C^2-smooth functions on finite subsets of $\mathbb{R}^2$

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1. The Whitney Extension Problem

Let $C^2(\mathbf{R}^2)$ be the space of two times continuously differentiable functions on $\mathbf{R}^2$ whose partial derivatives of the second order are bounded function on $\mathbf{R}^2$. We equip this space with the semi-norm

$$\|F\|_{C^2(\mathbf{R}^2)} :=$$

$$\sup_{z\in\mathbf{R}^2} \max\left\{\left|\frac{\partial^2 F}{\partial x^2}(z)\right|, \left|\frac{\partial^2 F}{\partial x \partial y}(z)\right|, \left|\frac{\partial^2 F}{\partial y^2}(z)\right|\right\}$$

Let $E \subset \mathbf{R}^2$ be a finite subset, and let $f : E \to \mathbf{R}^2$.

Problem. How can we extend a function $f : E \to \mathbf{R}$ to a function $F \in C^2(\mathbf{R}^2)$ with minimal $\|F\|_{C^2(\mathbf{R}^2)}$?

What is the order of magnitude of this minimal C^2-norm, i.e.,

$$\|f\|_{C^2(\mathbf{R}^2)|_E} = \inf\{\|F\|_{C^2(\mathbf{R}^2)} : F|_E = f\}?$$

Something of the history:

H. Whitney, TAMS, (1934);

[W1] Analytic extension of differentiable functions defined in closed sets.

An extension problem for jets:

Given a family of polynomials $\{P_x \in \mathcal{P}_1(\mathbf{R}^2) : x \in E\}$ find a function $F \in C^2(\mathbf{R}^2)$ such that the Taylor polynomial of the first order of F at x

$$T_x^1[F] = P_x \quad \textbf{for every} \quad x \in E$$

[W2] Differentiable functions defined in closed sets. I.

(A description of $C^2(\mathbf{R})|_E$ via divided differences of the second order of f on E.)

2. The finiteness principle.

The Whitney problem of characterization of the trace space $C^2(\mathbf{R}^2)|_E$: we have to restore in an optimal way all partial derivatives of the second order of a function $f : E \to \mathbf{R}$ using only the values of f on E.

In many cases Whitney-type problems (for different spaces of smooth functions) can be reduced to the same kinds of problems, but for finite sets with prescribed number of points.

Theorem 2.1 **(Sh. [1982])**

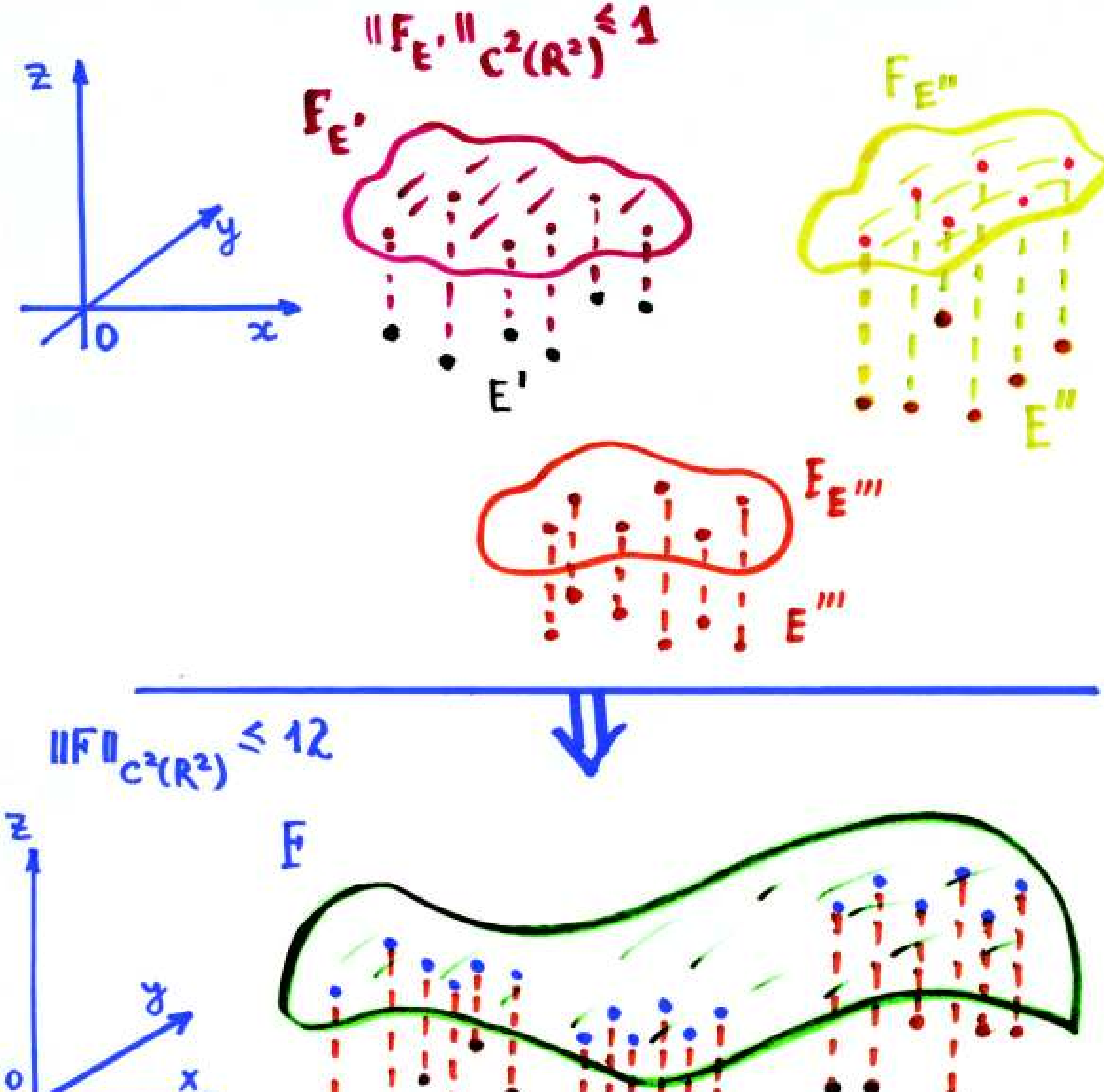
Let $E \subset \mathbf{R}^2$ be a finite set and let $f : E \to \mathbf{R}$.

