A numerical algorithm for general HJB equations : a jump-constrained BSDE approach

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Modeling volatility

of an asset price S_t

Constant	:	$\sigma > 0$
$\Downarrow$		
Deterministic	:	$\sigma(t)$
$\Downarrow$		
Local	:	$\sigma(t, S_t)$
$\Downarrow$		
Stochastic	:	$d\sigma_t = \ldots$
$\Downarrow$		
Uncertain	:	$\sigma \in [\sigma_{\min}, \sigma_{\max}]$

Uncertain Volatility Model

Example

$$dS_t = \sigma S_t dW_t$$

$$\sigma \in [\sigma_{\min}, \sigma_{\max}] \text{ uncertain}$$

Super-replication price

$$\text{Payoff } \Phi = \Phi(T, S_T)$$

$$P_0^+ = \sup_{\mathbb{Q} \in \mathbf{Q}} \mathbb{E}^{\mathbb{Q}} [\Phi(T, S_T)]$$

$$\mathbf{Q} = \left\{ \mathbb{Q} \leftrightarrow \sigma^{\mathbb{Q}} \,;\, \sigma_{\min} \leq \sigma^{\mathbb{Q}} \leq \sigma_{\max} \right\}$$

Stochastic control problem

with controlled driver & drift & volatility

Formulation

$$dX_s^{\alpha} = b\left(X_s^{\alpha}, \alpha_s\right) ds + \sigma\left(X_s^{\alpha}, \alpha_s\right) dW_s$$

$$v\left(t,x\right) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{t,x}\left[\int_t^T f\left(X_s^{\alpha}, \alpha_s\right) ds + g\left(X_T^{\alpha}\right)\right]$$

General HJB equation

$$\frac{\partial v}{\partial t} + \sup_{a \in A}\left\{b\left(x,a\right).D_x v + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^{\top}\left(x,a\right) D_x^2 v\right) + f\left(x,a\right)\right\} = 0 \quad \text{on } [0,T) \times \mathbb{R}^d$$

$$v\left(T,x\right) = g\left(x\right) \quad , x \in \mathbb{R}^d$$

STEP 1 : Randomization of controls

Poisson random measure $\mu_A(dt, da)$ on $\mathbb{R}_+ \times A$, $\perp\!\!\!\perp W$
associated to the marked point process $I \leftrightarrow (\tau_i, \zeta_i)_i$, valued in A

$$I_t = \zeta_i \,, \;\; \tau_i \leq t < \tau_{i+1}$$

Uncontrolled randomized problem

$$dX_s = b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s$$

$$v(t, x, a) = \mathbb{E}^{t,x,a}\left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right]$$

Linear FBSDE

$$Y_t = g(X_T) + \int_t^T f(X_s, I_s)\, ds - \int_t^T Z_s dW_s - \int_t^T \int_A U_s(a)\, \tilde{\mu}_A(ds, da)$$

$$Y_t \longleftrightarrow v(t, X_t, I_t)$$

STEP 2 : Constraint on jumps

$$U_t(a) = v(t, X_t, a) - v(t, X_t, I_{t-})$$

Now, how to retrieve HJB ?

$\Rightarrow$ Add the constraint $U_t(a) \leq 0 \quad \forall (t, a)$!

Jump-constrained BSDE

Minimal solution (Y, Z, U, K) of

$$Y_t = g(X_T) + \int_t^T f(X_s, I_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s$$
$$+ K_T - K_t - \int_t^T \int_A U_s(a)\, \tilde{\mu}_A(ds, da), \; 0 \leq t \leq T$$

subject to $U_t(a) \leq 0 \quad \forall (t, a)$

Link with general HJB equations

(X, I) Markov $\Rightarrow \exists v = v(t,x,a)$ s.t. $Y_t = v(t, X_t, I_t)$

Key Lemma

$v = v(t,x,a)$ does not depend on a!
$\hookrightarrow v = v(t,x)$

Theorem

$v = v(t,x)$ is solution of the HJB equation

$$\frac{\partial v}{\partial t} + \sup_{a\in A}\left\{b(x,a).D_x v + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^{\top}(x,a)D_x^2 v\right) + f\left(x,a,y,\sigma^{\top}(x,a).D_x v\right)\right\} = 0 \quad \text{on } [0,T)\times\mathbb{R}^d$$
$$v(T,x) = g(x) \quad , x\in\mathbb{R}^d$$

