Social Discounting and the Long Rate of Interest

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1. Social discounting

A recent article appearing in the *Financial Times* reports that Andrew Haldane, director of financial stability at the Bank of England, while addressing a conference on the role of higher education in boosting the economy, told delegates the following:

> We know that financial markets discount rather too heavily projects with a long life that yield returns in the distant future, to the extent that some of those projects may not be initiated in the first place.

Haldane's remarks are indicative of the importance of the unresolved issues—and indeed, the ongoing debates—concerning the form of the discount function that should be used in the cost/benefit analysis of proposals for long-term projects carried out for the benefit of society.

At the heart of the matter is the apparent inadequacy of the standard discounted utility model (which dates back to early work of Samuelson) as a basis for rational decision making when the beneficiaries of the future consumption are not the same as the beneficiaries of present consumption, and whose needs cannot or should not be neglected.

The use of the familiar exponential discount function for this purpose is problematic, since even for small values of the exponential discount rate the corrosive effect of continuous compounding is to reduce the present value of even substantial benefits secured for the distant future to very little.

As a consequence various alternative proposals as to how long-term social discounting should be carried out have been put forward.

In essence it seems that for the valuation of social projects some form of "hyperbolic" discounting is required, where the discount rate is a decreasing function of the time at which the benefits are received, with the effect of enhancing the importance of benefits accruing to the remote future.

But what is the justification for such an approach, and does it make good sense scientifically?

Numerous authors have contributed to this discussion, including for example Arrow *et al*. (1996), Azfar (1999), Gollier (2002a,2002b), Groom *et al*. (2005), Harvey (1986,1994), Henderson & Bateman (1995), Jouini *et al*. (2010), Lengwiler (2005), Laibson (1997), Lind (1997), Nocetti *et al*. (2008), Reinschmidt (2002), Schelling (1995), Strota (1956), Weitzman (1998,2001), and Yao (1999a,b), to name a few.

Social discounting

Apart from normative considerations regarding society as a whole, the view has also been put forward that social discounting might arise in part as a byproduct of the effects of aggregation in a heterogeneous population.

To see how this works, we construct the following simple model, which gives some insight into the nature of hyperbolic discounting.

Let R be a random variable taking values in $\mathbb{R}^+$, and consider the random discount function given by $\{\mathrm{e}^{-Rt}\}_{t\geq 0}$.

We interpret R as representing the discount rate associated with an individual chosen at random in a large heterogeneous population.

Then $t \mapsto \mathbb{E}[\mathrm{e}^{-Rt}]$ is a decreasing function on $\mathbb{R}^+$ taking the value unity at $t = 0$, and we can think of

$$P_{0t} = \int_0^\infty \mathrm{e}^{-rt}\mu(\mathrm{d}r) \tag{1}$$

as representing the "aggregate" discount function defined by the population.

Here $\mu(\mathrm{d}r) = \mathbb{P}(R \in \mathrm{d}r)$ is the probability measure on $\mathbb{R}^+$ associated with the random variable R.

Thus we can regard R as representing the various views held in the population as to what the appropriate rate of discount should be, and $\{P_{0t}\}_{t\geq 0}$ as the aggregate discount function obtained by averaging in an appropriate sense over the views of the various market participants.

There is, of course, no *a priori* reason why individuals should exhibit a strictly constant exponential rate of discounting, except perhaps a desire for time consistency, but this is a simplifying assumption of the argument.

Then depending on the distribution of R, we obtain the associated aggregate discount function. For example, suppose that

$$\mu(\mathrm{d}r) = \sum_i p_i \delta_{r_i}(\mathrm{d}r). \tag{2}$$

Here $\delta_{r_i}(\mathrm{d}r)$ is the Dirac measure centred at r_i, $i = 1, 2, \ldots, n$, and $p_1, p_2, \cdots, p_n$ are nonnegative weights satisfying $\sum_i p_i = 1$. Then we have

$$P_{0t} = \sum_i p_i \mathrm{e}^{-r_i t}. \tag{3}$$

A calculation shows that the associated asymptotic rate (long rate) is:

$$r_\infty = -\lim_{t\to\infty} \frac{1}{t} \ln P_{0t} = \min_i \, r_i. \tag{4}$$

We see that the aggregation of any finite number of exponential discounters is asymptotically exponential, and that the asymptotic rate is given by the *minimum* of the various individual rates under consideration. Weitzman (1998) argued on that basis that the far-distant future should be discounted at the lowest possible rate.