Suppose that the restriction $f|_{E'}$ to every $E' \subset E$ of $\operatorname{card} E' \leq 6$ can be extended to a function $F_{E'} \in C^2(\mathbf{R}^2)$ with the norm

$$\|F_{E'}\|_{C^2(\mathbf{R}^2)} \leq 1$$

Then f itself can be extended to a function $F \in C^2(\mathbf{R}^2)$ with

$$\|F\|_{C^2(\mathbf{R}^2)} \leq 12.$$



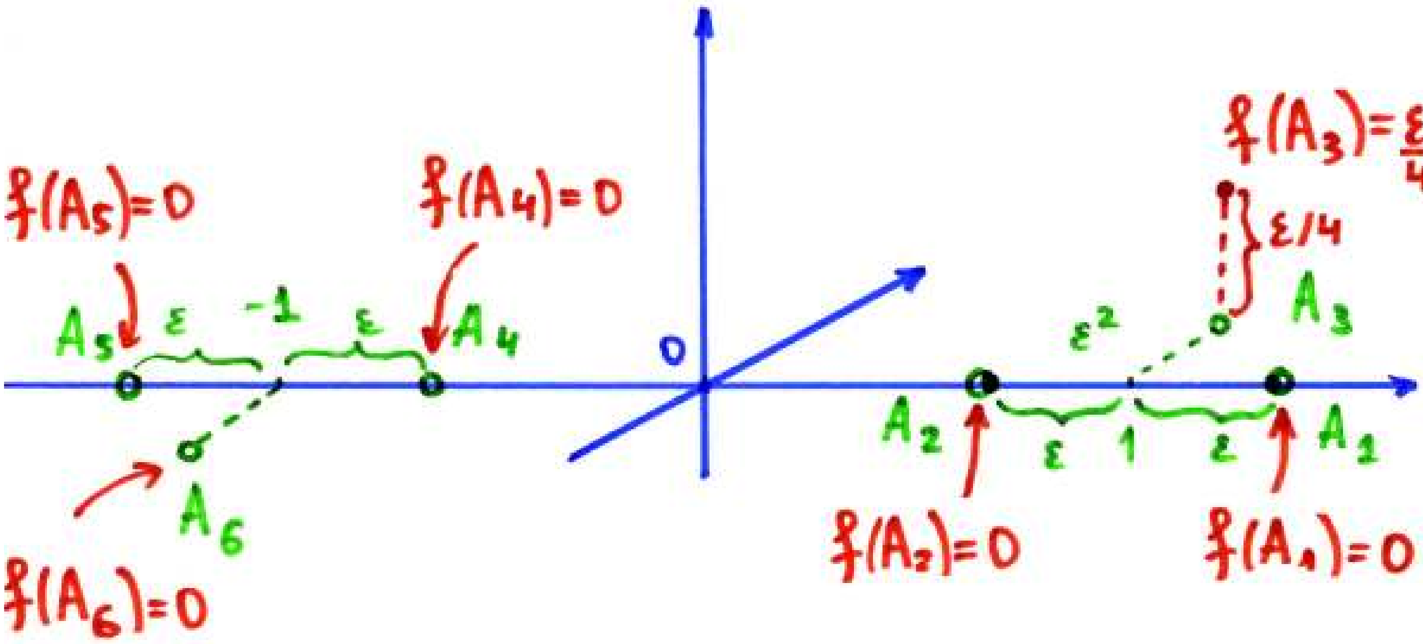
The finiteness number $N = 6$ **for the space** $C^2(\mathrm{R}^2)$ **is sharp.**