Proofs : cf. [Kharroubi, Pham, 2012]

Numerical scheme

$$U_t(a) = v(t, X_t, a) - v(t, X_t, I_{t-}) \leq 0 \quad \forall (t, a)$$

$\Rightarrow$ $v(t, X_t, I_{t-}) \geq \sup_{a \in A} v(t, X_t, a)$

$\Rightarrow$ Minimal solution $v(t, X_t, I_{t-}) = \sup_{a \in A} v(t, X_t, a)$

$$v(t, X_t, I_t) - v(t, X_t, I_{t-}) \longleftrightarrow K_t - K_{t-}$$

Forward-Backward numerical scheme

$$\begin{aligned} Y_N &= g(X_N) \\ \mathcal{Z}_i &= \mathbb{E}_i\left[Y_{i+1} \Delta W_i^\top\right] / \Delta_i \\ \mathcal{Y}_i &= \mathbb{E}_i\left[Y_{i+1} + f_i(X_i, I_i, Y_{i+1}, \mathcal{Z}_i)\Delta_i\right] \\ Y_i &= \sup_{A \in \mathcal{A}_i} \mathbb{E}_{i,A}[\mathcal{Y}_i] \end{aligned}$$

where $\mathbb{E}_{i,A}[.] := \mathbb{E}[.|X_i, I_i = A]$

Towards an implementable scheme

How to compute the conditional expectations ?
(quantification, Malliavin calculus, empirical regression,...)

$\Rightarrow$ cf. comparative tests in [Bouchard, Warin, 2012]

Conditional expectation approximation

$$\mathbb{E}\left[U \,|\mathcal{F}_{t_i}\right] = \arg \inf_{V \in \mathbf{L}\left(\mathcal{F}_{t_i}, \mathbb{P}\right)} \mathbb{E}\left[(V-U)^2\right]$$

$$\hat{\mathbb{E}}\left[U \,|\mathcal{F}_{t_i}\right] = \arg \inf_{V \in \mathcal{S}} \frac{1}{M} \sum_{m=1}^{M} (V_m - U_m)^2$$

where $\mathcal{S} \subset \mathbf{L}\left(\mathcal{F}_{t_i}, \mathbb{P}\right)$

Empirical regression schemes

First algorithm ("Tsitsiklis - van Roy")

$$\hat{Y}_N = g\left(X_N\right)$$

$$\hat{\mathcal{Y}}_i = \hat{\mathbb{E}}_i\left[\hat{Y}_{i+1} + f_i\left(X_i, I_i\right)\Delta_i\right]$$

$$\hat{Y}_i = \sup_{A\in\mathcal{A}_i} \mathbb{E}_{i,A}\left[\hat{\mathcal{Y}}_i\right]$$

Upward biased (up to Monte Carlo error & regression bias)

Second algorithm ("Longstaff - Schwartz")

$$\hat{\alpha}_i = \arg\sup_{A\in\mathcal{A}_i} \mathbb{E}_{i,A}\left[\hat{\mathcal{Y}}_i\right]$$

$$\hat{X}_{i+1} = b(\hat{X}_i, \hat{\alpha}_i)\Delta_i + \sigma(\hat{X}_i, \hat{\alpha}_i)\Delta W_i$$

$$\hat{v}\left(t_0, x_0\right) = \frac{1}{M}\sum_{m=1}^{M}\left[\sum_{i=1}^{N} f(\hat{X}_{i+1}, \hat{\alpha}_i)\Delta_i + g(\hat{X}_N)\right]$$

Downward biased (up to Monte Carlo error)

Uncertain correlation model

Model

$$dS_t^i = \sigma_i S_t^i dW_t^i \quad , \quad i = 1, 2$$
$$\left\langle dW_t^1, dW_t^2 \right\rangle = \rho dt$$
$$-1 \leq \rho_{\min} \leq \rho \leq \rho_{\max} \leq 1$$

