Lecture III: Stationary stochastic models

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Overview of the lecture

1. Examples of weather markets
   - Temperature
   - Wind
2. Continuous-time ARMA models
   - ...with seasonal volatility
   - Empirical analysis of temperature and wind data
3. Pricing of weather futures
   - CAT and wind index futures prices
   - The *modified* Samuelson effect
4. General Lévy semistationary (LSS) models
   - Applications to electricity
   - Futures pricing and relationship to spot

The temperature market

The temperature market

- Chicago Mercantile Exchange (CME) organizes trade in temperature derivatives:
    - Futures contracts on weekly, monthly and seasonal temperatures
    - European call and put options on these futures
- Contracts on several US, Canadian, Japanese and European cities
    - Calgary, Edmonton, Montreal, Toronto, Vancouver, Winnipeg

HDD, CDD and CAT

- HDD (heating-degree days) over a period $[\tau_1, \tau_2]$

$$\int_{\tau_1}^{\tau_2} \max\left(18 - T(u), 0\right) \, du$$

- HDD is the accumulated degrees when temperature $T(u)$ is below 18°C
- CDD (cooling-degree days) is correspondingly the accumulated degrees when temperature $T(u)$ is above 18°C
- CAT = cumulative average temperature
  - Average temperature here meaning the *daily* average

$$\int_{\tau_1}^{\tau_2} T(u) \, du$$

At the CME...

- Futures written on HDD, CDD, and CAT as index
  - HDD and CDD is the index for US temperature futures
  - CAT index for European temperature futures, along with HDD and CDD
- Discrete (daily) measurement of HDD, CDD, and CAT
- All futures are cash settled
  - 1 trade unit=20 Currency (trade unit being HDD, CDD or CAT)
  - Currency equal to USD for US futures and GBP for European
- Call and put options written on the different futures

The wind market

- The US Futures Exchange launched wind futures and options summer 2007
  - ... exchange closed before market started, though...
- Futures on a wind speed index (Nordix) in two wind farm areas
  - Texas and New York
  - Texas divided into 2 subareas, New York into 3
- The Nordix index aggregates the daily *deviation* from a 20 year mean over a specified period
  - Benchmarked at 100
- Futures are settled against this index
  - European calls and puts written on these futures

- Formal definition of the index:

$$N(\tau_1, \tau_2) = 100 + \sum_{s=\tau_1}^{\tau_2} W(s) - w_{20}(s)$$

- $W(s)$ is the wind speed on day s
  - Daily average wind speed
  - Typically measured at specific hours during a day
- $w_{20}(s)$ is the *20-year average* wind speed for day s
- $[\tau_1, \tau_2]$ measurement period, typically a month or a season

Stochastic models for temperature and wind

A continuous-time ARMA(p, q)-process

- Define the Ornstein-Uhlenbeck process $\mathbf{X}(t) \in \mathbb{R}^p$

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p\sigma(t)\,dB(t)\,,$$

- $\mathbf{e}_k$: k'th unit vector in $\mathbb{R}^p$, $\sigma(t)$ "volatility"
- A: $p \times p$-matrix

$$A = \begin{bmatrix} \mathbf{0} & & \mathbf{I} \\ -\alpha_p & \cdots & -\alpha_1 \end{bmatrix}$$

- Explicit solution of $\mathbf{X}(s)$, given $\mathbf{X}(t), s \geq t \geq 0$:

$$\mathbf{X}(s) = \exp\left(A(s-t)\right)\mathbf{X}(t) + \int_t^s \exp\left(A(s-u)\right)\mathbf{e}_p \sigma(u)\, dB(u)\,,$$

- Proof goes by applying the multidimensional Ito Formula on
  - Note: Only one Brownian motion B, and not a multidimensional one

$$f(s, \mathbf{X}(s)) = \exp(As)\mathbf{X}(s)$$

- Matrix exponential defined as:

$$\exp(At) = \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n$$

- Define a continuous-time ARMA(p, q)-process for $p > q \geq 0$
  - Named CARMA(p, q)

$$Y(t) = \mathbf{b}'\mathbf{X}(t)$$

- Vector $\mathbf{b} \in \mathbb{R}^p$ given as

$$\mathbf{b}' = (b_0, b_1, \ldots, b_{q-1}, 1, 0, \ldots)$$

- Special case $q = 0$, $\mathbf{b} = \mathbf{e}_1$: CAR$(p)$-model

$$X_1(t) = \mathbf{e}_1'\mathbf{X}(t)$$

- Y is stationary if and only if A has eigenvalues with negative real part

Why is X_1 a CAR(p) process?