Let $0 < \varepsilon < 1/4$ **and**

$$A_\varepsilon = \{(1-\varepsilon, 0), (-1, -\varepsilon^2), (-1+\varepsilon, 0), (1-\varepsilon, 0), (1, \varepsilon^2)(1+\varepsilon, 0)\}.$$

Define $f : A_\varepsilon \to \mathrm{R}$ **by**

$$f(1, \varepsilon^2) = \varepsilon,\ f(x) = 0,\ x \in A_\varepsilon \backslash \{(1, \varepsilon^2)\}$$



For <u>**each**</u> $A' \subset A_\varepsilon$, card $A' = 5$, $f|_{A'}$ **extends to an** $F_{A'} \in C^2(\mathrm{R}^2)$, $\|F_{A'}\|_{C^2(\mathrm{R}^2)} \leq 1$. **However,**

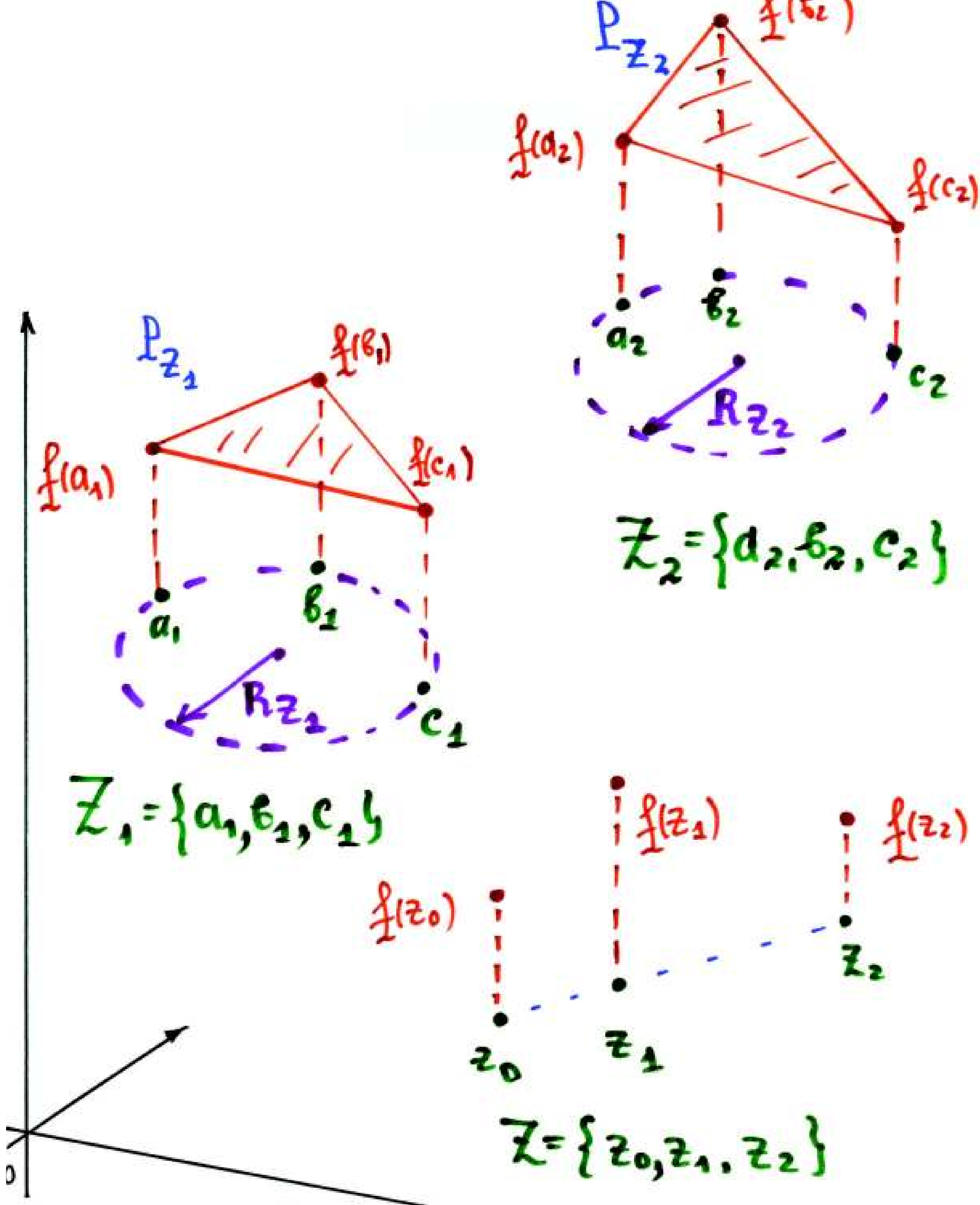
$\|F\|_{C^2(\mathrm{R}^2)} \geq 1/4\varepsilon$, $\forall F, F|_{A_\varepsilon} = f$

