# Part IV: Storage

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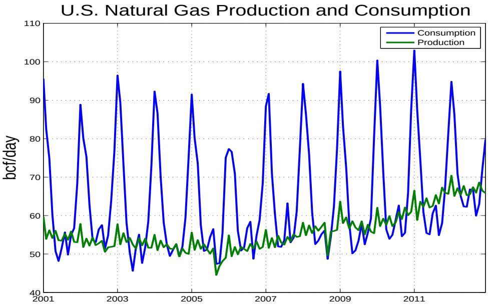
# Outline

- Physical Storage
- Virtual Storage
- Physical Storage
  - Calculating Instrinsic Value
  - General Valuation and Dynamic Programming
  - Alternative Methods
- Practical Considerations
- Structual Models—A Start

# Physical Storage

## Context

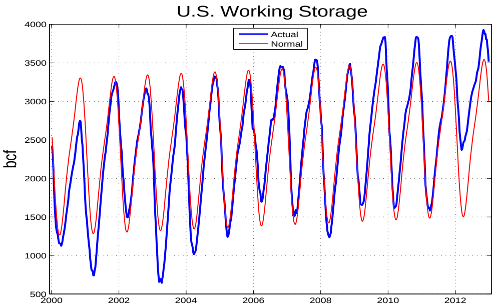
- Storage facilities resolve temporal imbalances between supply and demand.
  - Storage is to time as transportation/transmission is to pace.
- Recall: In 2012 the U.S. had roughly 4.2 tcf of storage capacity with annual consumption of approximately 25 tcf per year.



# Physical Storage

## Context

- Historically available storage capacity has been sufficient to resolve the seasonal variations in demand and the structural changes in supply.
  - Recall the historical inventory and the "normal" storage level obtained by Fourier + drift.



# Physical Storage

## Types of Storage

- Types of storage:
  - Aquifer storage—facilities consisting of large porous rock structures.
  - Reservoir storage—depleted natural gas or oil production fields suitably modified for natural gas storage.
  - Salt caverns—salt dome formations that can be engineered to store natural gas.
- There is actually a total of roughly 8.7 tcf of natural gas storage.
  - Of this roughly 4.5 tcf is "base gas"—gas that must be injected once and is required to maintain working pressure.
  - The remaining inventory is referred to as "working storage".

# Physical Storage

## Constraints

- Operational constraints:
  - Capacity constraints: Ensure that enough capacity exists to accommodate a storage contract.
    - This involves specifying bounds on contracted capacity: $S(t) \in [0, S_{\max}]$ where $S(t)$ denotes the inventory level at time $t$.
  - Rate constraints: Ensure that contracted injection and withdrawal rates can be achieved.
    - This is accomplished by constraints on the rates: $s(t) \in [s_*, s^*]$ where $s(t) \equiv \frac{dS}{dt}$ is the injection rate with e $s_* < 0$ and $s^* > 0$.
    - In practice, as inventory approaches the bounds 0 or $S_{\max}$, these constraints can change—as the tank gets full (empty) putting more in (taking more out) becomes harder. Such constraints take the form: $s_* = s_*(S(t))$ and $s^* = s^*(S(t))$.

# Physical Storage

## Constraints

- Operational constraints:
  - Inventory constraints: All storage contracts specify initial and final conditions.
    - The most common structure: $S(T) = S(0)$ where $T$ denotes the term of the contract.
    - Often the contract spans the start of the injection season 01Apr and terminates at the end of the withdrawal season 31Mar of the following year.
    - Some structures constrain inventory to specified inequalities at intermediate times: $S_{\vec{\tau}_*} \leq \vec{L}_*$ and $S_{\vec{\tau}^*} \geq \vec{L}^*$ for specific times $\vec{\tau}$ and limits $\vec{L}$.

# Physical Storage

## Constraints

- Cyclability:
  - The number of times it can fill and then empty, in the parlance "cycle" or "turn", in one year.
    - In the case of constant limits, one cycle time is:

$$\tau = \frac{S_{\max}}{|s_*|} + \frac{S_{\max}}{s^*}$$

    - The number of turns is $1/\tau$.
  - Most aquifer and reservoir storage is effectively "one turn storage", that is one cycle per year.
  - Salt cavern storage is much more flexible with some facilities able to turn several times a year.

# Virtual Storage

## CSOs

- The simplest storage structure is a modified CSO.
  - Consider options with the following payoff:

    $$\max\left[F(\tau, T+U) - e^{rU}F(\tau, T), 0\right]$$

    where we are assuming constant interest rates.
  - The owner of this option has the ability to:
    - Purchase natural gas at exercise time $\tau$ for delivery at time $T$ at price $F(\tau, T)$;
    - Funding the cost to time $T+U$;
    - Selling the same quantity forward at time $T+U$.

# Virtual Storage

## CSOs

- Exercise of CSO:
  - The holder of such an option will only exercise if the accrued cost of the purchase is less than the forward price at withdrawal

$$y(\tau, T, T+U) = \frac{1}{U} \log \left[ \frac{F(\tau, T+U)}{F(\tau, T)} \right] > r$$

- Storage value is driven by the difference between forward yields and financing costs.

