Scalar Curvature and Gauss-Bonnet Theorem for Noncommutative Tori

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Spectral Triples

Noncommutative geometric spaces are described by spectral triples:

$$(\mathcal{A}, \mathcal{H}, D),$$

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \qquad (*\text{-representation}),$$

$$D = D^* : Dom(D) \subset \mathcal{H} \to \mathcal{H},$$

$$D\,\pi(a) - \pi(a)\,D \in \mathcal{L}(\mathcal{H}).$$

Examples.

$$\big(C^\infty(M), L^2(M, S), D = \text{Dirac operator}\big).$$

$$\Big(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), \frac{1}{i}\,\frac{\partial}{\partial x}\Big).$$

Noncommutative Local Invariants

The local geometric invariants such as scalar curvature of $(A, \mathcal{H}, D)$ are detected by the high frequency behavior of the spectrum of D and the action of A via heat kernel asymptotic expansions of the form

$$\text{Trace}\left(a\, e^{-tD^2}\right) \ \sim_{t\searrow 0} \ \sum_{j=0}^{\infty} a_j(a, D^2)\, t^{(-n+j)/2}, \qquad a \in A.$$

Noncommutative 2-Torus $A_\theta = C(\mathbb{T}^2_\theta)$

It is the universal C^*-algebra generated by U and V s.t.

$$U^* = U^{-1},$$

$$V^* = V^{-1},$$

$$VU = e^{2\pi i\theta} UV,$$

where $\theta \in \mathbb{R}$ is fixed.

The geometry of the Kronecker foliation $dy = \theta dx$ on the ordinary torus $\mathbb{R}^2/\mathbb{Z}^2$ is closely related to the structure of this algebra.

A representation of A_θ:

$$U\xi(x) = e^{2\pi i x}\xi(x), \qquad V\xi(x) = \xi(x+\theta), \qquad \xi \in L^2(\mathbb{R}).$$

Action of $\mathbb{T}^2 = (\frac{\mathbb{R}}{2\pi\mathbb{Z}})^2$ on A_θ and Smooth Elements

- $$\alpha_s : A_\theta \to A_\theta, \quad s \in \mathbb{R}^2,$$
  $$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n, \qquad m, n \in \mathbb{Z}.$$

- $$A_\theta^\infty := \{a \in A_\theta; \qquad s \mapsto \alpha_s(a) \text{ is smooth from } \mathbb{R}^2 \text{ to } A_\theta\}$$
  $$= \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n \in A_\theta; \quad (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \Big\}.$$

- $$\delta_j = \frac{\partial}{\partial s_j} \Big|_{s=0} \alpha_s \; : A_\theta^\infty \to A_\theta^\infty.$$

The Derivations δ_1, δ_2 and the Volume Form

- $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$ are defined by:

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V,$$

$$\delta_i(a\,b) = \delta_i(a)\,b + a\,\delta_i(b), \qquad a, b \in A_\theta^\infty.$$

- Tracial state $\varphi_0 : A_\theta \to \mathbb{C}$ (analog of integration):

$$\varphi_0\,(1) = 1, \qquad \varphi_0\,(U^m V^n) = 0 \quad \text{if} \quad (m,n) \neq (0,0).$$

Conformal Structure on A_θ (Connes)

The Dolbeault operators associated with $\tau \in \mathbb{C}, \Im(\tau) > 0$ are

$$\partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)},$$
$$\bar{\partial} = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(0,1)}.$$

The conformal structure represented by τ is encoded in

$$\psi(a,b,c) = -\varphi_0\big(a\,\partial(b)\,\bar{\partial}(c)\big), \qquad a,b,c \in A_\theta^\infty,$$

which is a positive Hochschild cocycle.

Conformal Perturbation (Connes-Tretkoff)

Let $h = h^* \in A_\theta^\infty$ and replace the trace φ_0 by

$$\varphi : A_\theta \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-h}), \qquad a \in A_\theta.$$

φ is a KMS state with the modular group

$$\sigma_t(a) = e^{ith}\, a\, e^{-ith}, \qquad a \in A_\theta,$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-h}\, a\, e^{h}, \qquad a \in A_\theta.$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in A_\theta.$$

A Spectral Triple $(A_\theta^\infty, \mathcal{H}, D)$

$$\mathcal{H} := \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$D := \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$\partial_\varphi := \partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

