Lecture V: Heath-Jarrow-Morton modeling of energy markets

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Overview of the lecture

1. Introduction to the HJM approach
   - Modelling issues
2. Two *dynamic* approaches
   - Fixed-delivery forwards
   - Electricity forwards with a delivery period
3. Dynamic market models for electricity forwards
   - "LIBOR" models
4. *Direct* modeling of forwards
   - Based on ambit fields
   - Random field extension of Lévy semistationary processes

Itroduction to the HJM approach

- Use ideas from interest rate theory to model electricity markets
- Heath-Jarrow-Morton 1992:
  - Model the complete term structure dynamics of interest rates directly under the risk-neutral probability
  - Analogue in electricity: Model the term structure dynamics of forward/futures prices
  - Problem: Electricity forwards has a delivery period!
- Goal of modelling: Models which can be used for derivatives pricing and risk analysis

- Look at two alternative HJM-approaches for electricity markets
  1. The "fixed-delivery" approach: Modelling $f(t,u)$
  2. The direct approach: Modelling $F(t,\tau_1,\tau_2)$
- Problems with "fixed-delivery" $f(t,u)$:
  - No trade in such forwards
  - Data needs to be constructed
- Problems with direct modelling $F(t,\tau_1,\tau_2)$
  - Complicated to specify a no-arbitrage models
  - Due to overlapping delivery periods
- Using *market models* to resolve last issue
  - Motivated from LIBOR models
  - Model only the traded contracts

- Modelling issues include:
- Marginal:
  - Are logreturns normal, or leptokurtic?
  - Volatility term structure?
- Multivariate:
  - What is the dependency structure across contracts?
- Direct modeling of forwards, or implied through options
  - Latter raises the question of liquidity
  - Börger et al (2009): Implied volatility at EEX
  - B & K (2008): Direct modelling of NordPool

The “fixed-delivery” approach

- Assume a geometric Brownian motion dynamics for the *fixed-delivery* forwards (under Q)

$$df(t,u) = \sigma(t,u) f(t,u)\, dW(t)$$

- Forward with delivery over a period $[\tau_1, \tau_2]$

$$F(t,\tau_1,\tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(t,u)\, du$$

- **Q1:** What is the implied dynamics of the electricity forward?
- **Q2:** How to fit model to data?

- Q1: Use stochastic Fubini and integration-by-parts

$$
\begin{aligned}
F(t,\tau_1,\tau_2) &= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} f(s,u)\,du \\
&= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} f(0,u)\,du + \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\int_0^t \sigma(s,u)f(s,u)\,dW(s)\,du \\
&= F(0,\tau_1,\tau_2) + \frac{1}{\tau_2-\tau_1}\int_0^t\int_{\tau_1}^{\tau_2} \sigma(s,u)f(s,u)\,du\,dW(s) \\
&= F(0,\tau_1,\tau_2) + \int_0^t \sigma(s,\tau_2)F(s,\tau_1,\tau_2)\,dW(s) \\
&\quad - \int_0^t\int_{\tau_1}^{\tau_2} \frac{u-\tau_1}{\tau_2-\tau_1}\partial_u\sigma(s,u)F(s,\tau_1,u)\,du\,dW(s)
\end{aligned}
$$

- The dynamics becomes

$$dF(t,\tau_1,\tau_2) = \Big\{\sigma(t,\tau_2)F(t,\tau_1,\tau_2) - \int_{\tau_1}^{\tau_2}\frac{u-\tau_1}{\tau_2-\tau_1}\partial_u\sigma(t,u)F(t,\tau_1,u)\,du\Big\}\,dW(t)$$

- Electricity forward does not have the lognormal property
  - Note the infinite dimensional structure
- Options are written on $F(t,\tau_1,\tau_2)$
  - Model for fixed-delivery forwards $f(t,u)$ leads to options on the average
  - Approximations available?
  - Numerical procedures?