# Virtual Storage

## CSOs

- Consider a storage facility where the only constraint is $S(t) \in [0, S_{\max}]$.
  - There are no constraints on rates.
  - This type of structure is sometimes referred to as virtual storage and is clearly an idealization—the owner can toggle between empty and full instantaneously.
  - One purpose for considering virtual storage is that it is analytically tractable.
  - The value function does not depend on $S_t$ since this can be changed instantly.

# Virtual Storage

## CSOs

- Virtual storage is the sum of nearest-neighbor CSOs:

$$V\left[T_n, F(T_n, \cdot)\right] = S_{\max}\tilde{E}_{T_n}\left[\sum_{m=n}^{N_*} e^{(T_m - T_n)r} \max\left[F(T_m, T_{m+1}) - e^{r\Delta t}F(T_m, T_m), 0\right]\right]$$

  where the problem is defined on the time grid $T \in \{n\Delta T\}_{n=1}^{N_*}$.

- Evaluation of virtual storage is equivalent to CSO valuation.
- We will work under the Gaussian exponential framework:

$$dF(t, T) = F(t, T)\sum_{j=1}^{J} \sigma_j(T) e^{-\beta_j(T-t)} dB_t^{(j)}$$

# Virtual Storage

## Valuation

- Valuation proceeds in the usual CSO fashion:
  - The $m^{th}$ option value is:

$$V_{n,m} = e^{-(T_m - T_n)r}\left[F(T_n, T_{m+1})\Phi(d_1) - e^{r\Delta T}F(T_n, T_m)\Phi(d_2)\right]$$

where

$$d_{1,2} = \frac{\log\left[\frac{F(T_n,T_{m+1})}{e^{r\Delta T}F(T_n,T_m)} \pm \frac{1}{2}\tilde{\sigma}^2\right]}{\tilde{\sigma}}$$

where:

$$\tilde{\sigma}^2 = (T_m - T_n)\left[\bar{\sigma}^2_{m,m+1} - 2\rho_{m,m+1}\bar{\sigma}^2_{m,m}\bar{\sigma}^2_{m,m+1} + \bar{\sigma}^2_{m,m}\right]$$

  - $\bar{\sigma}^2_{m,m}$ and $\bar{\sigma}^2_{m,m+1}$ are the term volatilities of contracts $T_m$ and $T_{m+1}$ over the time interval $[0, T_m]$ respectively;
  - $\rho_{m,m+1}$ is the term correlation between the two contracts over the same interval.

# Virtual Storage

## Valuation

- Taking the continuous time limit ($\Delta T \to 0$) yields:

$$V(t, \vec{Y}_t) = \int_t^{T_*} e^{-r(T-t)} F(t,T) v(t,T) \left[ h(t,T)\Phi(h(t,T)) + \phi(h(t,T)) \right] dT$$

where:

- The variance term is:

$$v^2(t,T) = \sum_j \left[ \beta_j - \frac{\sigma_j'}{\sigma_j}(T) \right]^2 \frac{\sigma_j^2}{2\beta_j} \left[ 1 - e^{-2\beta_j(T-t)} \right]$$

- The "carry" term is:

$$h(t,T) = \frac{1}{v(t,T)} \left[ \frac{\partial \log F(t,U)}{\partial U} \Big|_{U=T} - r \right]$$

- Note that $\vec{Y}_t$ are the state-variables—the OU processes driving the forward curve.

# Virtual Storage

## Valuation

- The previous formula is the closest to basic Black that you will find in storage valuation.
- The valuation formula permits fast exploration of basic attributes.
- Example: Intrinsic value:
  - Intrinsic value is generally not simple to calculate.
  - Here the result can obtained directly.
  - The optimal inventory path in the zero-vol case is:

$$S_{\text{Intrinsic}}(t, T) = \begin{cases} S_{\max} & \text{if } \left.\frac{\partial \log F(t,U)}{\partial U}\right|_{U=T} - r > 0 \\ 0 & \text{otherwise} \end{cases}$$

  - This yields an intrinsic value of:

$$\int_t^{T_*} S_{\max} \left[ \left.\frac{\partial \log F(t,U)}{\partial U}\right|_{U=T} - r \right]^+ dT$$

# Virtual Storage

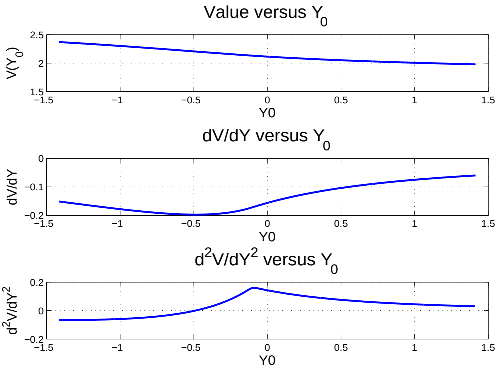
## Valuation

- Simple case in which $V(t, \bar{Y}(t))$ does not depend on $t$ is the infinite horizon setting with initial forward curve of exponential form.
  - $T_* = \infty$
  - $F(0, T) = F_\infty e^{\sum_j e^{-\beta_j T} Y_j(0)}$.
- The next slide shows results in the one-factor setting:
  - The top figure is the value versus $Y(0)$.
  - Negative values of $Y(0)$ correspond to contango—the value function must be a decreasing function of $Y(0)$ in order to be an increasing function of the forward yield.
  - The middle figure shows the change in value with respect to $Y(0)$; as this is a single factor model the plot shows $\Delta_Y = \frac{\partial V}{\partial Y}$ versus $Y(0)$.
  - The bottom plot of $\Gamma = \frac{\partial^2 V}{\partial Y^2}$ versus $Y(0)$ illustrates an important point: As with vanilla options, the convexity of the structure decreases at extreme values of backwardation or contango.