Anti-Unitary Equivalence of the Laplacians

$$D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}.$$

Lemma: Let

$$k = e^{h/2}.$$

We have

$$\begin{aligned} \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi \qquad &\sim \qquad k\bar{\partial}\partial k : \mathcal{H}_0 \to \mathcal{H}_0, \\ \partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)} \qquad &\sim \qquad \bar{\partial} k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}. \end{aligned}$$

Derivation of the Asymptotic Expansion

Approximate e^{-tD^2} by pseudodifferential operators:

$$e^{-tD^2} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (D^2 - \lambda)^{-1} \, d\lambda,$$

$$B_\lambda (D^2 - \lambda) \approx 1,$$

$$\sigma(B_\lambda) = b_0 + b_1 + b_2 + \cdots .$$

Connes' pseudodifferential calculus (1980)

- Symbols $\rho : \mathbb{R}^2 \to A_\theta^\infty \Rightarrow P_\rho : A_\theta^\infty \to A_\theta^\infty$

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \, \rho(\xi) \, \alpha_s(a) \, ds \, d\xi, \qquad a \in A_\theta^\infty.$$

- Differential operators:

$$\rho(\xi_1, \xi_2) = \sum a_{ij} \, \xi_1^i \, \xi_2^j, \quad a_{ij} \in A_\theta^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \, \delta_1^i \, \delta_2^j.$$

- ΨDO's on A_θ^∞ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \, \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \, \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Symbol of the first Laplacian

$$\sigma(k\bar{\partial}\partial k) = a_2(\xi) + a_1(\xi) + a_0(\xi),$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\Re(\tau)\xi_1\xi_2 k^2,$$

$$a_1(\xi) = 2\xi_1 k\delta_1(k) + 2|\tau|^2 \xi_2 k\delta_2(k) + 2\Re(\tau)\xi_1 k\delta_2(k) + 2\Re(\tau)\xi_2 k\delta_1(k),$$

$$a_0(\xi) = k\delta_1^2(k) + |\tau|^2 k\delta_2^2(k) + 2\Re(\tau)k\delta_1\delta_2(k).$$

$$b_n = -\sum_{\substack{2+j+\ell_1+\ell_2-k=n,\\ 0\le j<n,\, 0\le k\le 2}} \frac{1}{\ell_1!\ell_2!}\partial_1^{\ell_1}\partial_2^{\ell_2}(b_j)\delta_1^{\ell_1}\delta_2^{\ell_2}(a_k)b_0, \qquad n>0.$$

$$b_0 = a_2'^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\Re(\tau)\xi_1\xi_2 k^2 - \lambda)^{-1}.$$

Weyl's law for $\mathbb{T}_\theta^2$

Theorem. (Khalkhali-F.) Let

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}$$

be the eigenvalue counting function of D^2. We have

$$N(\lambda) \sim \frac{\pi}{\Im(\tau)}\, \varphi_0(e^{-h})\, \lambda \qquad (\lambda \to \infty).$$

Equivalently:

$$\lambda_j \sim \frac{\Im(\tau)}{\pi\, \varphi(1)}\, j \qquad (j \to \infty).$$

Connes' trace theorem for $\mathbb{T}^2_\theta$

Classical symbols: $\rho : \mathbb{R}^2 \to A_\theta^\infty$

$$\rho(\xi) \sim \sum_{i=-0}^{\infty} \rho_{m-i}(\xi) \qquad (\xi \to \infty),$$

$$\rho_{m-i}(t\,\xi) = t^{m-i}\,\rho_{m-i}(\xi), \qquad t > 0, \quad \xi \in \mathbb{R}^2.$$

Theorem. (Khalkhali-F.) For any classical symbol ρ of order -2 on A_θ, we have

$$P_\rho \in \mathcal{L}^{1,\infty}(\mathcal{H}_0),$$

and

$$\mathrm{Tr}_\omega(P_\rho) = \frac{1}{2} \int_{\mathbb{S}^1} \varphi_0\big(\rho_{-2}(\xi)\big)\, d\xi.$$

$$b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 +$$
$$\partial_2(b_0)\delta_2(a_1)b_0 + \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 +$$
$$(1/2)\partial_{11}(b_0)\delta_1^2(a_2)b_0 + (1/2)\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0).$$