- Consider $p = 3$
- Do an Euler approximation of the $\mathbf{X}(t)$-dynamics with time step 1
  - Substitute iteratively in $X_1(t)$-dynamics
  - Use $B(t+1) - B(t) = \epsilon(t)$
- Resulting discrete-time dynamics

$$X_1(t+3) \approx (3-\alpha_1)X_1(t+2) + (2\alpha_1 - \alpha_2 - 1)X_1(t+1) + (\alpha_2 - 1 + (\alpha_1 + \alpha_3))X_1(t) + \sigma(t)\epsilon(t).$$

- Empirical analysis suggests the following models for temperature and wind:
- Temperature dynamics $T(t)$ defined as

$$T(t) = \Lambda(t) + X_1(t)$$

- Wind dynamics $W(t)$ defined as

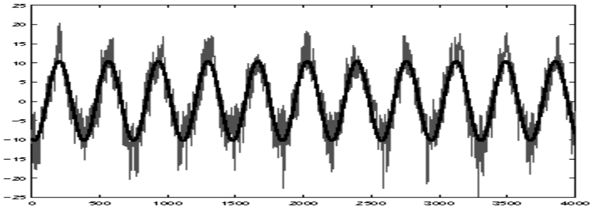
$$W(t) = \exp\left(\Lambda(t) + X_1(t)\right)$$

- $\Lambda(t)$ some deterministic seasonality function

Empirical analysis of temperature and wind data

Empirical study of Stockholm temperature data

- Daily average temperatures from 1 Jan 1961 till 25 May 2006
  - 29 February removed in every leap year
  - 16,570 recordings
- Last 11 years snapshot with seasonal function



- Fitting of model goes stepwise:
  1. Fit seasonal function $\Lambda(t)$ with least squares
  2. Fit AR(p)-model on deseasonalized temperatures
  3. Fit seasonal volatility $\sigma(t)$ to residuals

1. Seasonal function

- Suppose seasonal function with trend

$$\Lambda(t) = a_0 + a_1\, t + a_2 \cos\left(2\pi(t - a_3)/365\right)$$

- Use least squares to fit parameters
  - May use higher order truncated Fourier series
- Estimates: $a_0 = 6.4, a_1 = 0.0001, a_2 = 10.4, a_3 = -166$
  - Average temperature increases over sample period by 1.6°C

2. Fitting an auto-regressive model

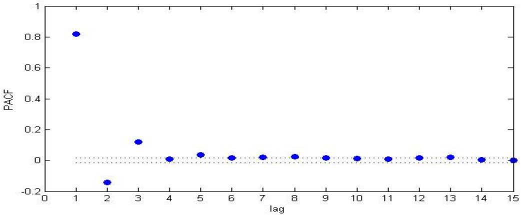
- Remove the effect of $\Lambda(t)$ from the data

$$Y_i := T(i) - \Lambda(i)\,, i = 0, 1, \ldots$$

- Claim that AR(3) is a good model for Y_i:

$$Y_{i+3} = \beta_1 Y_{i+2} + \beta_2 Y_{i+1} + \beta_3 Y_i + \sigma_i \epsilon_i\,,$$

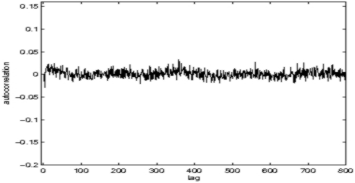
- The partial autocorrelation function for the data suggests AR(3)

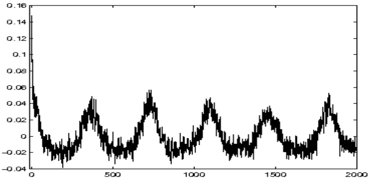


- Estimates $\beta_1 = 0.957, \beta_2 = -0.253, \beta_3 = 0.119$ (significant at 1% level)
- R^2 is 94.1% (higher-order AR-models did not increase R^2 significantly)

3. Seasonal volatility

- Consider the residuals from the AR(3) model
- Close to zero ACF for residuals
- Highly seasonal ACF for *squared* residuals