# Virtual Storage

## Valuation

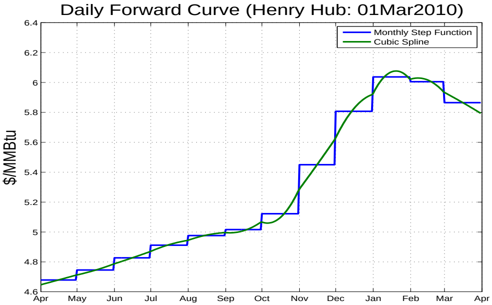
- Results in the one-factor case:



# Virtual Storage

## Daily Forward Curves

- To value natural gas storage for a "high-turn" facility, daily forward curve construction is required.
  - Using a step function for each month results in massive (an unrealistic) value at the monthly boundaries.
  - The daily forward construction must be consistent with traded monthly forwards—as shown in the figure below.



# Virtual Storage

## Daily Forward Curves

- This interpolate was constructed by calculating the cumulative forward value of the commodity:

$$C(\bar{N}_M) = \sum_{m=1}^{M} N_m F_m$$

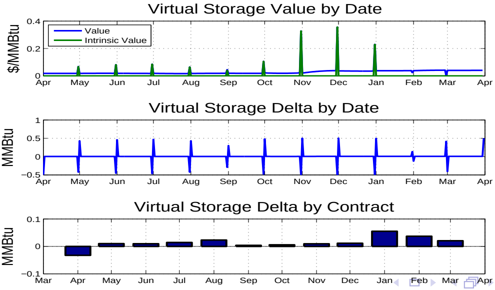
  - $N_m$ is the number of days in month $m$;
  - $F_m$ is the month forward;
  - $\bar{N}_M = \sum_{m=1}^{M} N_m$ is the number of delivery days through month $m$.
  - T $\{\bar{N}_m, C(T_m)\}_{m=1}^{M_{\max}}$ was interpolated using a cubic spline to obtain $C(n)$ for all delivery days $n \in [1, N_{\max}]$.
  - $C(\cdot)$ was then differenced to yield a daily forward price $F(0, n)$.
  - Consistency with the monthly forwards is guaranteed:

$$N_m F_m = C(N_m) - C(N_{m-1}) = \sum_{d \in m} F(0, d)$$

# Virtual Storage

## Valuation

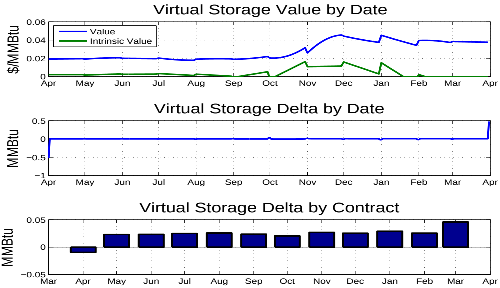
- Working problem: Value a virtual storage contract from 01Apr2010 to 31Mar2011 as of pricing date 01Mar2010.
- As default parameters we will use: $\vec{\beta} = [0.30, 40]$.
- Effect of daily forward curves:
  - This plot shows the value and $\Delta$'s using the step function.
  - Note the effects of discontinuities in the forward curve of value and exposures at the daily level.



# Virtual Storage

## Valuation

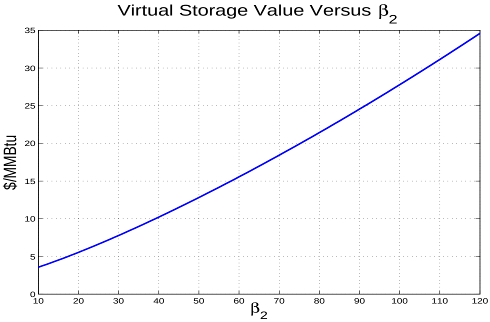
- Effect of daily forward curves:
  - This plot shows the value and $\Delta$s using the cubic spline method.
  - The results are more reasonable.
  - However, the fact that a cubic spline has continuity up to only the first derivative is clear in the kinks in value and $\Delta$.



# Virtual Storage

## Valuation

- Effect of mean-reversion rates:
  - This plot shows the value as a function of $\beta_2$ holding $\beta_1$ constant.
  - The higher the mean-reversion rate the more value due to the higher value of $\sigma_2$ required to calibrate to daily vols.