Connes' Rearrangement Lemma

For any $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$ and $\rho_1, \ldots, \rho_\ell \in A_\theta^\infty$

$$\int_0^\infty u^{|m|-2}(e^h u + 1)^{-m_0} \prod_1^\ell \rho_j \, (e^h u + 1)^{-m_j} \, du$$

$$= e^{-(|m|-1)h} \, F_m(\Delta, \ldots, \Delta)\Big(\prod_1^\ell \rho_j\Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \Big(x \prod_1^j u_k + 1\Big)^{-m_j} \, dx.$$

Conformal Geometry of $\mathbb{T}_\theta^2$ with $\tau = i$ (Cohen-Connes)

Let

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \qquad \text{be the eigenvalues of } \partial_\varphi^* \partial_\varphi,$$

and

$$\zeta(s) = \sum \lambda_j^{-s}, \qquad \Re(s) > 1.$$

Then

$$\zeta(0) + 1 =$$

$$\varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big),$$

where

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1+u^{1/2})\mathcal{L}_2(u) + (1+u^{1/2})^2\mathcal{L}_3(u),$$

$$\mathcal{L}_m(u) = (-1)^m (u-1)^{-(m+1)} \Big(\log u - \sum_{j=1}^{m} (-1)^{j+1} \frac{(u-1)^j}{j} \Big).$$

The Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$

Theorem. (Connes-Tretkoff; Khalkhali-F.) For any $\theta \in \mathbb{R}$, complex parameter $\tau \in \mathbb{C} \setminus \mathbb{R}$ and Weyl conformal factor $e^h, h = h^* \in A_\theta^\infty$, we have

$$\zeta(0) + 1 = 0.$$

Final Part of the Proof

$$\zeta(0) + 1 =$$

$$\frac{2\pi}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big) + \frac{2\pi|\tau|^2}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)$$

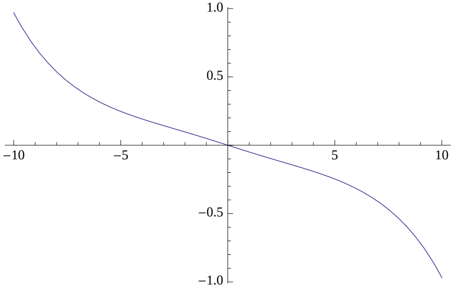
$$+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big),$$

where

$$K(x) = -\frac{\left(3x - 3\sinh\left(\frac{x}{2}\right) - 3\sinh(x) + \sinh\left(\frac{3x}{2}\right)\right)\mathsf{csch}^2\left(\frac{x}{2}\right)}{3x^2}$$

is an odd entire function, and $\nabla = \log\Delta$.

$$K(x) = -\frac{x}{20} + \frac{x^3}{2240} - \frac{23x^5}{806400} + O\left(x^6\right).$$



Scalar Curvature for $(A_\theta^\infty, \mathcal{H}, D)$

It is the unique element $R \in A_\theta^\infty$ such that

$$\zeta_a(0) + \varphi_0(a) = \varphi_0(a\,R), \qquad a \in A_\theta^\infty,$$

where

$$\zeta_a(s) := \mathrm{Trace}(a\,|D|^{-2s}), \qquad \mathrm{Re}(s) \gg 0.$$

Equivalently, consider small-time heat kernel expansions:

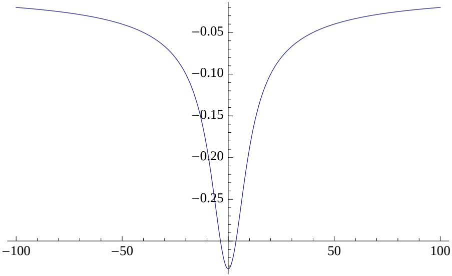
$$\mathrm{Trace}(a\,e^{-tD^2}) \sim \sum_{n\geq 0} B_n(a, D^2)\,t^{\frac{n-2}{2}}, \qquad a \in A_\theta^\infty.$$

Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^2$

Theorem. (Connes-Moscovici; Khalkhali-F.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, R is equal to