- One approximative approach is to *assume* that the implied $F(t,\tau_1,\tau_2)$ is GBM!
  - Bjerksund et. al 2000.
- Approximation of the dynamics

$$dF(t,\tau_1,\tau_2) = \left\{\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\sigma(t,u)\,du\right\} F(t,\tau_1,\tau_2)\,dW(t)$$

- Prices for call options are simple to calculate for this model
  - Black-76 with time-dependent volatility
- "Industry standard" (VizRisk)

- Q2: How to fit the "fixed-delivery model" to data?
- First, we have dynamics under Q
  - Data are measured under P
  - ...at least the real forwards with delivery period
- Use Girsanov transform

$$dW(t) = \lambda\, dt + dB(t)$$

- P-dynamics of $f(t,u)$

$$df(t,u) = \lambda\sigma(t,u)f(t,u)\, dt + \sigma(t,u)f(t,u)\, dB(t)$$

- GBM, with a market "price of risk" λ

- Note that λ can be time-dependent
  - ... and even stochastic
  - ... but NOT dependent on u, time of delivery
- Why?
  - There must exist one risk-neutral Q such that $f(t,u)$ is a Q-martingale
  - This is given by λ
  - λ u-dependent does not give one Q for the whole market
  - ..and arbitrage exists
- Can be resolved by introducing more noise (BMs)

- Fitting to data
    - Standard approach is to smoothen the electricity forward curve
- One idea:
    - Today's forward curve is factorized into a seasonal and a correction term

$$f(u) := f(0,u) = \Lambda(u) + \epsilon(u)$$

- Alternatively:
    - Find numerical average of $f(t,u)$ over delivery period
    - Theoretical $F(t,\tau_1,\tau_2)$
    - Find optimal parameters by minimizing distance to data

$$\min_{\lambda,\sigma} \|F(t,\tau_1,\tau_2) - \widehat{F}(t,\tau_1,\tau_2)\|$$

- Drawbacks with the fixed-delivery HJM-approach:
  1. Model of non-existing forwards
  2. Estimation uses data which must be transformed (smoothed)
  3. Implied electricity forward dynamics is very involved, even for a GBM-model

The direct approach

- Model the electricity forward dynamics directly
- No-arbitrage condition: Overlapping forwards must satisfy

$$F(t,\tau_1,\tau_N)=\sum_{i=1}^{N-1}\frac{\tau_{i+1}-\tau_i}{\tau_N-\tau_1}F(t,\tau_i,\tau_{i+1})$$

- If market trades in forwards with *all* possible delivery periods

$$F(t,\tau_1,\tau_N)=\frac{1}{\tau_N-\tau_1}\int_{\tau_1}^{\tau_N}F(t,u,u)\,du$$

- Consider GBM model

$$dF(t,\tau_1,\tau_2) = \Sigma(t,\tau_1,\tau_2)F(t,\tau_1,\tau_2)\,dW(t)$$

- Explicit dynamics

$$F(t,\tau_1,\tau_2) = F(0,\tau_1,\tau_2)\exp\left(\int_0^t \Sigma(s,\tau_1,\tau_2)\,dW(s) - \frac{1}{2}\int_0^t \Sigma^2(s,\tau_1,\tau_2)\,ds\right)$$

- GBM-model does not in general satisfy the condition no-arbitrage condition

- The exception is constant volatility, $\sigma := \Sigma$

$$\frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N} F(t,u,u)\,du = \frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N} F(0,u,u)\,du$$

$$+ \frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N}\int_0^t \sigma F(s,u,u)\,dW(s)\,du$$

$$= \frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N} F(0,u,u)\,du$$

$$+ \int_0^t \sigma \frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N} F(s,u,u)\,du\,dW(s)$$

- Hence, if initial forward curve satisfies the no-arbitrage condition, we recover the no-arbitrage condition

$$F(t,\tau_1,\tau_N) = \frac{1}{\tau_N - \tau_1}\int_{\tau_1}^{\tau_N} F(t,u,u)\,du$$

- What if volatility is delivery-period dependent?
  - Differentiate no-arbitrage condition wrt τ_n on both sides

$$F(t,\tau_1,\tau_n)\left(\frac{1}{\tau_n - \tau_1} - \frac{1}{2}\int_0^t \partial_{\tau_n}\Sigma^2(s,\tau_1,\tau_n)\,ds + \int_0^t \partial_{\tau_n}\Sigma(s,\tau_1,\tau_n)\,dW(s)\right) = \frac{1}{\tau_n - \tau_1}F(t,\tau_n,\tau_n)$$

- RHS is positive, while LHS may be arbitrary negative
  - No-arbitrage is violated