# Physical Storage: Valuation

## The General Problem

- Optimization must now be phrased in terms of maximizing the expected value of futures cash flows as one injects (spends cash) or withdraws (receives cash).
- Formally, the valuation problem is:

$$V[0, S(0), F(0,\cdot)] = \sup_{s(\cdot)\in\mathcal{A}} \tilde{E}\left\{\int_0^{T_*} d(0,t)[-s(t)F(t,t) - \kappa(s(t), S(t), F(t,t))]dt\right\}$$

where:

- $d()$ and $F()$ are the discount factor and forward curve respectively.
- $S(t)$ is the current inventory level and $s(t) = S'(t)$.
- $\kappa$ denotes costs associated with injection and withdrawal. For example, a cost that is a fraction of the fuel charge would take the form $\kappa = k|s(t)|F(t,t)$.
- $\mathcal{A}$ denotes allowed controls (for example: $0 \leq S(t) \leq S_{\max}$).

# Physical Storage: Valuation

## The General Problem

- The constraints $\mathcal{A}$ on $s(t)$ and $S(t)$ are what make the valuation problem challenging.
  - In what follows we will work with the following constraints:
    - $0 \leq S(t) \leq S_{\max}$
    - $s_* \leq s(t) \leq s^*$ with constant bounds.
  - The value function now depends on $S_t$.
    - Rephrased, your decision to change inventory by $s(t)dt$ effects the future value of the structure.
    - Computationally this adds the $S$-dimension to the problem rendering numerical analysis more challenging.
  - In general the state-space is of very high dimension due to the forward price curve: $V(t, S_t, F(t, \cdot))$
    - Most approaches are predicated on the Gaussian exponential framework which reduce this space of $\bar{Y}_t$.
    - Even two-factor implementations are computationally challenging.

# Physical Storage: Valuation

## Intrinsic Value

- The first step in storage valuation is establishing intrinsic value.
  - Net of hedging costs this functions practically as a lower bound on a bid.
- Intrinsic value is the zero-volatility solution; the value that can be captured by a single static hedge placed at the time of pricing.
  - Since there is no randomness, the value function takes the form: $V_I = V_I\left[S(0)\right]$ with implicit dependence on the forward curve $F(0, T)$.
- There are two approaches to calculating intrinsic value
  - Reduction of the problem to a linear programming task (when possible);
  - Dynamic progamming which is more generally applicable but much slower.

# Physical Storage: Valuation

## Intrinsic Value: Linear Programming

- Linear progamming can be applied if the cost function is independent of inventory level $S$ and linear in the absolute rate:

$$\kappa = \begin{cases} -k_* s(t) F(t,t) & \text{if } s(t) < 0 \\ k^* s(t) F(t,t) & \text{if } s(t) > 0. \end{cases}$$

- The intrinsic value $V_I\left[S(0)\right]$ is the maximum value over portfolios of forward spreads between all pairs of delivery days $(i,j)$, subject to the constraints of the storage facility:

$$V_I\left[S(0)\right] = \sup_{\substack{\vec{v}\in\mathcal{A} \\ \vec{v}\geq 0}} \sum_{i,j} v_{i,j} \left[ e^{-rT_j} F_j - e^{-rT_i} F_i - K_{i,j} \right]$$

  - The portfolio is defined by $\vec{v}$, with $v_{i,j}$ the notional of the $(i,j)$ spread.
  - $K_{i,j}$ denotes the discounted cost per unit notional of injection at time $T_i$ and withdrawal at time $T_j$.

- By considering all pairs $(i,j)$ with $\bar{v} \geq 0$ the problem is amenable to linear programming methods.

# Physical Storage: Valuation

## Intrinsic Value: Dynamic Programming

- Dynamic progamming is very general but more challenging.
- The idea: In any possible state $[t, S(t)]$ one should take the best possible action.
  - We will work on the discrete time grid:
  $$t \in \mathcal{T} \equiv [0, \Delta t, 2\Delta t, \ldots, T_*]$$
  - Notation:
  $$t_n = n\Delta t \qquad \text{and} \qquad p_n = F(0, t_n)$$
  - We will use typical terminal condition for a storage contract: $V(T_*, S) \equiv 0$ for all $S$ (you lose whatever inventory you injudiciously left in the facility at the end of the contract)

# Physical Storage: Valuation

## Intrinsic Value: Dynamic Programming

- With the final condition as the "starting point" for a back-propagation solution, the single-step optimization at each state is:

$$V\left(t_n, S\right) = \max_{x \in \mathcal{A}_{\Delta t}} \left[ c\left(t_n, x, S, p_n\right) + e^{-r\Delta t} V\left(t_{n+1}, S + x\right) \right]$$

  where

$$c(t, x, S, p) = -xp - \kappa\left(x, S, p\right)$$

  - We have accounted for the fact that the allowed set of transitions depends on $\Delta t$ using the notation $\mathcal{A}_{\Delta t}$. For example, given $\Delta t$ the rate constraint becomes: $s_* \Delta t \leq x \leq s^* \Delta t$.

- This optimization is nothing more than:

  "Make the most money you can (inclusive of future value) given the constraints on your actions."