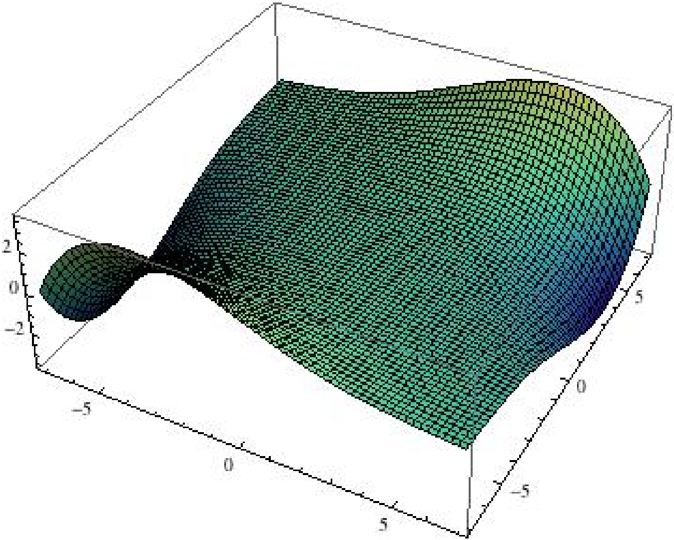
$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$
$$+R_2(\nabla,\nabla)\Big(\delta_1(\frac{h}{2})^2 + |\tau|^2\,\delta_2(\frac{h}{2})^2 + \Re(\tau)\,\big\{\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})\big\}\Big)$$
$$+i\,W(\nabla,\nabla)\Big(\Im(\tau)\,[\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})]\Big).$$

$$R_1(x) = \frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}.$$

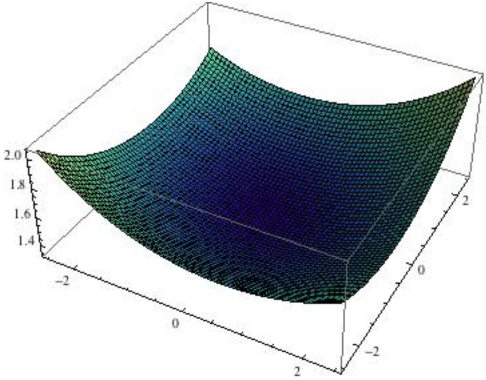


$R_2(s,t) =$

$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



$$W(s,t) = \frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



Symbol of the second Laplacian

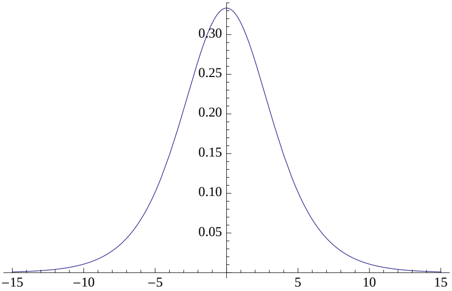
$$\sigma(\partial^* k^2 \partial) = c_2(\xi) + c_1(\xi),$$

where

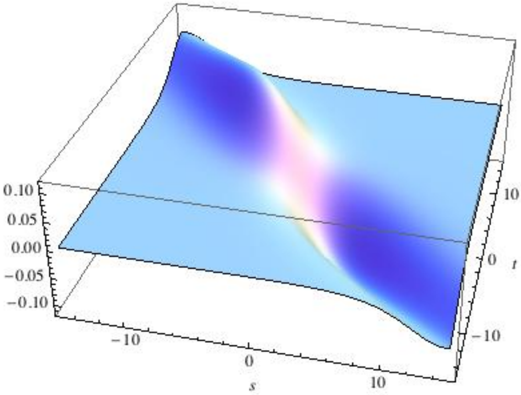
$$c_2(\xi) = \xi_1^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + |\tau|^2 \xi_2^2 k^2,$$

$$c_1(\xi) = (\delta_1(k^2) + \tau\delta_2(k^2))\xi_1 + (\bar{\tau}\delta_1(k^2) + |\tau|^2 \delta_2(k^2))\xi_2.$$

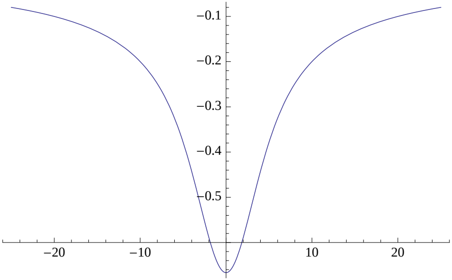
$$K_1(x) = \frac{2e^{x/2}\left(e^x(x-2)+x+2\right)}{\left(e^x-1\right)^2 x}$$