Super-replication price

$$\text{Payoff } \Phi = \Phi\left(T, S_T^1, S_T^2\right)$$
$$P_0^+ = \sup_{\mathbb{Q} \in \mathbf{Q}} \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(T, S_T^1, S_T^2\right)\right]$$
$$\mathbf{Q} = \left\{\mathbb{Q} \leftrightarrow \rho^{\mathbb{Q}} ; \rho_{\min} \leq \rho^{\mathbb{Q}} \leq \rho_{\max}\right\}$$

Call spread on spread $S_1(T) - S_2(T)$

$$\Phi = (\mathbf{S_1(T)} - \mathbf{S_2(T)} - \mathbf{K_1})^+ - (\mathbf{S_1(T)} - \mathbf{S_2(T)} - \mathbf{K_2})^+$$

$S_1(0)$	$S_2(0)$	σ_1	σ_2	ρ_{min}	ρ_{max}	K_1	K_2	T
50	50	0.4	0.3	-0.8	0.8	-5	5	0.25

Regression basis

$$\phi(t, s_1, s_2, \rho) = (K_2 - K_1) \times \mathcal{S}(\beta_0 + \beta_1 s_1 + \beta_2 s_2 + \beta_3 \rho + \beta_4 \rho s_1 + \beta_5 \rho s_2)$$

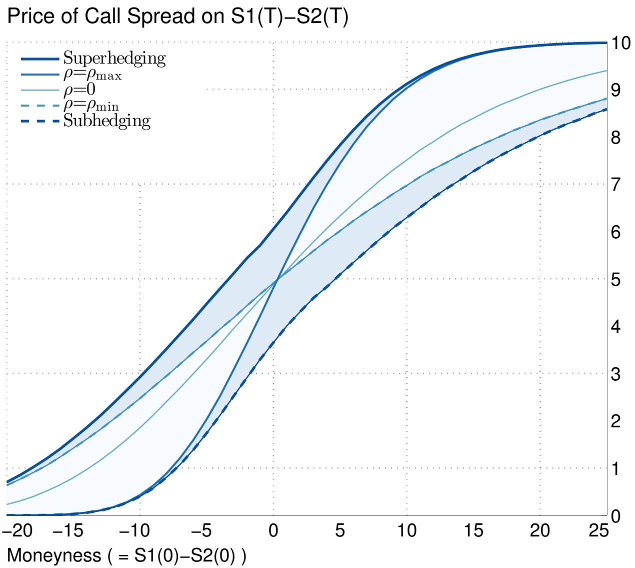
$$\mathcal{S}(x) = 1/(1 + \exp(-x))$$

⇒ Bang-bang optimal control

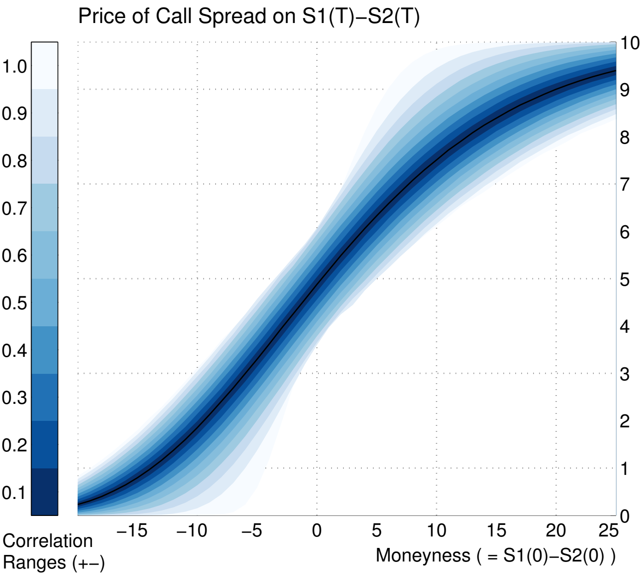
$$\rho^*(t, s_1, s_2) = \arg\max_{\alpha} \phi(t, s_1, s_2, \rho) = \rho_{max} \text{ if } \beta_3 + \beta_4 s_1 + \beta_5 s_2 \geq 0$$