- In conclusion: Hard to state models which are
  - realistic,
  - easy to estimate,
  - and satisfy the no-arbitrage condition
- A practical approach: Model only the existing forwards in the market
  1. Single out the "smallest" forwards (the building blocks)
  2. Model these
  3. Forwards with larger delivery period are modelled by the no-arbitrage relation

Market models

Market models

- Suppose $[\tau_1^i, \tau_2^i]$ is a sequence of delivery periods for the building block forwards
- Suppose each forward is modelled as a GBM

$$dF^i(t) = \Sigma^i(t)F^i(t)\,dW^i(t)$$

- The volatilities Σ^i will depend on the start and end of delivery period
- W^i are Brownian motions
  - With a correlation structure between the contracts
- Options priced using Black-76, with time-dependent volatility

- Fitting to data:
  - We need the P dynamics
- Use Girsanov again

$$dW^i(t) = \lambda^i \, dt + dB^i(t)$$

- Correlation structure of W^i preserved in B^i
- The market price of risk is delivery-dependent
  - ...may also be a stochastic process

$$dF^i(t) = \lambda^i \Sigma^i(t) F^i(t) \, dt + \Sigma^i(t) F^i(t) dW^i(t)$$

Discussion on empirics....

- Questions:
  1. Volatility term structure?
  2. Are logreturns normally distributed?
  3. What is correlation structure for the W^i?
- Presentation of some empirical findings
  - Work by B & Koekebakker (2008)

Case study from Nordpool

- Suppose $W^i = W$, e.g., one common risk factor for all contracts
- Fitted to data from NordPool
  - Extracting only non-overlapping forwards
  - Using more than 10.000 price data
- Assumed constant market price of risk
  - Model for F^i specified under Q
  - Add a constant drift for model under P

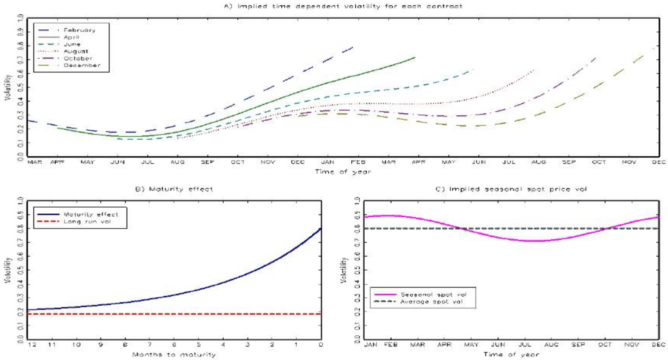
- Volatility with seasonality and maturity effect

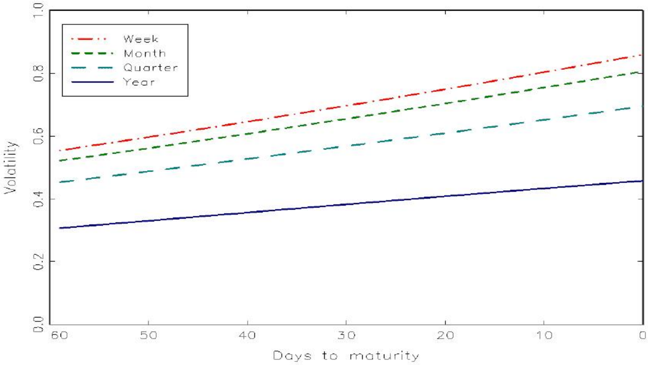
$$\Sigma(t,\tau_1,\tau_2)=\frac{\sigma}{a(\tau_2-\tau_1)}\left\{e^{-a(\tau_1-t)}-e^{-a(\tau_2-t)}\right\}+s(t)$$

- $s(t)$ seasonality function
    - Sine-function
- Mean reversion volatility coming from the average of

$$\sigma(t,u)=\sigma e^{-a(u-t)}$$

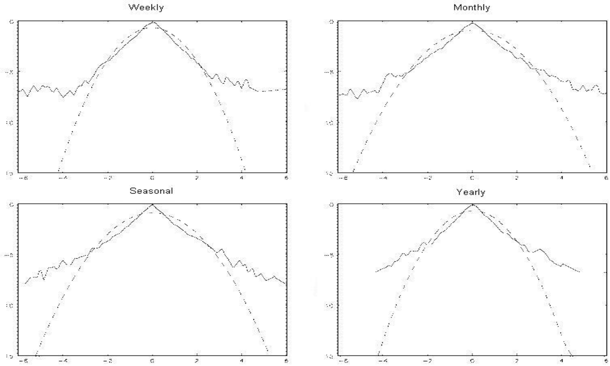
- Includes a Samuelson effect for the electricity forwards