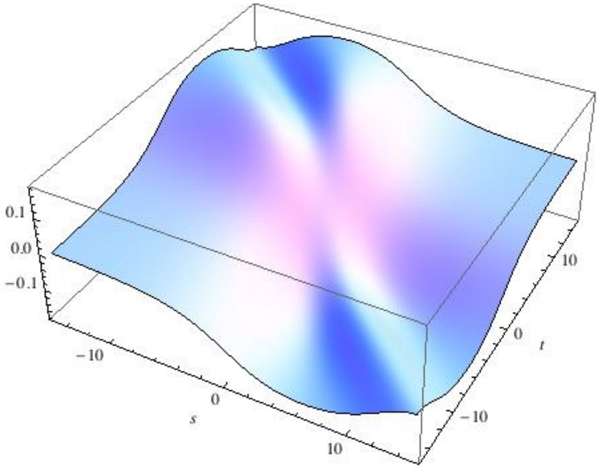
$$H_1(s,t) =$$
$$-\frac{\operatorname{csch}\left(\frac{s}{2}\right)\operatorname{csch}\left(\frac{t}{2}\right)\operatorname{csch}^2\left(\frac{s+t}{2}\right)(-(s-t)(-\sinh(s+t)+s+\sinh(s)+t+\sinh(t))-t(s+t)\cosh(s)+s(s+}{st(s+t)}$$



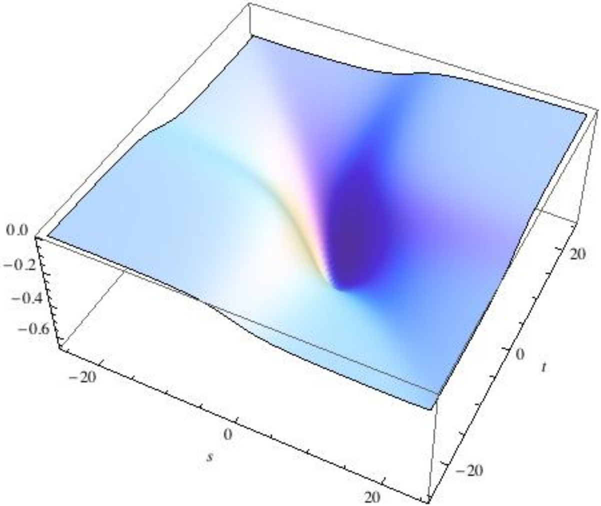
$$K_2(x) = -\frac{4e^x(\sinh(x) - x)}{\left(e^{x/2} - 1\right)^2 \left(e^{x/2} + 1\right)^2 x}$$



$$H_2(s,t) = \cosh\left(\frac{s+t}{2}\right) \times$$
$$-\frac{\mathsf{csch}\left(\frac{s}{2}\right)\mathsf{csch}\left(\frac{t}{2}\right)\mathsf{csch}^2\left(\frac{s+t}{2}\right)(-(s-t)(-\sinh(s+t)+s+\sinh(s)+t+\sinh(t))-t(s+t)\cosh(s)+s(s+}{st(s+t)}$$



$$W(s,t) = -\frac{4(-\sinh(s+t)+s\cosh(t)+t\cosh(s)-s+\sinh(s)-t+\sinh(t))}{st(-\sinh(s+t)+\sinh(s)+\sinh(t))}$$



Noncommutative 4-Torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal C^*-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

Perturbed Laplacian on $\mathbb{T}_\theta^4$

$$d = \partial \oplus \bar\partial : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

Lemma. (Khalkhali-F.) Up to an anti-unitary equivalence $\triangle_\varphi$ is given by

$$e^h \bar\partial_1 e^{-h} \partial_1 e^h + e^h\, \partial_1 e^{-h} \bar\partial_1\, e^h + e^h \bar\partial_2 e^{-h}\, \partial_2 e^h + e^h \partial_2 e^{-h} \bar\partial_2 e^h,$$

where ∂_1, ∂_2 are analogues of the Dolbeault operators.