$$= \rho_{min} \text{ else}$$

Results

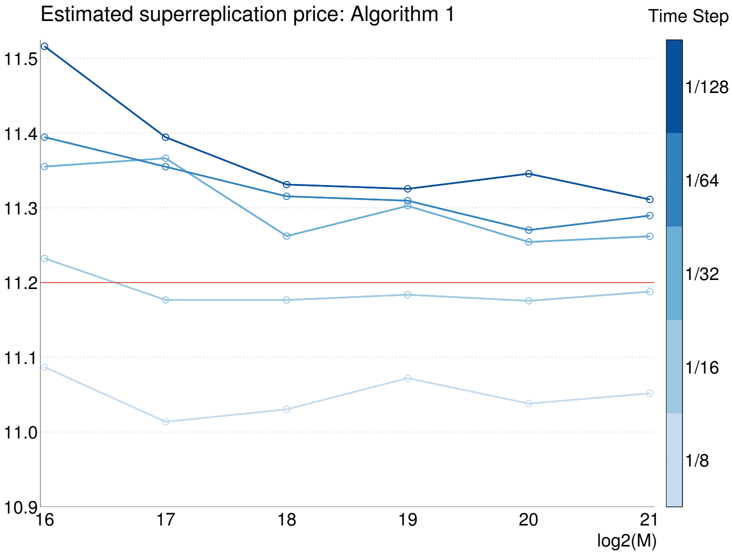

Price of Call Spread on S1(T)–S2(T)
Superhedging
$\rho=\rho_{\max}$
$\rho=0$
$\rho=\rho_{\min}$
Subhedging
10
9
8
7
6
5
4
3
2
1
0
–20 –15 –10 –5 0 5 10 15 20 25
Moneyness (= S1(0)–S2(0))

Impact of correlation range



Call Spread $(S(T)-K_1)^+ - (S(T)-K_2)^+$

$S(0)=100$, $K_1=90$, $K_2=110$, $T=1$, uncertain $\sigma \in [0.1, 0.2]$



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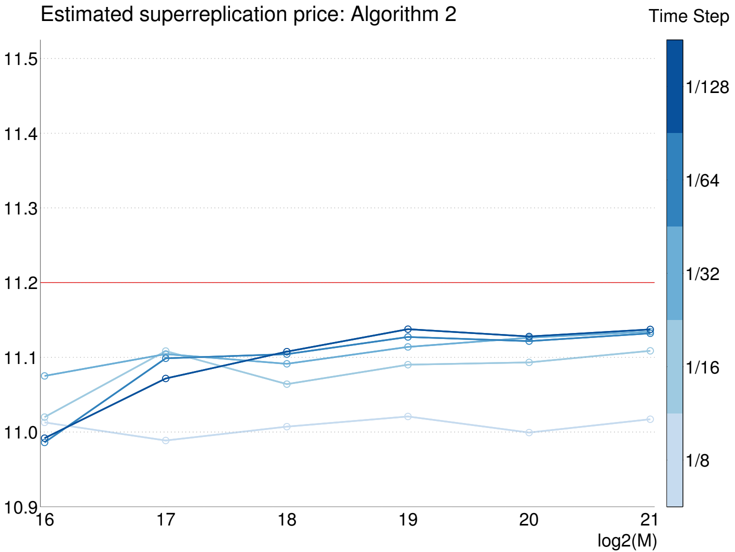
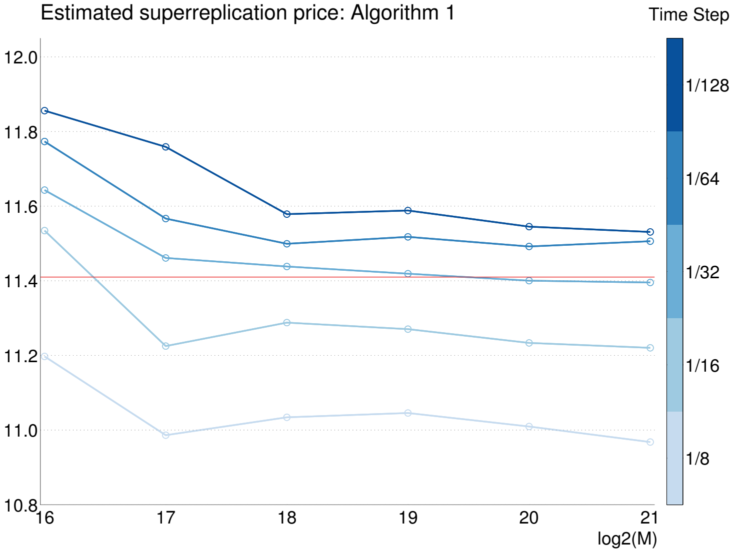