# Physical Storage: Valuation

## Intrinsic Value: Dynamic Programming

- Dynamic programming problems are usually solved on a discretized state-space for storage:

$$S \in \mathcal{S} \equiv [0, \Delta S, 2\Delta S, \ldots, S_{\max}]$$

- Optimization becomes:

$$V\left(t_n, S_k\right) = \max_{\hat{k} \in \mathcal{A}_{\Delta t, \Delta S}} \left[ -(\hat{k} - k)\Delta S p_n - \kappa\left((\hat{k} - k)\Delta S, S_k, p_n\right) + e^{-r\Delta t} V\left(t_{n+1}, S_{\hat{k}}\right) \right]$$

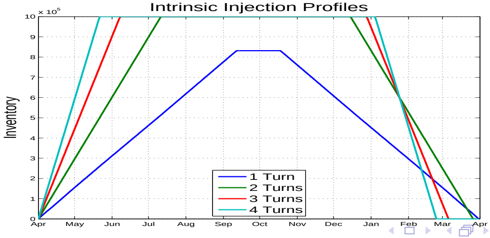
  - Iterating yields the value function $V(t_n, S_k)$ and the optimal control at each state: $k^*(n, k)$
  - Starting at an initial storage level $S_0 = k_0 \Delta S$, the optimal injection and withdrawal path is then obtained by iterating forward in time:

$$k_{n+1} = k^*\left[n, k_n\right]$$

# Physical Storage: Valuation

## Intrinsic Value

- Continuing with our working problem using:
  - $S_{\max} = 10^6$ MMBtu ( 1bcf)
  - $[s_*, s^*] = [-\alpha, \alpha] * S_{\max}$
  - $\kappa(s(t), S_u, F(t,t)) = 0.01F(t,t)$
- Below is the optimal instrinsic inventory trajectory indexed by turns.
  - Turn number $N$ and $\alpha$ are related via: $\alpha(N) = 2S_{\max}N$.
  - Note the anticipated change in profile with change in injection rate, ultimately yielding profiles that do not utilize the full capacity.



# Physical Storage: Valuation

General Valuation

- The dynamic programming approach is easily extended to the fully stochastic problem in which forward curves are driven by discrete set of random factors.
- In the setting of the Gaussian factor models this becomes:

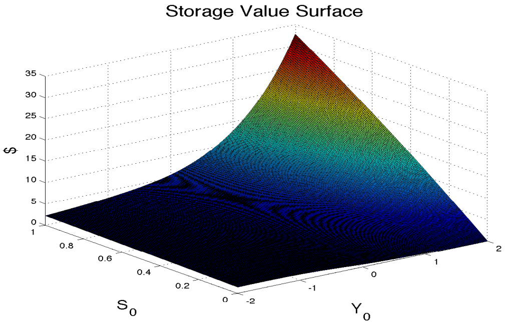
$$V\left(t_n, S, \vec{Y}\right) = \max_{x \in \mathcal{A}_{\Delta t}} \left( c\left(t_n, x, S, p_n(\vec{Y})\right) + e^{-r\Delta t} E\left[ V\left(t_{n+1}, S + x, \vec{Y}_{t_{n+1}}\right) \middle| \mathcal{F}_{t_n} \right] \right)$$

- Now the interpretation is make the most "expected money" given the state you are in.
- This is solved on discrete lattices (usually) and can be very time consuming.
- In the one-factor results that follow the lattice-size was roughly $6 \times 10^6$ (Time x Storage x Y).

# Physical Storage: Valuation

## General Valuation

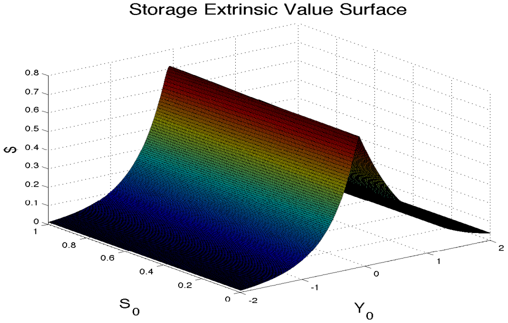
- Parameters:
  - For any $\beta$ we will fix the stationary variance at $\frac{\sigma^2}{2\beta} \equiv C = .5$.
  - Default paramters $r = .05$ and $\beta = 1$.
- The following figure shows $V(S_0, Y_0)$.
  - Value increases with $Y_0$—more backwardation means that you can sell inventory at higher prices.
  - Value increases with $S_0$—more inventory is better than less.



# Physical Storage: Valuation

## General Valuation

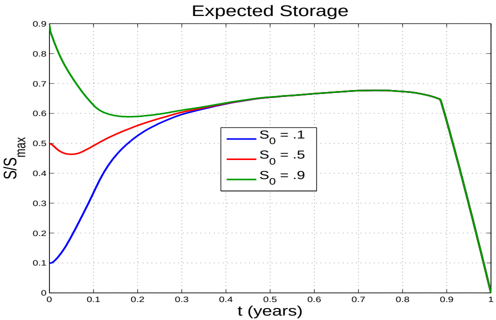
- A more interest picture is afforded by purely the extrinsic value.
  - Extrinsic value is highest for values of $Y_0$ near zero—when the forward curve is flat.
  - Extrinsic value decreases (slightly) with $S_0$.