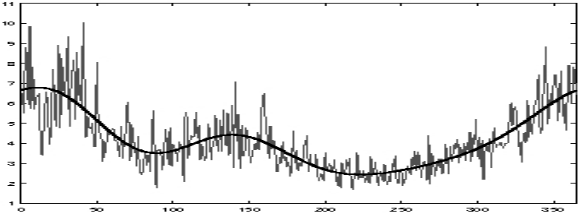


- Suppose the volatility is a truncated Fourier series

$$\sigma^2(t) = c + \sum_{i=1}^{4} c_i \sin(2i\pi t/365) + \sum_{j=1}^{4} d_j \cos(2j\pi t/365)$$

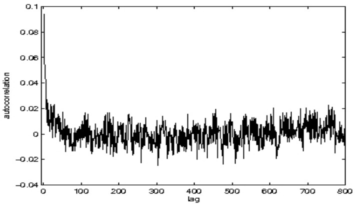
- This is calibrated to the daily variances
    - 45 years of daily residuals
    - Line up each year next to each other
    - Calculate the variance for each day in the year

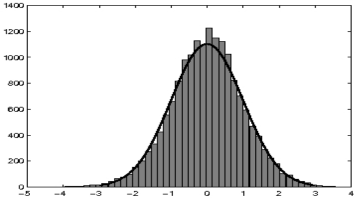
- A plot of the daily empirical variance with the fitted squared volatility function
- High variance in winter, and early summer
- Low variance in spring and late summer/autumn



- Similar observations in other studies
  - Several cities in Norway and Lithuania
  - Calgary and Toronto: Swishchuk and Cui (2013)
  - German and Asian cities: Benth, Härdle and Lopez-Cabrera (2011,2012)
  - Seasonality in ACF for squared residuals observed in Campbell and Diebold (2005) for several US cities

- Dividing out the seasonal volatility from the regression residuals
- ACF for squared residuals non-seasonal
  - ACF for residuals unchanged
  - Residuals become (close to) normally distributed



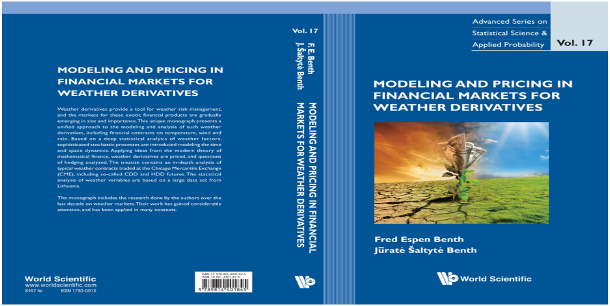


- Conclusion: fitted an AR(3)-model with seasonal variance to deseasonalized daily temperatures
- Apply the link between CAR(3) and AR(3) to derive the continuous-time parameters α_1, α_2 and α_3

$$\alpha_1 = 2.043, \alpha_2 = 1.339, \alpha_3 = 0.177$$

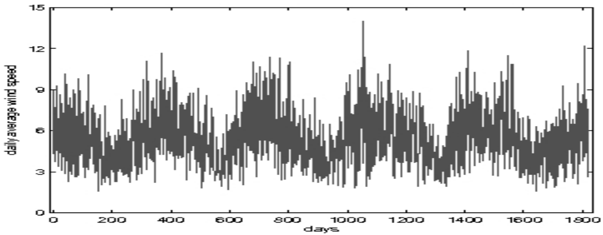
- Seasonality Λ and variance σ given
- The fitted CAR(3)-model is stationary (to a normal distribution)
  - Eigenvalues of A have negative real parts

Commercial break



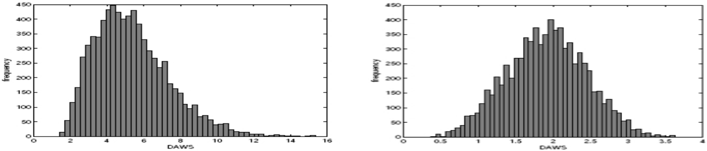
Empirical study of New York wind speed data

- Daily average wind speed data from New York wind farm region 1 from Jan 1 1987 till Sept 7 2007.
- 7,550 daily recordings, after leap year data were removed
- Figure shows 5 years from 1987



- Fitting wind speed model to data follows (almost) the same scheme as temperature
  1. Logarithmic transformation of data to symmetrize
  2. Fit seasonal function
  3. Find AR(p)-model to deseasonalized data
  4. Find volatility structure of residuals

1. Symmetrization of data



- Wind speed histogram (left), logarithmic transformed speeds (right)