Outperformer Spread $(S_2(T)-K_1S_1(T))^+-(S_2(T)-K_2S_1(T))^+$

$S_i(0)=100$, $K_1=0.9$, $K_2=1.1$, $T=1$, uncertain $\sigma_i\in[0.1,0.2]$, $\rho=-0.5$



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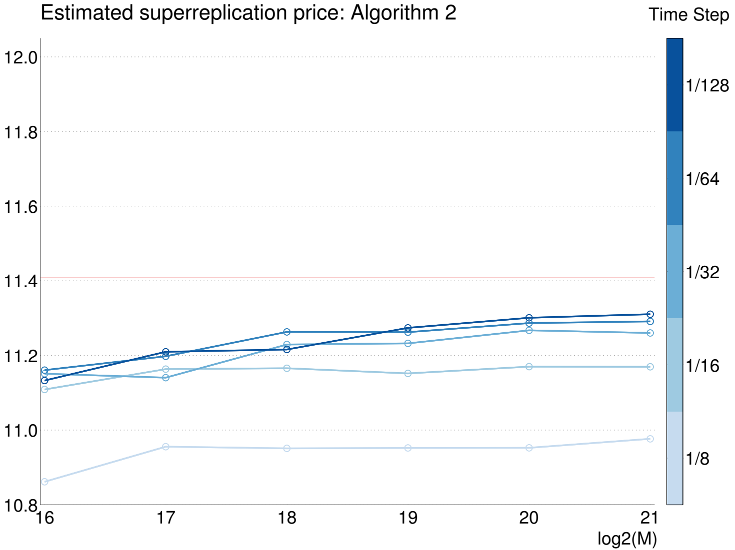


Summary

1. Probabilistic representation of stochastic control problem with controlled volatility : **jump-constrained BSDE**
2. **Numerical scheme** for jump-constrained BSDEs
3. Application to pricing under **uncertain volatility**
4. Extension : stochastic games, HJB-Isaacs equations

Thank you !

Questions ?

References

J-P. Lemor and E. Gobet and X. Warin (2006)
Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations

J. Guyon and P. Henry-Labordère (2011)
Uncertain volatility model : a Monte Carlo approach

E. Gobet and P. Turkedjiev (2011)
Approximation of discrete BSDE using least-squares regression

B. Bouchard and X. Warin (2012)
Monte Carlo valorisation of American options : facts and new algorithms to improve existing methods

I. Kharroubi and H. Pham (2012)
Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE

S. Alanko and M. Avellaneda (2013)
Reducing variance in the numerical solution of BSDEs

Example

A linear-quadratic stochastic control problem

Problem

$$
\begin{aligned}
dX_s^{\alpha} &= \left(-\mu_0 X_s^{\alpha} + \mu_1 \alpha_s\right) ds + \left(\sigma_0 + \sigma_1 \alpha_s\right) dW_s \\
X_0^{\alpha} &= 0
\end{aligned}
$$

$$
v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[-\lambda_0 \int_t^T (\alpha_s)^2 \, ds - \lambda_1 \left(X_T^{\alpha}\right)^2\right]
$$

Set of parameters

μ_0	μ_1	σ_0	σ_1	λ_0	λ_1	T
0.02	0.5	0.2	0.1	20	200	2

Numerical parameters

$n = 52$ time steps

$M = 10^6$ Monte Carlo simulations

Regression basis

$$\phi(t, x, \alpha) = \beta_0 + \beta_1 x + \beta_2 \alpha + \beta_3 x\alpha + \beta_4 x^2 + \beta_5 \alpha^2$$