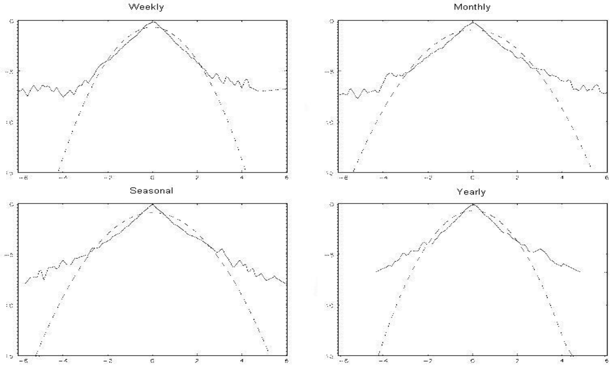

A) Implied time dependent volatility for each contract
February
April
June
August
October
December
Volatility
Time of year
B) Maturity effect
Maturity effect
Long run vol
Months to maturity
C) Implied seasonal spot price vol
Seasonal spot vol
Average spot vol
Time of year



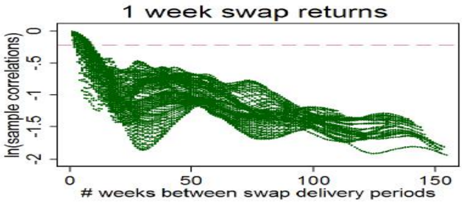
Week
Month
Quarter
Year
Volatility
0.0
0.2
0.4
0.6
0.8
1.0
60
50
40
30
20
10
0
Days to maturity

- Distributions of electricity forwards log-returns are non-normal
  - Tails are heavy
  - Symmetric




Weekly
Monthly
Seasonal
Yearly

- Correlation among forwards with one week delivery
- Estimated from smoothed forward curves
  - week contracts are extracted over regular times
- Correlation not stationary as a function of distance-between-delivery



- To specify a market model for all forwards including correlation:
  - Need non-stationary correlation structures
- Correlation function of
  - Time to delivery
  - Time of the year when delivery takes place
  - Length of delivery

Ambit processes and forward price modelling –HJM modelling–

Recall empirics of forwards: Lecture I

- Complex statistical dependencies between different forward contracts
  - Monthly, quarterly, yearly....
  - Different times to maturity
- Can we explain most of the uncertainty by a noise few factors?
  - For fixed-income markets: PCA indicate ~ 3 factors for explaining about 95-99% of the uncertainty
  - Electricity different!
- Koekebakker and Ollmar 2005: 10 factors not enough to capture 95% of the uncertainty in the forward market
  - A lot of idiosyncratic risk
  - Points towards random field models

Definition of ambit process

$$Y(t,x) = \int_{-\infty}^{t} \int_{\mathbb{R}_+} k(t-s,x,y)\sigma(s,y)\, L(ds,dy)$$

- L is a *Lévy basis*
- k non-negative deterministic function, $k(u,x,y) = 0$ for $u < 0$.
- Stochastic volatility process σ independent of L, stationary

- L is a *Lévy basis* on $\mathbb{R}^d$ if
  1. the law of $L(A)$ is infinitely divisible for all bounded sets A
  2. if $A \cap B = \emptyset$, then $L(A)$ and $L(B)$ are independent
  3. if $A_1, A_2, \ldots$ are disjoint bounded sets, then

$$L(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i)\,, a.s$$

- We choose $d = 2$
  - First coordinate being time t
  - Second being time-to-maturity x
- We restrict to zero-mean, and square integrable Lévy bases

- Stochastic integration in ambit process: use the Walsh-definition
  - Extension of Itô integration theory to temporal-spatial setting
  - In time: integration "as usual"
  - In space: exploit independence and additivity properties
  - Isometry by square-integrability hypothesis
  - Key object: (orthogonal) martingale measures
- Suppose k and σ integrable
  - Essentially square-integrability in time and space