2. Seasonal function

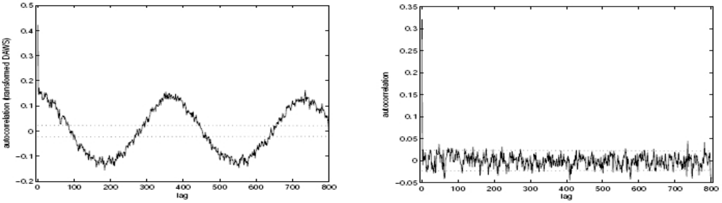
- Seasonality function with annual and biannual periodicity

$$\Lambda(t) = a_0 + a_1\cos(2\pi t/365) + a_2\sin(2\pi t/365) + a_3\cos(4\pi t/365) + a_4\sin(4\pi t/365)$$

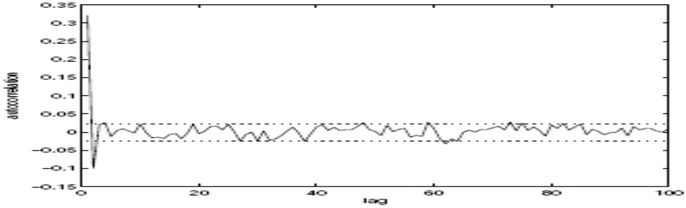
- Nonlinear least squares (using matlab) on transformed data gives

$$a_0 = 1.91, a_1 = 0.26, a_2 = 0.08, a_3 = -0.04, a_4 = -0.07$$

- Consider the ACF *before* and *after* estimated seasonality has been removed
- We see (right plot) that the ACF of deseasonalized data does not show any periodic pattern



3. Fitting an AR(p)-model



- Partial ACF for deseasonalized data suggests a higher-order AR(MA) structure
  - AR(4) best according to Akaike's Information Criterion
  - ...best among ARMA($p \leq 5, q \leq 5$)

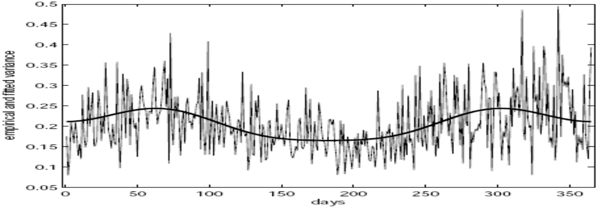
- Estimated regression parameters in the AR(4) model

$$z_t = \beta_1 z_{t-1} + \beta_2 z_{t-2} + \beta_3 z_{t-3} + \beta_4 z_{t-4}$$

$$\beta_1 = 0.355, \beta_2 = -0.104, \beta_3 = 0.010, \beta_4 = 0.027$$

- All except β_3 are found to be significant

4. Volatility structure



- Estimated daily empirical variance, and fitted a truncated Fourier series
    - ...as for temperature

$$\sigma^2(t) = c_0 + \sum_{k=1}^{3} c_k \cos(2\pi kt/365)$$

- Estimated parameters (nonlinear least squares)

$$c_0 = 0.208, c_1 = 0.033, c_2 = -0.019, c_3 = -0.010$$

Relation to CAR(4)-model $X_1(t)$

- Using Euler approximation on dynamics of $X_1(t)$

$$X_1(t) \approx (4-\alpha_1)X_1(t-1) + (3\alpha_1 - \alpha_2 - 6)X_1(t-2) + (4 + 2\alpha_2 - \alpha_3 - 3\alpha_1)X_1(t-3) + (\alpha_3 - \alpha_4 - \alpha_2 + \alpha_1 - 1)X_1(t-4)$$

- Knowing the β's yield

$$\alpha_1 = 3.645, \alpha_2 = 5.039, \alpha_3 = 3.133, \alpha_4 = 0.712$$

- Eigenvalues of A have negative real part, thus stationary dynamics

Weather futures pricing

CAT temperature futures

- CAT-futures price $F_{\mathrm{CAT}}(t, \tau_1, \tau_2)$ at time $t \leq \tau_1$
  - No trade in settlement period

$$F_{\mathrm{CAT}}(t, \tau_1, \tau_2) = \mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2} T(u)\, du \,|\, \mathcal{F}_t\right]$$

- Constant interest rate r, and settlement at the end of index period, τ_2
- Q is the pricing measure
  - Not unique since market is incomplete
  - Temperature is not tradeable!