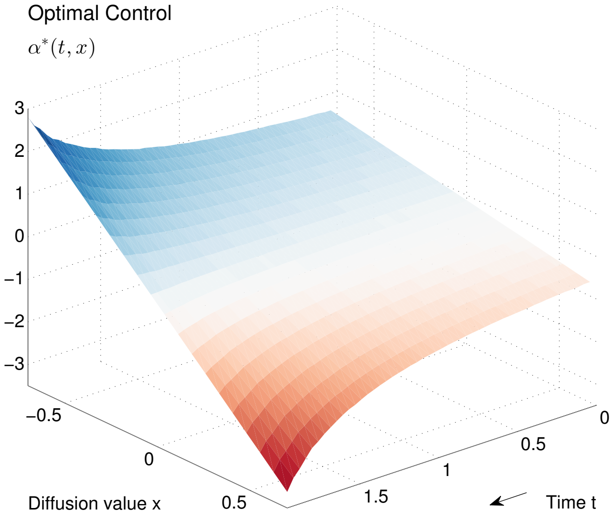
⇒ Linear optimal control

$$\alpha^*(t, x) = \arg\max_{\alpha} \phi(t, x, \alpha) = A(t)\, x + B(t)$$

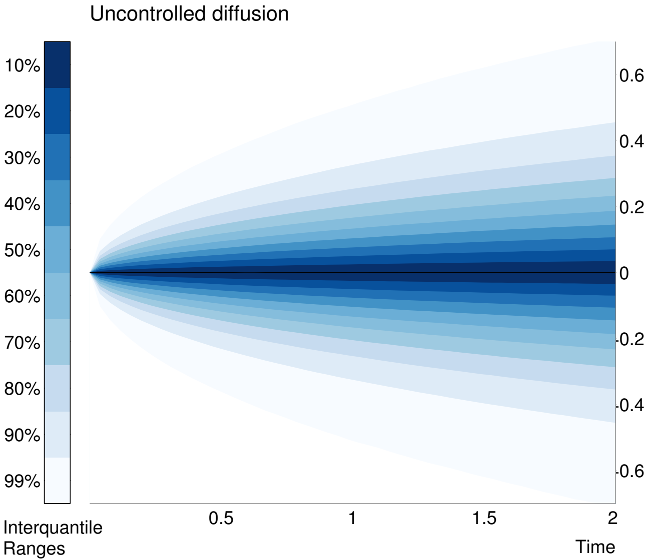
$$A(t) = -0.5\, \beta_3 / \beta_5$$

$$B(t) = -0.5\, \beta_2 / \beta_5$$

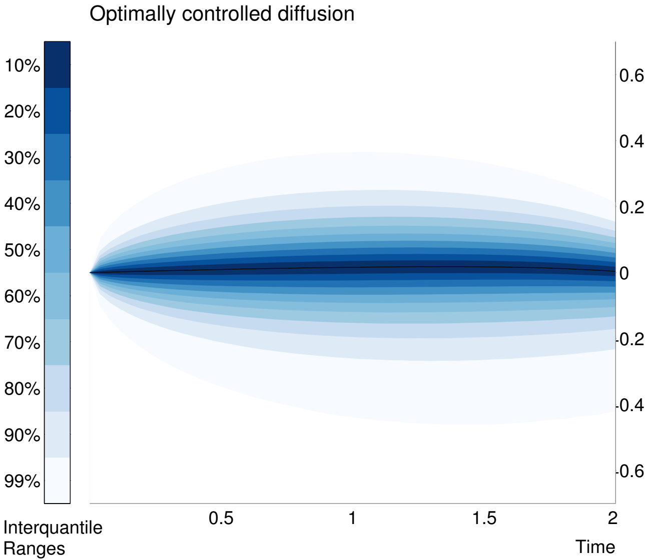
Estimated optimal controls



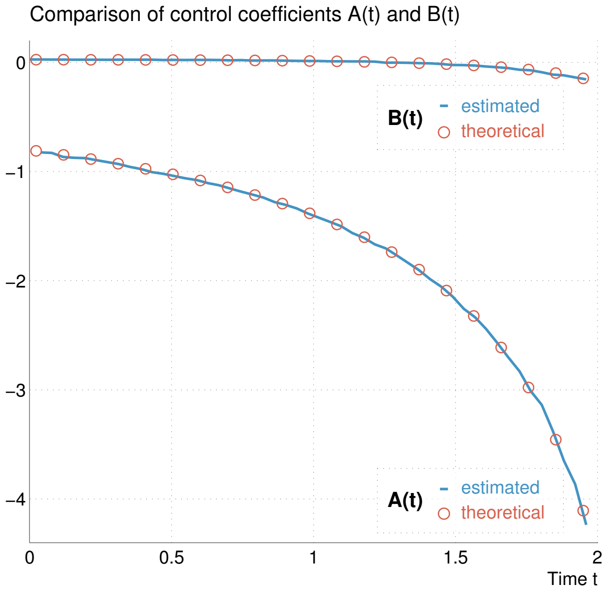
Control impact (1/2) : no control



Control impact (2/2) : optimal control



Accuracy



Value Function

$\hat{v}(0,0) = -5.761$

$v(0,0) = -5.705$

Relative Error : 1%

Convergence rate

1. Localizations
2. Theoretical regressions
3. Empirical regressions

Assumptions

$\exists p \geq 1,\ L_g, L_f, C_{f,0} > 0$ s.t. $\forall i = 0, \ldots, N-1$

$$|g(x) - g(x')| \leq L_g |x^p - x'^p|$$
$$|f_i(x,a,y,z) - f_i(x',a',y',z')| \leq L_f(|x^p - x'^p| + |a^p - a'^p| + |y - y'| + |z - z'|)$$
$$|f_i(0,0,0,0)| \leq C_{f,0}$$