# Physical Storage: Valuation

## General Valuation

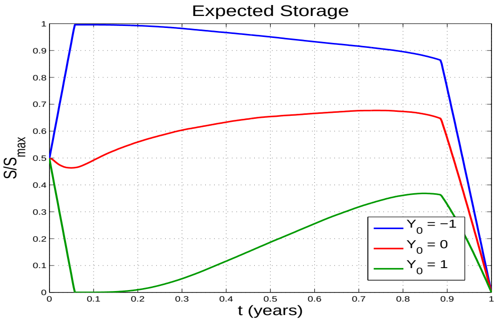
- This plot shows expected storage levels varying $S_0$ with $Y_0 = 0$



# Physical Storage: Valuation

## General Valuation

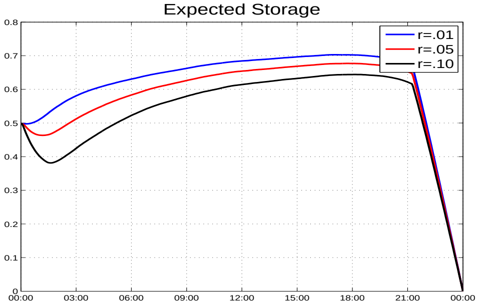
- This plot shows expected storage levels varying $Y_0$ with $S_0 = 0.5$.



# Physical Storage: Valuation

## General Valuation

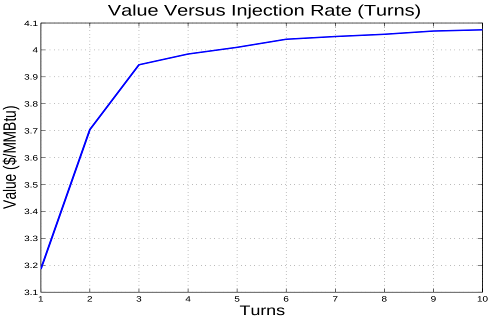
- This plot shows expected storage levels varying interest rates for $Y_0 = 0$ with $S_0 = 0.5$.
- The higher the rate, the more expensive to fund inventory and the lower the expected inventory level.



# Physical Storage: Valuation

## General Valuation

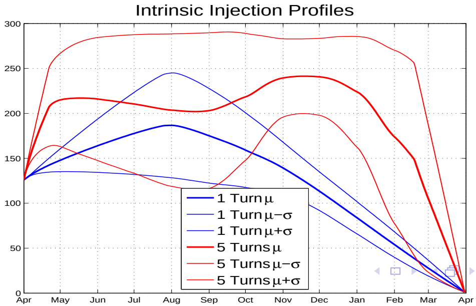
- The value of storage versus turns exhibits a similar profile as was seen for intrinsic value.
- The asymptotic value is the virtual storage value.



# Physical Storage: Valuation

## General Valuation

- Distributional features vary with the flexibility of the facility.
- This shows the expected inventory and one standard deviation envelopes around it for one and five turn storage.
  - The drop in expected inventory in the July time-frame for the high-turn facility, shows that it is able to exploit relatively nuanced changes in forward yields.
  - The standard deviation of the inventory is meaningfully higher for the high turn facility—manifestation of the greater response to local fluctuations in spot forward yields.



# Physical Storage: Alternative Methods

## Numerical challenges

- The results above were in a one-factor setting.
- Even in a two-factor the numerical challenges are daunting.
- Moreover, it is not clear just how realistic the multi-factor Gaussian exponential framework is (more on this later).
- These considerations have spawned a variety of approaches to approximate methods of valuation, such as:
  - Longstaff-Schwartz type methods.[1]
  - Rolling Intrinsic[2]
  - CSO subordination

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[1] See F.A. Longstaff and E.S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach", The Review of Financial Studies, 2001 and A. Boogert and C. de Jong, "Gas Storage Valuation Using a Monte Carlo Method", The Journal of Derivatives (2008)

[2] See, for example, Breslin, Clewlow, Elbert, Kwok and Strickland, "Gas Storage: Overview and Static Valuation", Energy Risk (2008)

# Physical Storage: Alternative Methods

## Rolling Intrinsic

- The approach is simulation-based and very robust.
- The algorithm:
  - Consider a storage transaction spanning $[T_*, T^*]$.
  - The intrinsic value of the storage transaction at time $t = 0 < T_*$ is:

$$V_I(0) = \max_{w \in \mathcal{A}} \int_{T_*}^{T^*} e^{-rT} w(T) F(0, T) dT$$

    with the minimizing volumes $w_0^*(\cdot)$.
  - Note: $w(T) > 0$ corresponds to a forward *sale* of natural gas at time $T$.
  - The actual forward hedge required to "lock in" intrinsic value is $-w_0^*(T)$ for delivery time $T$.

# Physical Storage: Alternative Methods

## Rolling Intrinsic

- The algorithm: (cont)
  - At a later time $t > 0$, the same calculation results, in general, in a different solution $w_t^*(T)$ with value $V_I(t)$.
  - If $V_I(t) > V_I(0)$, then it makes sense to rebalance the IV hedge, extracting the difference:

$$\int_{T_*}^{T^*} e^{-r(T-t)} \left[w_t^*(T) - w_0^*(T)\right] F(t, T) dT$$

  - Note that when $t$ exceeds the first flow date $T_*$ of the transaction, the current intrinsic value refers only to the forward (unrealized) value.