On the other hand, suppose we model R by setting

$$\mu(\mathrm{d}r) = \mathbb{1}\{r > 0\} L^{-1} \mathrm{e}^{-r/L} \mathrm{d}r \tag{5}$$

for some mean rate $L > 0$. Then we find that

$$P_{0t} = \frac{1}{1 + Lt}. \tag{6}$$

In other words, the effect of spreading the discount rate by use of an exponential distribution is that the resulting aggregate discount function is of the *hyperbolic* type, with a fixed "simple" (or Libor) rate L.

Thus, if we know that the market consists of exponential discounters, and if all we know, beyond that, is that their mean rate of discount is L, then from an information-theoretic perspective the best (or least biased) model is a discount function of the *hyperbolic* type indicated above.

As another example of such probability-weighted discounting (Brody & Hughston 2001, 2002; Weitzman 2001), consider the case for which R has a gamma distribution:

$$\mu(\mathrm{d}r) = \mathbb{1}\{r > 0\} \frac{1}{\Gamma[\lambda]} \theta^{\lambda} r^{\lambda - 1} \mathrm{e}^{-\theta r} \mathrm{d}r, \tag{7}$$

where $\theta, \lambda > 0$.

A calculation shows that the discount function takes the form of a Pareto tail distribution, given by $P_{0t} = [\theta/(\theta + t)]^{\lambda}$.

Then if we set $\theta = \lambda/L$ we obtain

$$P_{0t} = \left[\frac{1}{1 + (Lt/\lambda)} \right]^{\lambda}. \tag{8}$$

Thus we obtain a discount function of the so-called *generalised hyperbolic* type (Harvey 1986, 1994; Loewenstein & Prelec 1992).

For each maturity date $t > 0$ we have a flat term structure with a constant annualised rate of interest L, assuming compounding at the frequency λ over the life of the bond (λ need not be an integer).

Interest rates models for social discounting?

This leads us to ask whether one can construct interest rate models from a modern perspective that incorporate the principles of social discounting.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathbb{P}$ denotes the real-world measure, and $\{\mathcal{F}_t\}$ is taken to satisfy the "usual conditions".

Price processes are modelled by $\{\mathcal{F}_t\}$-semimartingales and are assumed to have the càdlàg property.

We assume the existence of an established numeraire currency in units of which valuations are expressed.

We assume that the market is arbitrage-free, but not necessarily complete.

To ensure the absence of arbitrage we assume the existence of an established pricing kernel $\{\pi_t\}_{t\geq 0}$ satisfying $\pi_t > 0$ for $t \geq 0$, and with the property that if an asset with value process $\{S_t\}_{t\geq 0}$ delivers a single random cash flow H_T at T, and derives its value from that cash flow, then its value at $t < T$ is given by

$$S_t = \frac{1}{\pi_t}\,\mathbb{E}_t[\pi_T H_T], \tag{9}$$

and for $t \geq T$ its value is zero.

In the case of a discount bond (or zero-coupon bond) which generates a single cash flow of unity at T, the price at t is thus given by

$$P_{tT} = \frac{1}{\pi_t}\,\mathbb{E}_t[\pi_T] \tag{10}$$

for $t < T$ and $P_{tT} = 0$ for $t \geq T$, with $\lim_{t \to T} P_{tT} = 1$.

Then for each fixed $T \geq 0$ the bond price process $\{P_{tT}\}$ is defined for all $t \geq 0$.

The initial bond price is given by P_{0T}.

The asymptotic properties of the discount bond system for large T are best pursued by consideration of the various interest rates associated with it.

We recall the relevant definitions.

The so-called "exponential" (or continuously compounded) rate R_{tT} (expressed on an annualised basis) is defined for $0 \leq t < T$ by

$$R_{tT} = -\frac{1}{T-t}\ln P_{tT} \quad \text{or} \quad P_{tT} = \exp\left[-(T-t)R_{tT}\right]. \tag{11}$$

The rate R_{tT} can be interpreted as the yield, at time t, expressed on a continuously compounded basis, offered by a zero-coupon bond of maturity T.