A class of risk neutral probabilities

- Parametric sub-class of risk-neutral probabilities Q^θ
- Defined by Girsanov transformation of $B(t)$

$$dB^\theta(t) = dB(t) - \theta(t)\,dt$$

- $\theta(t)$ deterministic *market price of risk*
- Dynamics of $\mathbf{X}(t)$ under Q^θ:

$$d\mathbf{X}(t) = (A\mathbf{X}(t) + \mathbf{e}_p\sigma(t)\theta(t))\ dt + \mathbf{e}_p\sigma(t)\,dB^\theta(t)\,.$$

- $X_1(s) = \mathbf{e}_1'\mathbf{X}(s)$ conditioned on $\mathbf{X}(t) = \mathbf{x}$, $t \leq s$ is normally distributed under Q^θ
- Mean:

$$\mu_\theta(t,s,\mathbf{x}) = \mathbf{e}_1' \exp(A(s-t))\mathbf{x} + \int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p \sigma(u)\theta(u)\,du$$

- Variance:

$$\Sigma^2(t,s) = \int_t^s \sigma^2(u)\{\mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\}^2\,du$$

- CAT-futures price

$$
\begin{aligned}
F_{\mathrm{CAT}}(t,\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} \Lambda(u)\,du + \mathbf{a}(t,\tau_1,\tau_2)\mathbf{X}(t) \\
&\quad + \int_t^{\tau_1} \theta(u)\sigma(u)\mathbf{a}(t,\tau_1,\tau_2)\mathbf{e}_p\,du \\
&\quad + \int_{\tau_1}^{\tau_2} \theta(u)\sigma(u)\mathbf{e}_1' A^{-1}\left(\exp\left(A(\tau_2-u)\right) - I_p\right)\mathbf{e}_p\,du
\end{aligned}
$$

with I_p being the $p \times p$ identity matrix and

$$
\mathbf{a}(t,\tau_1,\tau_2) = \mathbf{e}_1' A^{-1}\left(\exp\left(A(\tau_2-t)\right) - \exp\left(A(\tau_1-t)\right)\right)
$$

- Time-dynamics of F_{CAT} (applying Ito's Formula)

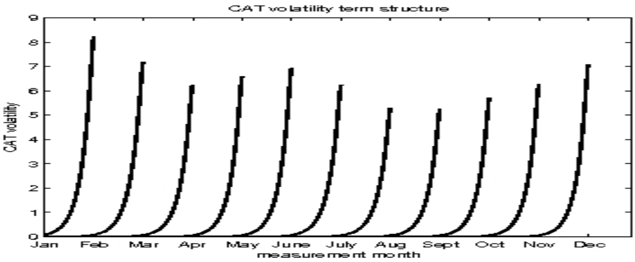
$$dF_{\mathrm{CAT}}(t,\tau_1,\tau_2)=\Sigma_{\mathrm{CAT}}(t,\tau_1,\tau_2)\,dB^{\theta}(t)$$

where

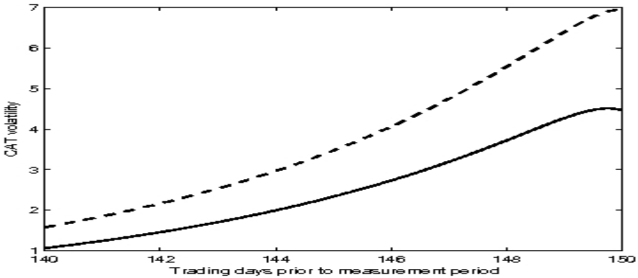
$$\Sigma_{\mathrm{CAT}}(t,\tau_1,\tau_2)=\sigma(t)\mathbf{e}_1'A^{-1}\left(\exp\left(A(\tau_2-t)\right)-\exp\left(A(\tau_1-t)\right)\right)\mathbf{e}_p$$

- Σ_{CAT} is the CAT volatility term structure

- Seasonal volatility, with maturity effect
- Plot of the volatility term structure as a function of t up to start of measurement period
  - Monthly contracts
  - Parameters taken from Stockholm for CAR(3)



- The Samuelson effect
  - The volatility is decreasing with time to delivery
  - Typical in mean-reverting markets
- AR(3) has memory
  - Implies a modification of this effect
  - Plot shows volatility of CAT with monthly vs. weekly measurement period



- Estimation of the market price of risk θ
  - Necessary for option pricing
  - Constant, or time-dependent?
- Calibrate theoretical futures curve to observed

$$\min_{\theta} \sum_i |F_{\mathrm{IND}}(0, \tau_1^i, \tau_2^i) - \widehat{F}_{\mathrm{IND}}^i|^2$$

- IND=HDD, CDD, CAT
- Empirical study for Berlin: see recent paper by Härdle and Lopez Cabrera (2012)