# Physical Storage: Alternative Methods

## CSO Subordinators

- Calendar spreads were useful for computing intrinsic value.
- In a similar train of thought CSO's could be used to establish bounds on storage value.
  - Denote the present value of the forwards spreads and CSOs as:

$$\begin{aligned} S_{i,j} &= e^{-rT_j}F_j - e^{-rT_i}F_i - K_{i,j} \\ C_{i,j} &= \tilde{E}\left[\left(e^{-rT_j}F_j - e^{-rT_i}F_i - K_{i,j}\right)^+\right] \\ P_{i,j} &= \tilde{E}\left[\left(e^{-rT_i}F_i - e^{-rT_j}F_j - K_{j,i}\right)^+\right] \end{aligned} \tag{1}$$

    assuming that $i < j$.
  - A lower bound on the value of a storage asset is:

$$V\left[S(0), F(0,\cdot)\right] = \sup_{\substack{\vec{v}\in\mathcal{A} \\ \vec{\alpha},\vec{\beta},\vec{v}\geq 0}} \left( \sum_{1\leq i<j\leq J} \left[\alpha_{i,j}C_{i,j} + \beta_{i,j}P_{i,j}\right] + \sum_{1\leq i<j\leq J} v_{i,j}S_{i,j} \right)$$

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CSO Subordinators

- The hard part is define $\mathcal{A}$—the allowed set of spread options.
- While each option may or may not be exercised, it must be assumed to be exercised in order to ensure that the constraints are satisfied under all realizations of forward prices and all possible option exercise events.
- The set of constraints is lengthy but constructable and the optimization problem is amenable to linear programming **once you know the values of the pairwise CSOs**.

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## CSO Subordinators

- While compelling in simplicity the solution can change suddenly.
  - Condensing our notation above into the values of the tradables involved and the parameters over which optimization occurs:

$$\vec{\pi} \equiv \left[\vec{\alpha}, \vec{\beta}, \vec{v}\right] \qquad \vec{\mathcal{V}} \equiv \left[\vec{C}, \vec{P}, \vec{S}\right]$$

    the optimal holding $\vec{\pi}_* \left[F(t,\cdot)\right]$ implies a lower bound on the value:

$$V_* \left[F(t,\cdot)\right] \equiv \vec{\pi}_*^\dagger \left[F(t,\cdot)\right] \vec{\mathcal{V}} \left[F(t,\cdot)\right]$$

  - Many implementations calculate Greeks on the static portfolio:

$$\frac{\partial V_*}{\partial F(t,T)} = \vec{\pi}_*^\dagger \left[F(t,\cdot)\right] \frac{\partial \vec{\mathcal{V}}}{\partial F(t,T)}$$

  - The proper calculation should be:

$$\frac{\partial V_*}{\partial F(t,T)} = \vec{\pi}_*^\dagger \left[F(t,\cdot)\right] \frac{\partial \vec{\mathcal{V}}}{\partial F(t,T)} + \frac{\partial \vec{\pi}_*}{\partial F(t,T)}^\dagger \left[F(t,\cdot)\right] \vec{\mathcal{V}}$$

    The second term above is not a smooth function of the market data.

# Physical Storage: Alternative Methods

## Structural Models

- Inventory-based models have been around for awhile.[3]
- Following Pirrong:
  - Spot price formation occurs through a market equilibrium based upon information about supply and demand drivers and storage capacity.
  - Supply and demand functions $p_t = \mathcal{S}(q_t^S)$ and $p_t = \mathcal{D}(q_t^D)$ are specified.
  - The equilibrium price is at the intersection of these two functions.
  - An alternative characterization is that the equilibrium price maximizes the sum of the consumer and producer surpluses defined respectively as:

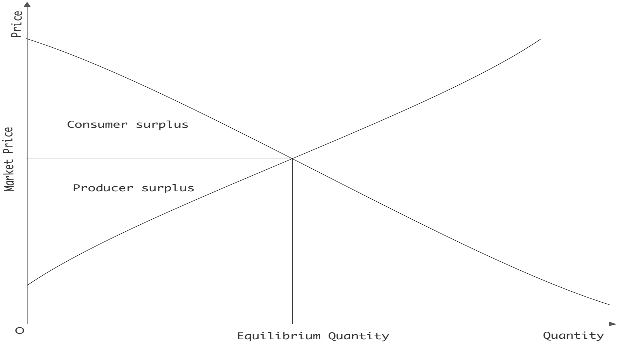
$$\begin{aligned} \mathcal{D}^*(q) &= \int_0^q \mathcal{D}(\hat{q})d\hat{q} - q\mathcal{D}(q) \\ \mathcal{S}^*(q) &= q\mathcal{D}(q) - \int_0^q \mathcal{S}(\hat{q})d\hat{q} \end{aligned} \quad (2)$$

  - The consumer surplus $\mathcal{D}^*(q)$ can be interpreted as the value which would be obtained by all consumers, when quantity $q$ is consumed at the demand curve price $\mathcal{D}(q)$.