Forward modelling by ambit processes

- Extension of the HJM approach
  - by direct modelling rather than as the solution of some dynamic equation
- Simple arithmetic model could be (in the risk-neutral setting)

$$F(t,x)=\int_{-\infty}^{t}\int_{0}^{\infty}k(t-s,x,y)\sigma(s,y)L(dy,ds)$$

- x is "time-to-maturity"

Martingale condition

- Forwards are tradeable
- $t \mapsto F(t, \tau - t)$ must be a martingale for $t \leq \tau$

Theorem

$F(t, \tau - t)$ is a martingale if and only if

$$k(t-s, \tau - t, y) = \widetilde{k}(s, \tau, y)$$

- Examples of kernel functions satisfying the martingale condition

  Example1: weighted exponential kernel function (motivated by OU spot models)

$$k(t-s,x,y)=\sum_{i=1}^{n} w_i \exp\left(-\alpha_i(t-s+x+y)\right)$$

  Example 2: "Spatial" Bjerksund kernel function

$$k(t-s,x,y)=h(y)\times\frac{a}{t-s+x+b}$$

- The "classical" case: the Musiela SPDE specification
  - $L = W$, Wiener case for simplicity

$$dF(t,x) = \frac{\partial F(t,x)}{\partial x}\,dt + h(t,x)\,dW(t)$$

- Solution of the SPDE with $x = \tau - t$

$$F(t,\tau - t) = F_0(\tau) + \int_0^t h(s,\tau - s)\,dW(s)$$

- Martingale condition is satisfied, of course!

Spatial correlation of ambit fields

- Spatial correlation for ambit fields with no stochastic volatility

$$\mathrm{corr}(F(t,x),F(t,y)) = \frac{\int_{-\infty}^{t}\int_{0}^{\infty} k(t-s,x,z)k(t-s,y,z)\,dz\,ds}{\sqrt{\int_{-\infty}^{t}\int_{0}^{\infty} k^2(t-s,x,z)\,dz\,ds}\sqrt{\int_{-\infty}^{t}\int_{0}^{\infty} k^2(t-s,y,z)\,dz\,ds}}$$

- An example with the spatial Bjerksund kernel function

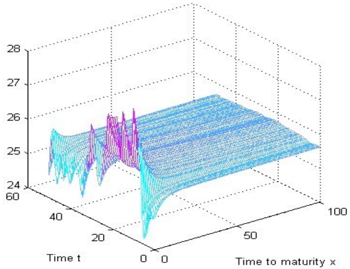
$$k(t-s,x,z) = h(z)\times\frac{a}{t-s+x+b}$$

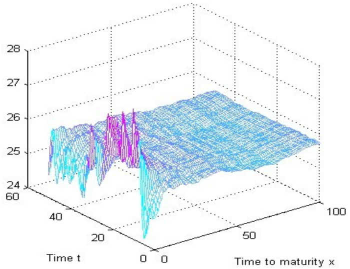
- Let $y = x+\Delta$, for $\Delta > 0$, and $r = \Delta/(x+b)$

$$\mathrm{corr}(F(t,x),F(t,x+\Delta)) = \frac{\sqrt{1+r}}{r}\ln(1+r) > 0$$

- Correlation tends to 1 as $r \downarrow 0$
  - Corresponds to either $\Delta \downarrow 0$
    - ...forwards of same time to maturity
  - ...or to $x \to \infty$
    - ...forwards in the long end are perfectly correlated
- Correlation tends to 0 as $r \to \infty$
  - Corresponds to $\Delta \to \infty$
    - ...forwards are uncorrelated far apart
- Correlation is monotonely decreasing with r
  - ...or, correlation is *increasing* with time-to-maturity x
  - More idiosyncratic risk in short end than in long end of the market

- Simulation example
  - Forward prices in Musiela parametrization $f(t,x)$
- Parameters taken from an empirical study of EEX prices
  - Random field generated as conditional Gaussian field, with variance given by inverse Gaussian
  - Exponential spatial correlation





Conclusions

- Discussed two dynamic HJM approaches for electricity forwards
    - Fixed-delivery modeling $f(t, u)$
    - Market modeling $F(t, \tau_1, \tau_2)$
- Both have their theoretical and practical problems
- Market models as an alternative
- Direct HJM-modelling of forwards
    - Based on ambit fields
    - Martingale condition on kernel function

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