Bounded control domain $A \subset \mathbb{R}^{d'}$: $\exists \bar{A} > 0$ s.t. $\forall a \in A,\ |a| \leq \bar{A}$

Theoretical regression (1/2)

Definition

$$\hat{\lambda}_i(U) := \arg \inf_{\lambda \in \mathbb{R}^B} \mathbb{E}\left[\left(\lambda . p(X_i, I_i) - U\right)^2\right]$$

$$\mathcal{P}_i(U) := \hat{\lambda}_i(U) . p(X_i, I_i)$$

Associated scheme

$$\hat{Y}_N = g(X_N)$$

$$\hat{\lambda}_i^Y = \text{regression coefficients at time } t_i$$

$$\hat{Y}_i = \sup_{A \in \mathcal{A}_i} \hat{\lambda}_i^Y . p_i(X_i, A)$$

Problem : $\hat{Y}_i$ is not itself the projection of some random variable...

Theoretical regression (2/2)

Alternative definition

$$\hat{\lambda}_{i,A}(U) := \arg \inf_{\lambda \in \mathbb{R}^B} \mathbb{E}\left[\left(\lambda.p(X_i, A) - U_A\right)^2\right]$$

$$\mathcal{P}_{i,A}(U) := \hat{\lambda}_i(U).p(X_i, A)$$

Regression error

$$\max\left\{\left|Y_i - \hat{Y}_i\right|^2, \Delta_i\left|Z_i - \hat{Z}_i\right|^2\right\}$$