Next, we introduce the so-called "simple" (or "Libor") rates L_{tT} for $0 \leq t < T$, which are quoted on an annualised basis, by setting

$$L_{tT} = \frac{1}{T-t}\left(\frac{1}{P_{tT}} - 1\right) \quad \text{or} \quad P_{tT} = \frac{1}{1+(T-t)L_{tT}}. \tag{12}$$

In general, the relation between L_{tT} and R_{tT} is tenor dependent.

In particular, we have

$$R_{tT} = \frac{1}{T-t}\ln\left(1+(T-t)L_{tT}\right) \quad \text{or} \quad L_{tT} = \frac{1}{T-t}\left(\mathrm{e}^{(T-t)R_{tT}} - 1\right). \tag{13}$$

The implication of this tenor dependence is that while for fixed finite tenor the relation between these rates is monotonic, this ceases to be the case in the limit of very long maturity.

We observe in particular that in the limit of large tenor the asymptotic properties of the long exponential rate and the long simple rate differ.

If $R_{t\infty}$ is finite and greater than zero, then $L_{t\infty}$ is infinite, whereas if $L_{t\infty}$ is finite and greater than zero, then $R_{t\infty}$ is zero.

Asymptotic properties

More specifically, the long exponential rate is defined by $R_{t\infty} := \lim_{T\to\infty} R_{tT}$, if this limit exists.

Similarly, we define the long simple rate (or long Libor rate) by $L_{t\infty} := \lim_{T\to\infty} L_{tT}$, if this limit exists.

The limiting rates are understood as taking values in the extended real numbers.

There is a problematic feature of the long exponential rate process that has attracted a good deal of attention in connection with the analysis of long-term investment.

This is the so-called Dybvig-Ingersoll-Ross theorem [2], the continuous-time version of which can be stated as follows:

Proposition 1. (Long exponential rates can never fall) *If* $R_{t\infty} < \infty$ *for all* $t \geq 0$*, then for all* s, t *such that* $0 \leq s \leq t$ *it holds that* $R_{t\infty} \geq R_{s\infty}$*.*

The DIR theorem implies that it is impossible to construct viable "long rate" models, where the long exponential rate of interest acts as a state variable.

This is born out by the fact that in many of the well known interest rate models the long exponential rate is simply a constant.

But the DIR theorem is not applicable to long Libor rates, and the behaviour of such rates is very different.

This is in fact already apparent in deterministic interest rate systems.

First we note the following:

Proposition 2. *In a deterministic interest-rate model, if the long exponential rate exists and is finite, then it is constant.*

Proof. By the definition of exponential rates we have $P_{tT} = \exp[-(T-t)R_{tT}]$ for $0 \leq t < T < \infty$ and $P_{0t} = \exp[-tR_{0t}]$ for $t \geq 0$.

In the absence of arbitrage, in the case of a deterministic interest-rate system we have $P_{tT} = P_{0T}/P_{0t}$.

It follows that

$$R_{tT} = \frac{TR_{0T} - tR_{0t}}{T-t}. \tag{14}$$

Assuming that the limit $R_{0\infty} = \lim_{T\to\infty} R_{0T}$ exists and that $R_{0\infty} < \infty$, we see that $R_{t\infty} = R_{0\infty}$ for all $t \geq 0$. □

On the other hand, the Libor rate system is "asymptotically free".

More precisely, we have the following:

Proposition 3. *In a deterministic interest-rate system, if the long Libor rate exists and is finite, then it is given by* $L_{t\infty} = P_{0t}L_{0\infty}$*, where* $L_{0\infty} = 1/(\lim_{t\to\infty} tP_{0t})$*.*

Proof. By the definition of the Libor (or simple) interest-rate system we have the relations $P_{tT} = 1/[1 + (T - t)L_{tT}]$ for $0 \le t < T < \infty$ and $P_{0t} = 1/[1 + tL_{0t}]$ for $t \ge 0$.

In the absence of arbitrage, in the case of a deterministic interest-rate system we have $P_{tT} = P_{0T}/P_{0t}$.

It follows that

$$L_{tT} = \frac{1}{T-t}\left[\frac{TL_{0T} - tL_{0t}}{1 + tL_{0t}}\right]. \tag{15}$$

If the limit $L_{0\infty} = \lim_{T\to\infty} L_{0T} = 1/(\lim_{t\to\infty} tP_{0t})$ exists and is finite, then $L_{t\infty} = \lim_{T\to\infty} L_{tT} = L_{0\infty}/(1 + tL_{0t})$ for all $t \ge 0$, and we deduce that $L_{t\infty} = P_{0t}L_{0\infty}$. □

Interest Rate Models for Social Discounting

Thus even in a deterministic model the long Libor rate process is dynamic, and indeed it contains the information of the entirety of the initial term structure.