Scalar Curvature for $\mathbb{T}^4_\theta$

It is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that

$$\mathrm{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\,R), \qquad a \in C^\infty(\mathbb{T}^4_\theta),$$

where

$$\zeta_a(s) := \mathrm{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

Final Formula for the Scalar Curvature of $\mathbb{T}^4_\theta$

Theorem. (Khalkhali-F.) We have

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

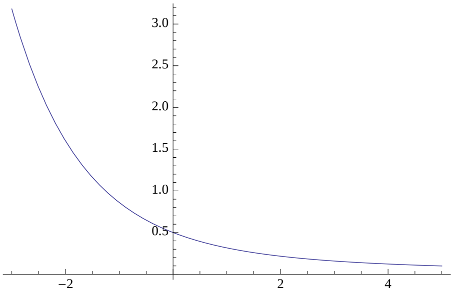
where

$$\nabla(a) = [-h, a], \qquad a \in C(\mathbb{T}^4_\theta),$$

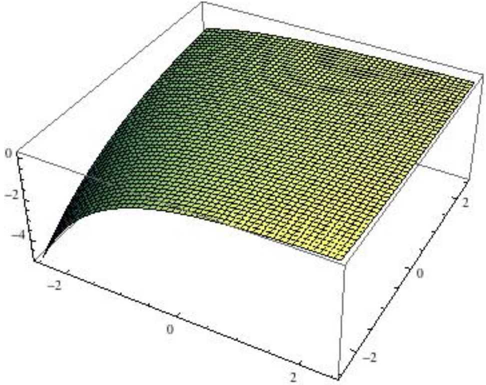
$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\,((-e^s-3)\,s\,(e^t-1) + (e^s-1)\,(3e^t+1)\,t)}{4\,s\,t\,(s+t)}.$$

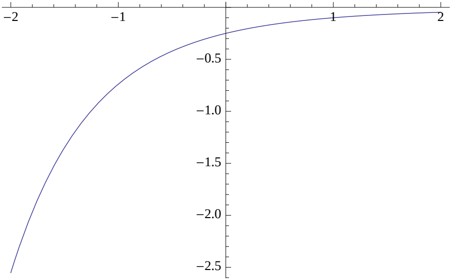
$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



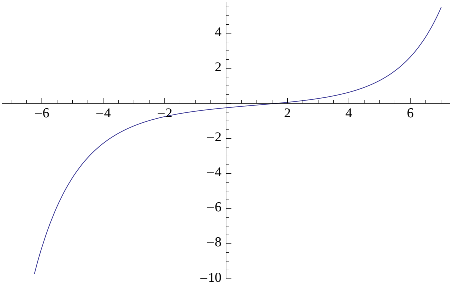
$$H(s,t) = \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^3\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^2}{80}+O\left(t^3\right)\right)$$
$$+s^2\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^2}{144}+O\left(t^3\right)\right)+O\left(s^3\right).$$



$$
\begin{aligned}
H(s,s) &= -\frac{e^{-2s}\left(e^{s}-1\right)^{2}}{4s^{2}} \\
&= -\frac{1}{4}+\frac{s}{4}-\frac{7s^{2}}{48}+\frac{s^{3}}{16}-\frac{31s^{4}}{1440}+\frac{s^{5}}{160}+O\left(s^{6}\right).
\end{aligned}
$$



$$\begin{aligned} G(s) \quad := \quad H(s,-s) \quad &= \quad \frac{-4s - 3e^{-s} + e^{s} + 2}{4s^2} \\ &= \quad -\frac{1}{4} + \frac{s}{6} - \frac{s^2}{48} + \frac{s^3}{120} - \frac{s^4}{1440} + \frac{s^5}{5040} + O\left(s^6\right). \end{aligned}$$



Einstein-Hilbert Action for $\mathbb{T}_\theta^4$

Theorem. (Khalkhali-F.) We have the local expression (up to a factor of π^2)

$$\begin{aligned} \varphi_0(R) &= \frac{1}{2}\sum_{i=1}^{4} \varphi_0\Big(e^{-h}\delta_i^2(h)\Big) \\ &\quad + \sum_{i=1}^{4} \varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\, \delta_i(h)\Big). \end{aligned}$$

Extremum of the Einstein-Hilbert Action

Theorem. (Khalkhali-F.) For any Weyl factor $e^{-h} \in C^\infty(\mathbb{T}^4_\theta)$

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if h is a constant.

Proof.

$$\varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h} T(\nabla)(\delta_i(h))\, \delta_i(h)\big),$$

where

$$T(s) = \frac{1}{2}\frac{e^{-s}-1}{-s} + G(s) = \frac{-2s + e^{s} - e^{-s}(2s+3) + 2}{4s^2}.$$

$$T(s) = \frac{1}{4} - \frac{s}{12} + \frac{s^2}{16} - \frac{s^3}{80} + \frac{s^4}{288} - \frac{s^5}{2016} + O\left(s^6\right).$$