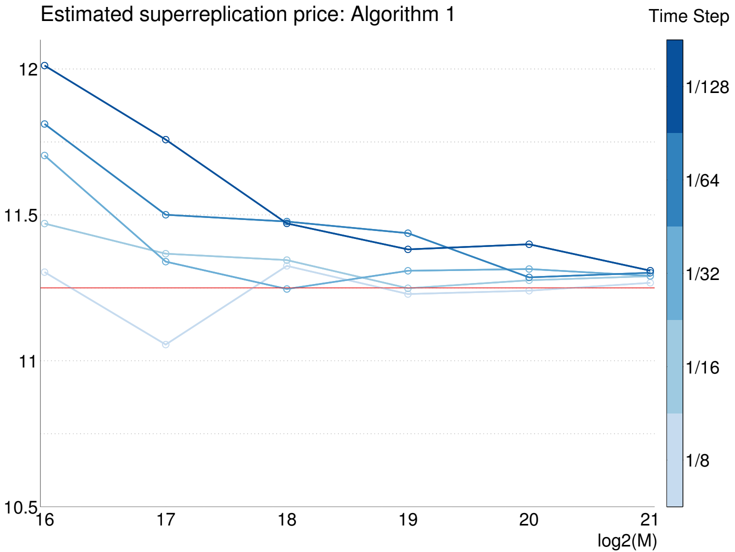
$$\leq e^{C(T-t_i)} \sum_{k=i}^{N-1} \left\{\mathbb{E}\left[\sup_{A \in \mathcal{A}_k} \left|\mathcal{Y}_{k,A} - \mathcal{P}^Y_{k,A}(\mathcal{Y}_{k,A})\right|^2\right]\right.$$

$$\left. + C\Delta_k \mathbb{E}\left[\sup_{A \in \mathcal{A}_k} \left|\mathcal{Z}_{k,A} - \mathcal{P}^Z_{k,A}(\mathcal{Z}_{k,A})\right|^2\right]\right\}$$

But its empirical version cannot be (efficiently) implemented...

Outperformer $(S_1(T) - S_2(T))^+$

$S_i(0) = 100$, $T = 1$, uncertain $\sigma_i \in [0.1, 0.2]$, $\rho = 0$



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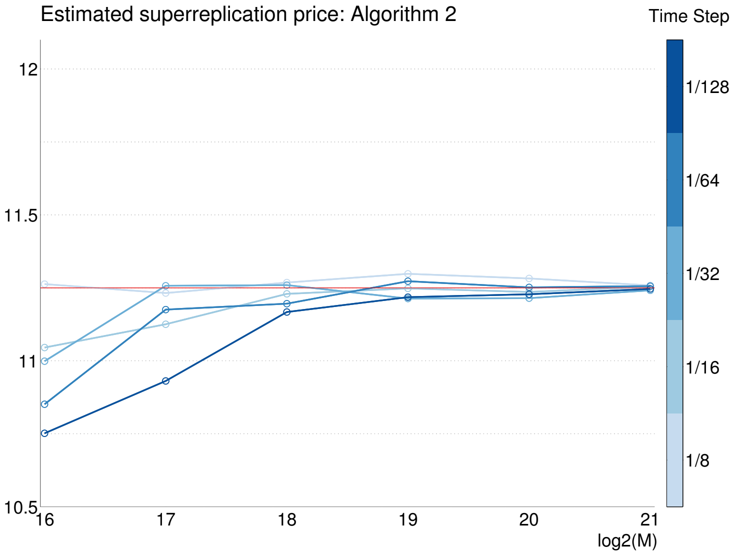

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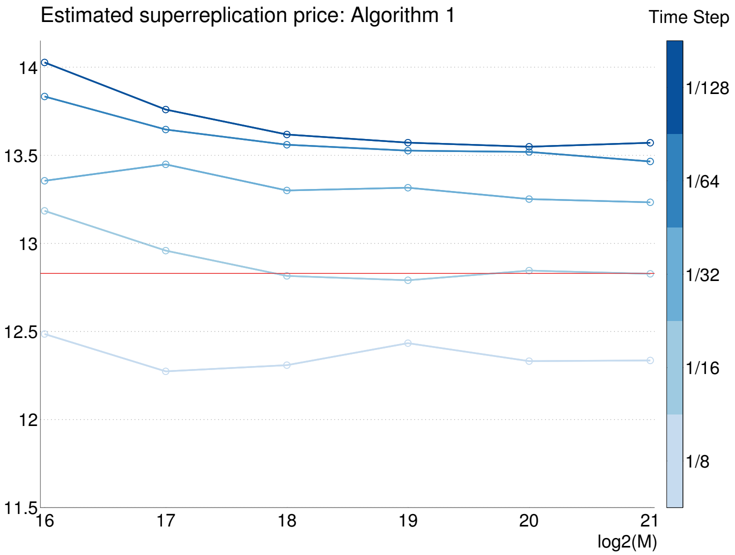

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