[3] See Deaton and Laroque (1991), Routledge, Seppi and Spatt (2000) and Pirrong, "Commodity Price Dynamics" (2012)

# Physical Storage: Alternative Methods

## Structural Models



# Physical Storage: Alternative Methods

## Structural Models

- When agents make inter-temporal decisions involving inventory of a commodity that is produced and consumed, it is the present value of integrated future total surplus that must be maximized.
  - Denote the level of inventory at time $t$ by $I_t$.
  - Assume that the demand and supply functions are functions of stochastic processes $\vec{X}_t$ and $\vec{Y}_t$.
  - The optimization problem is:

$$V\left(I_t, \vec{X}_t, \vec{Y}_t\right) \equiv \sup_{[q_D(\cdot), q_S(\cdot)] \in \mathcal{A}} \tilde{E}\left[\int_t^{\infty} e^{-r(T-t)}\left[\mathcal{D}^*(q_D(T), \vec{X}_T) - \mathcal{S}^*(q_S(T), \vec{Y}_T)\right] \Big| \mathcal{F}_t\right] dT$$

  - Constraints $\mathcal{A}$ on the allowed actions $q_D$ and $q_S$ always include:
    - Bounds on inventory: $I(t) \in [0, S_{\max}]$ for all $t$.
    - Conservation of commodity: $\frac{dI}{dt} = q_S(t) - q_D(t)$.
    - In some cases this last constraint is modified to have losses of inventory while in storage.
  - Consistency of spot price formation also requires:

$$p_t = D\left[q_D(t), \vec{Y}_t\right] \qquad p_t = S\left[q_S(t), \vec{X}_t\right]$$

# Physical Storage: Alternative Methods

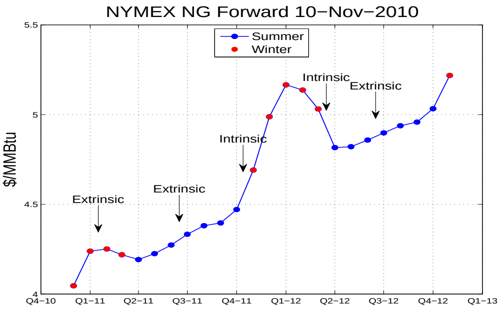
## Structural Models

- Various authors have studied models of this form, or closely related variations.
- The results of such these studies suggest that models of this form can generate many of the expected results, notably:
  - The instantaneous variance of returns (spot or forward) decreases with inventory $I_t$.
  - This effect decreases (in general) with tenor, with the ratio of spot to forward returns variances greatest at high demand and low inventory.
  - Returns correlation between spot and forward contracts is low at high demand and low inventory, and high at low demand and high inventory.
  - In some cases (see Routledge, Seppi and Spatt) contango of implied volatilities at short tenors—a phenomenon occasionally observed in market data.
- The main problem: Numerical efficiency, which currently precludes effective calibration to historical and forward market data.

# Practical Considerations

## Limited Market Data

- Consider the NG forward curve on 10Nov2010.
  - Seasonality puts the intrinsic value primarily by injecting in the summer and withdrawing in the winter.
  - Extrinsic value is primarily intra-seasonal.
  - This is where correlation information is particularly important.



# Practical Considerations

## Limited Market Data

- Now consider the following broker chat on the same date:
- What's wrong with this picture?

Insert: CSO Broker Chat (10Nov2010)

MARKET LOG FOR MARKET ID:—
11/10/10 8:19 AM v11/f12 -.50 call cso .0825-.095
11/10/10 8:34 AM V/F -50call CSO .075 / .10
11/10/10 8:37 AM V/F -50call CSO .075 @ .09
11/10/10 10:57 AM v11/f12 -.50 call cso @ .0875
11/10/10 10:58 AM v11/f12 -.50 call cso .075-.0875
11/10/10 1:43 PM v11/f12 -.50 call cso 6.5/8.25
11/10/10 1:44 PM v11/f12 -.50 call cso 7/8
11/11/10 8:46 AM v1/f2 -.50 call cso: .07 / .085
11/11/10 1:47 PM v11/f12 -.50 call cso .07-.085
11/11/10 1:52 PM v1/f2 -.50 call cso: .07 / .075
11/11/10 1:52 PM v1/f2 -.50 call cso: .07 / .075
11/11/10 1:59 PM v11/f12 -.50 call cso .07-.075
11/11/10 2:11 PM v11/f12 -.50 call cso .0725-.075
11/12/10 10:16 AM
V/F -1.00/-1.25 1x2 put spr CSO .01/.03 to the 2
V/F -.1.25 put CSO .0675/.075
V/F -1.00 put cso .115/.135
V/F -.50 call CSO .065/.075

# Practical Considerations

## Conclusion

- All of the valuation methods described boil down in essence to CSO valuation.
  - Absence of CSO data makes calibration and hedging challenging.
  - In the absence of such market data models must accurately represent correlation structure.
- The core (and largely unresolved) problem:
  - Correlation structure depends upon inventory.
  - Inventory depends on the actions of storage owners.
  - Recall this figure showing copper PCA results versus inventory.