theta \nearrow \theta_{\max}} m(\theta)$. This gives $m_+ = -1$

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2. $\frac{1}{1-\theta x} 1_{(-\infty,-1/4)}(x)$ is positive for $\theta \in (0, \infty) \cup (-\infty, -4)$. The domain of means can be extended to $\mathbf{m}_+ = \lim_{\theta \nearrow -4} m(\theta)$. This extends the domain of means up to $\mathbf{m}_+ = -1/2$

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- But $\int Q_m(dx) < 1$ for $m > 1/2$.
- $Q_m(dx) = \frac{m^2}{(m^2+m-x)} \mu(dx) + \frac{(1+2m)_+}{(m+1)^2} \delta_{m+m^2}$ is well defined and parameterized by the mean for all $m \in (-\infty, \infty)$.

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Kernels $e^{\theta x}$ and $1/(1-\theta x)$ generate NEF and CSK families

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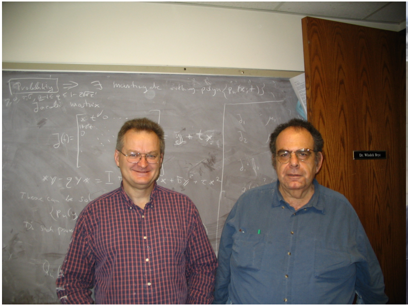
- The generating measure of a NEF is not unique.
- A CSK family in parameterizations by the mean may be well defined beyond the "domain of means"
- For CSK family, the variance function may be undefined. Instead of the variance function [Bryc and Hassairi, 2011] look at the "pseudo-variance" function $m \mapsto mV(m)/(m-m_0)$ which is well defined for more measures μ.



Thank you

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Back

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