This leads us to ask for general conditions on the pricing kernel sufficient to ensure that the resulting interest-rate system is socially efficient.

Suppose we wish to construct models for which the discount functions are asymptotically generalised hyperbolic (or tail-Pareto) with parameter $\lambda > 0$.

This notion can be formalised more precisely as follows:

Definition. *A pricing kernel* $\{\pi_t\}_{t\geq 0}$ *satisfying* $\lim_{t\to\infty} \mathbb{E}[\pi_t] = 0$ *will be said to be socially efficient with index* $\lambda > 0$ *if it holds that* $\mathrm{(a)}\ \lim_{t\to\infty} t^{\lambda}\pi_t > 0$ *and* $\mathrm{(b)}\ \lim_{t\to\infty} \mathbb{E}[t^{\lambda}\pi_t] < \infty$*.*

Then we have:

Proposition 4. (Socially-efficient discount bond systems.) *If a pricing kernel is socially efficient with index* $\lambda > 0$*, then for all* $t \geq 0$ *the associated discount bond system satisfies*

$$0 < \lim_{T\to\infty} T^{\lambda} P_{tT} < \infty. \tag{16}$$

Examples of Dynamic Social Discounting Models

In fact, it turns out that one can construct rather explicit examples of dynamic models admitting generalized hyperbolic long-rate structures.

For simplicity, we consider the hyperbolic case ($\lambda = 1$).

The generalization to other values of λ is straightforward.

The models that we shall consider have the property that the resulting long Libor rate processes are fully dynamic, and can be used as state variables.

We proceed as follows.

Let us write Γ^+ for the space of strictly positive functions $f : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ such that $t \in \mathbb{R}^+ \mapsto f_t \in \mathbb{R}^+ \setminus \{0\}$, with the property that $\{f_t\}_{t\geq 0} \in \mathrm{C}^1$.

Let $\{M_t\}$ be a positive martingale normalised to unity at $t = 0$.

Let $\{a_t\}$, $\{b_t\}$ be elements of Γ^+ satisfying $\lim_{t\to\infty} t a_t = a$, $\lim_{t\to\infty} t b_t = b$ for finite a, b such that $a + b > 0$.

Let the initial discount function $P_{0t} = a_t + b_t$ be given for $t \geq 0$.

Then we have:

Proposition 5. (Existence of long-rate state-variable models.) *The pricing kernel defined by* $\pi_t = a_t + b_t M_t$ *determines an arbitrage-free one-factor interest-rate model, for which one can choose the relevant state variable to be either the short rate, given by*

$$r_t = -\frac{a'_t + b'_t M_t}{a_t + b_t M_t}, \qquad (17)$$

or alternatively the long simple rate, given by

$$L_{t\infty} = \frac{a_t + b_t M_t}{a + b M_t}. \qquad (18)$$

Proof. Under the stated assumptions we find that the discount bond system takes the form

$$P_{tT} = \frac{a_T + b_T M_t}{a_t + b_t M_t}. \qquad (19)$$

A calculation using the relation $r_t = -(\partial_u P_{tu})|_{u=t}$ then shows that the short rate is given by (17), and a further calculation using the relation $\{L_{t\infty}\} = 1/\lim_{T\to\infty} T P_{tT}$ shows that the long rate is given by (18).

Since r_t and $L_{t\infty}$ are both rational functions of M_t, we can invert these relations to obtain M_t as a function of r_t and to obtain M_t as a function of $L_{t\infty}$, hence allowing us to express P_{tT} both as a function of r_t and as a function of $L_{t\infty}$. $\square$

In fact, we find that the discount bond price, when it expressed as a function of the short rate, takes the form

$$P_{tT} = \frac{(a_T b'_t - b_T a'_t) + (a_T b_t - b_T a_t) r_t}{a_t b'_t - b_t a'_t}, \tag{20}$$

and when it is expressed as a function of the long rate, takes the form

$$P_{tT} = \frac{(a_T b - b_T a) + (a_t b_T - b_t a_T) L_{t\infty}^{-1}}{(a_t b - b_t a)}. \tag{21}$$

Thus we deduce that the bond price is linear in the short rate, and inverse-linear in the long rate.