Wind futures pricing

- Recall the Nordix index for wind speed

$$N(\tau_1, \tau_2) = 100 + \sum_{s=\tau_1}^{\tau_2} W(s) - w_{20}(s)$$

- Arbitrage-free pricing dynamics (analogous to temperature futures)

$$\begin{aligned} F(t, \tau_1, \tau_2) &= \mathbb{E}_Q \left[N(\tau_1, \tau_2) \,|\, \mathcal{F}_t \right] \\ &= 100 + \sum_{s=\tau_1}^{\tau_2} \mathbb{E}_Q \left[W(s) \,|\, \mathbf{X}(t) \right] - w_{20}(s) \end{aligned}$$

- Choose $Q = Q^\theta$ as for temperature futures

- Calculation of futures price:

$$f(t,s) \triangleq \mathbb{E}_{Q^\theta}\left[W(s) \mid \mathcal{F}_t\right] = \exp\left(\Lambda(s) + \mu_\theta(t,s,\mathbf{X}(t)) + \frac{1}{2}\Sigma^2(t,s)\right)$$

- Recalling μ_θ and $\Sigma(t,s)$ from the temperature calculations
- Dynamics of $f(t,s)$ (using Ito's Formula again)

$$\frac{df(t,s)}{f(t,s)} = \sigma(t)\left\{\mathbf{e}_1' \exp(A(s-t))\mathbf{e}_p\right\} dB^\theta(t)$$

- The term $v^2(t,s) = \mathbf{e}_1' \exp(A(s-t))\mathbf{e}_p$ models the *modified* Samuelson effect
- Consider $p = 1$, i.e., AR(1)-model

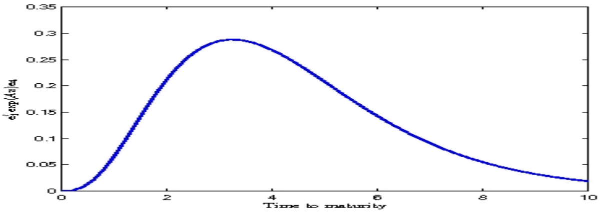
$$v^2(t,s) = \mathbf{e}_1' \exp(A(s-t))\mathbf{e}_1 = \exp(-\alpha_1(s-t))$$

- When $s \downarrow t$, $v^2(s,t) \to 1$
  - $v^2(s,t)$ increases to 1 when "time-to-maturity" $s-t$ goes to zero
  - Samuelson effect again...
- $v^2(s,t)$ is the scaling of volatility, which goes to 1 in the AR(1)-case

- Consider $p > 1$

$$\lim_{s \downarrow t} v^2(s,t) = \mathbf{e}_1' \mathbf{I} \mathbf{e}_p = 0$$

- Volatility of f is scaled to zero when "time-to-maturity" goes to zero
- The uncertainty of the futures price $f(t,s)$ goes to zero close to maturity!
  - ...and *not* at its maximum as for AR(1)-models
  - ...which has the Samuelson effect



- AR(4) means that wind speed has a memory up to 4 days
- Close to maturity we can predict the wind speed at maturity very good
  - ...which obviously reduces the uncertainty

LSS processes

Definition of LSS process

$$Y(t) = \int_{-\infty}^{t} g(t-s)\sigma(s)\,dL(s)$$

- L a (two-sided) Lévy process (with finite variance)
- σ a stochastic volatility process
- g kernel function defined on $\mathbb{R}_+$
- Integration in semimartingale (Ito) sense
  - σ typically assumed to be independent of L, with finite variance and stationary
  - usually σ is again an LSS process....
  - g square-integrable on $\mathbb{R}_+$
- Y is stationary whenever σ is

Models of temperatures and wind *in stationarity*

- Temperature model

$$T(t) = \Lambda(t) + \int_{-\infty}^{t} g(t-s)\sigma(s)\, dB(s)$$

- σ deterministic *seasonal volatility*, Λ seasonal mean function, B Brownian motion, and

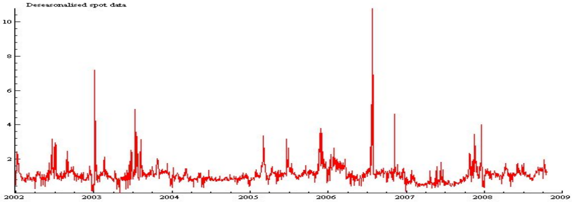
$$g(u) = \mathbf{e}_1' \mathrm{e}^{Au} \mathbf{e}_3$$

- Stochastic model for New York daily averaged wind speeds

$$W(t) = \exp\left(\Lambda(t) + \int_{-\infty}^{t} g(t-s)\sigma(s)\, dB(s)\right)$$

- g is a CAR(4)-kernel

Electricity spot



- Electricity spot given by an arithmetic two-factor model (B., Kluppelberg, Müller and Vos, 2011)

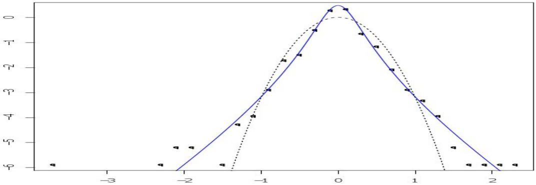
$$S(t) = \Lambda(t) + X(t) + \int_{-\infty}^{t} g(t-s)\, dL(s)$$

- $g(u) = (b_0, 1)e^{Au}\mathbf{e}_2$, CARMA(2,1)-kernel, L a stable Lévy process
- X long-term factor modelled as a NIG Lévy process

- Recent paper by Barndorff-Nielsen, B., and Veraart (2013):

$$\ln S(t) = \Lambda(t) + \int_{-\infty}^{t} g(t-s)\sigma(s)\,dB(s)$$

- EEX: $Y(t) := \ln S(t) - \Lambda(t)$:
  - is stationary with p-value smaller than 0.01 (augmented Dickey-Fuller unit root text)
- Deseasonalized data has a normal inverse Gaussian (NIG) stationary distribution (generalized hyperbolic)



- Question: How to choose σ and g such that $Y \sim$GH?
- Assume a "gamma"-kernel g: For $\lambda > 0$ and $\frac{1}{2} < \nu < 1$,

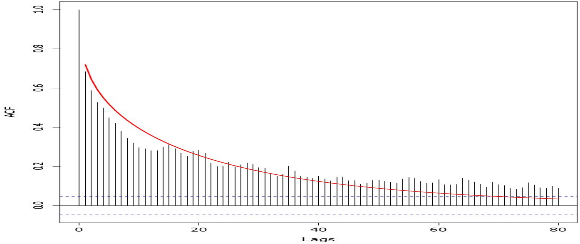
$$g(u) \sim u^{\nu-1} \exp(-\lambda u)$$

- $\sigma^2(t)$ chosen as LSS process again

$$\sigma^2(t) = \int_{-\infty}^{t} h(t-s)\, dU(s)\,, h(t) \sim t^{1-2\nu} \mathrm{e}^{-\lambda t}$$

- U a subordinator process
  - specificied so that $\sigma^2(t)$ has generalized inverse Gaussian stationary distribution
- Idea in construction:
  - Separately model stationary distribution and ACF structure (and stochastic volatility)

- Fitted ACF function vs. empirical
    - EEX deseasonalized log-spot price data



Forward pricing under LSS models

- Focus on the case of power
- Forward price of a contract delivering electricity spot $S(t)$ over the time interval $[\tau_1, \tau_2]$

$$F(t,\tau_1,\tau_2) = \mathbb{E}_Q\left[\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} S(\tau)\,d\tau \Big| \mathcal{F}_t\right] = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} f(t,\tau)\,d\tau$$

  - Weather: In case of PRIM index, $S(\tau)$ is temperature at time τ

- Q chosen via the Esscher transform (or Girsanov for Brownian models)
  - Measure change only for positive times, $t \geq 0$
  - Preserves independent increment property (and Lévy property for constant θ)

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta(s)\, dL(s) - \int_0^t \phi_L(\theta(s))\, ds\right)$$

- ϕ_L log-moment generating function of L
  - supposed to exist
- θ market price of risk
  - to be estimated/calibrated
- Similar change of measure for the stochastic volatility σ

Theorem

The forward price is

- *Geometric LSS case*

$$f(t,\tau) = \Lambda(\tau)\mathbb{E}_Q\left[\exp\left(\int_t^\tau \phi_L^Q(g(\tau-u)\sigma(u))\,du\right) \mid \mathcal{F}_t\right] \times \exp\left(\int_{-\infty}^t g(\tau-u)\sigma(u)\,dL(u)\right)$$

- *Arithmetic LSS case*

$$f(t,\tau) = \Lambda(\tau) + \int_{-\infty}^t g(\tau-u)\sigma(u)\,dL(u) + \mathbb{E}_Q[L(1)]\int_t^\tau g(\tau-u)\mathbb{E}_Q[\sigma(u)\mid\mathcal{F}_t]\,du$$

Proof(outline) Split into

$$\int_{-\infty}^{\tau} g(\tau-u)\sigma(u)\,dL(u) = \int_{-\infty}^{t} g(\tau-u)\sigma(u)\,dL(u) + \int_{t}^{\tau} g(\tau-u)\sigma(u)\,dL(u)$$

1. Apply $\mathcal{F}_t$-measurability on the first integral on the RHS.
2. Condition on σ using independence
3. Apply the tower property of conditional expectation.