This establishes the fact that one can construct fully dynamic arbitrage-free term structure models admitting the long rate as a state variable, and indeed it seems to be a characteristic property of social discounting that this possibility is admitted.

As a somewhat more realistic dynamical model of the term structure, an explicit example of an arbitrage-free two-factor state-variable model based on both the short rate and the long rate can be constructed as follows.

Let $\{M_t\}$ and $\{N_t\}$ be a pair of positive martingales normalised to unity at $t = 0$. Let $\{a_t\}$, $\{b_t\}$, $\{c_t\}$ be elements of Γ^+ satisfying $\lim_{t\to\infty} ta_t = a$, $\lim_{t\to\infty} tb_t = b$, $\lim_{t\to\infty} tc_t = c$ for finite a, b, c such that $a + b + c > 0$. Let the initial term structure $P_{0t} = a_t + b_t + c_t$ be given for $t \geq 0$. Then we have:

Proposition 6. (Existence of long-rate/short-rate two-factor state-variable models.) *The pricing kernel defined by*

$$\pi_t = a_t + b_t M_t + c_t N_t \tag{22}$$

determines an arbitrage-free two-factor interest rate model, for which the state variables include the short rate, given by

$$r_t = -\frac{a'_t + b'_t M_t + c'_t N_t}{a_t + b_t M_t + c_t N_t}, \tag{23}$$

and the long simple rate, given by

$$L_{t\infty} = \frac{a_t + b_t M_t + c_t N_t}{a + bM_t + cN_t}. \tag{24}$$

In this case we find that the discount bond system is given by

$$P_{tT} = \frac{a_T + b_T M_t + c_T N_t}{a_t + b_t M_t + c_t N_t}, \tag{25}$$

and a calculation establishes that r_t is of the form (23), and $L_{t\infty}$ is of the form (24). Since r_t and $L_{t\infty}$ are rational functions of M_t and N_t, we can invert these relations to obtain M_t and N_t in terms of r_t and $L_{t\infty}$, thus allowing us to express P_{tT} in terms of r_t and $L_{t\infty}$. □

In fact, we find that the discount bond price takes the following form when it is expressed as a function of the long rate and the short rate:

$$P_{tT} = F_{tT} + G_{tT} r_t + H_{tT} L_{t\infty}^{-1}, \tag{26}$$

where the three deterministic coefficients appearing here are given by:

$$F_{tT} = \frac{(b'_t c_t - c'_t b_t) a_T + (c'_t a_t - a'_t c_t) b_T + (a'_t b_t - b'_t a_t) c_T}{(b_t c - c_t b) a'_t + (c_t a - a_t c) b'_t + (a_t b - b_t a) c'_t}, \tag{27}$$

$$G_{tT} = \frac{(b c_t - c b_t) a_T + (c a_t - a c_t) b_T + (a b_t - b a_t) c_T}{(b_t c - c_t b) a'_t + (c_t a - a_t c) b'_t + (a_t b - b_t a) c'_t}, \tag{28}$$

$$H_{tT} = \frac{(b c'_t - c b'_t) a_T + (c a'_t - a c'_t) b_T + (a b'_t - b a'_t) c_T}{(b_t c - c_t b) a'_t + (c_t a - a_t c) b'_t + (a_t b - b_t a) c'_t}. \tag{29}$$

We observe that in this model the discount function P_{tT} is given by a function of the state variables r_t and $L_{t\infty}$ which is linear in r_t and inverse linear in $L_{t\infty}$.

It is remarkable that such a simple and tractable expression emerges for the bond price.

And it is evident from the construction that an n-factor version of the model can be developed by essentially the same approach.

The initial yield curve in this model is freely specifiable at the short end, and is asymptotically hyperbolic at the long end, and hence suitable for as an input for a dynamic model of the valuation of long-term claims in situations where social discounting is needed.

More generally, as this example illustrates, one concludes that it is indeed possible to construct fully dynamic interest rate models that are compatible with the principles of social discounting.

As a consequence, we are able to provide the basis for a consistent arbitrage-free valuation framework for the cost-benefit analysis and risk management of long-term social projects, such as those associated with sustainable energy, resource conservation, and climate change.

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