- Note: Spot and forward...

$$(\ln)S(t) \sim \int_{-\infty}^{t} g(t-u)\sigma(u)\,dL(u) = Y(t)$$

$$(\ln)f(t,T) \sim \int_{-\infty}^{t} g(\tau-u)\sigma(u)\,dL(u) := Y(t,T-t)$$

- Analyse the spot-forward connection by Laplace transform
  - Let $x = \tau - t$, time-to-maturity
  - Suppose integral from zero

$$\int_0^\infty \int_0^t g(x+t-s)\sigma(s)\,dL(s)\mathrm{e}^{-\theta t}\,dt$$
$$= \mathcal{L}(g(\cdot+x))(\theta)\int_0^\infty \mathrm{e}^{-\theta s}\sigma(s)\,dL(s)$$

- Suppose there exists some "nice" $h(t,x)$ such that

$$\mathcal{L}(g(\cdot+x))(\theta)=\mathcal{L}(h(\cdot,x))(\theta)\mathcal{L}(g)(\theta)$$

- Forward price becomes a *weighted average of past* spot prices

$$Y(t,x)=\int_0^t h(t-s,x)Y(s)\,ds$$

- LSS processes Y have a memory (moving-average process)
  - Forward prices depends on past and present spot prices....
  - ...and not only the present spot price!

Case I: CARMA(p,0)-kernel

- Recall $g(u) = \mathbf{e}_1 \mathrm{e}^{Au} \mathbf{e}_p$

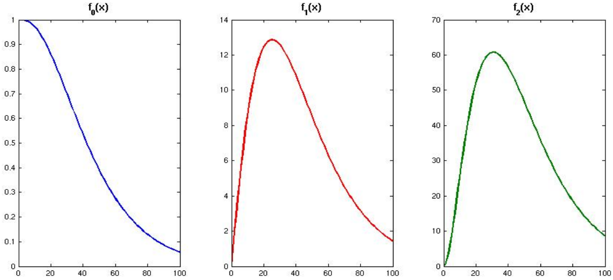
$$Y(t,x) = \sum_{i=1}^{p} f_i(x) Y^{(i-1)}(t)$$

- $Y^{(k)}$ kth derivative
  - LSS with CAR(p)-kernel has $p-1$-times continuously differentiable paths
  - Implied by g being differentiable of all orders, $g^{(k)}(0) = 0$ for $k \leq p-1$ and semimartingale representation of $Y(t)$.

- Forward curve shapes $f_i(x)$, $i = 0, 1, \ldots, p-1$

$$f_i(x) = \mathbf{e}_1' \mathrm{e}^{Ax} \mathbf{e}_{i+1}$$

- Plot of the three first



Case II: gamma kernel

- Recall $g(u) \sim u^{\nu-1}\exp(-\lambda u)$, $0.5 < \nu < 1$
- We obtain

$$Y(t,x) = \int_0^t h(t-s,x) Y(s)\, ds$$

for

$$h(t,x) \sim \left(\frac{x}{t}\right)^{\nu} \frac{1}{x+t} e^{-\lambda(t+x)}$$

- Forward price is a weighted average of past spot prices

Conclusions

- CAR(p) model for the daily temperature and wind speed dynamics
  - Auto-regressive process, with
  - Seasonal mean
  - seasonal volatility
- Allows for analytical futures prices
  - HDD/CDD, and CAT temperature futures
  - Nordix wind futures
  - Futures contracts with "delivery" over months or seasons
  - Seasonal volatility with a modified Samuelson effect: volatility may even decrease close to maturity
- Problem: understand the market price of weather risk

- General stationary models: LSS processes
  - Includes CARMA processes
  - Extends to more general mean-reversion dynamics
- Forward pricing under LSS
  - Forward expressable as an average of past spot prices
  - CARMA: factor shapes associated to the spot and its derivatives

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