Theorem 2.2 **For every finite set $E \subset \mathbf{R}^2$ and for every $f : E \to \mathbf{R}$**

$$\|f\|_{C^2(\mathbf{R}^2)|_E} \sim$$

$$\sup_{\substack{z_0,z_1,z_2 \in E \\ z_1 \in (z_0,z_2)}} \left| \frac{\frac{f(z_0)-f(z_1)}{\|z_0-z_1\|} - \frac{f(z_1)-f(z_2)}{\|z_1-z_2\|}}{\|z_0 - z_2\|} \right| +$$

$$\sup_{Z_1,Z_2 \subset E} \frac{\|\nabla P_{Z_1}[f] - \nabla P_{Z_2}[f]\|}{R_{Z_1} + R_{Z_2} + \operatorname{diam}(Z_1 \cup Z_2)}$$



P_{Z_2}
$f(b_2)$
$f(a_2)$
$f(c_2)$
a_2
b_2
c_2
R_{Z_2}
$Z_2=\{a_2, b_2, c_2\}$
P_{Z_1}
$f(b_1)$
$f(a_1)$
$f(c_1)$
a_1
b_1
c_1
R_{Z_1}
$Z_1=\{a_1, b_1, c_1\}$
$f(z_1)$
$f(z_2)$
$f(z_0)$
z_2
z_1
z_0
$Z=\{z_0, z_1, z_2\}$

3. A geometrical approach to the Whitney problem: main ideas.

Theorem (E. Helly,1913).

Let $\mathcal{K}$ be a family of convex sets in $\mathbf{R}^n$. Suppose that $\mathcal{K}$ is finite or that each member of $\mathcal{K}$ is compact.

If every $n+1$ members of $\mathcal{K}$ have a common point, then there is a point common to all members of $\mathcal{K}$.

The Whitney Extension Problem for the space $C^2(\mathbf{R}^2)$

Let $E \subset \mathbf{R}^2$ be finite and let $f : E \to \mathbf{R}$.

Theorem (Whitney). **(Necessity)**

Suppose $\exists\ F \in C^2(\mathbf{R}^2)$, $F|_E = f$.

Let $\vec{g} = \nabla F|_E$ and $\lambda = \|F\|_{C^2(\mathbf{R}^2)}$. Then for every $x, y \in E$

$$|f(y) - (f(x) + \langle \vec{g}(x), y - x\rangle)| \le C\lambda\, \|x - y\|^2$$

and

$$\|\vec{g}(x) - \vec{g}(y)\| \le C\lambda\, \|x - y\|$$

where C is an absolute constant.

The first inequality is an estimate of the Taylor reminder of F of the first order at points x, y.

(Sufficiency). **Let** $f : E \to \mathbf{R}$. **Suppose that** $\exists\ \lambda > 0$ **and a mapping** $\vec{g} : E \to \mathbf{R}^n$ **such that for every** $x, y \in E$

$$|f(y) - (f(x) + \langle \vec{g}(x), y - x \rangle)| \leq \lambda \, \|x - y\|^2$$

and

$$\|\vec{g}(x) - \vec{g}(y)\| \leq \lambda \, \|x - y\|$$

Then $\exists\ F \in C^2(\mathbf{R}^2)$ **such that** $F|_E = f$**,** $\nabla F|_E = \vec{g}$**, and**

$$\|F\|_{C^2(\mathbf{R}^2)} \leq C \, \lambda$$

The conditions

$$|f(x) - f(y) - \langle \vec{g}(x), x - y \rangle| \leq \lambda \|x - y\|^2$$

and

$$\|\vec{g}(x) - \vec{g}(y)\| \leq \lambda \|x - y\|$$

where $x, y \in E$, are a chain (system) of inequalities.

Our goal is to find the minimal $\lambda > 0$ (up to an absolute constant) such that this system has a solution with respect to $\vec{g} : E \to \mathbf{R}^2$.

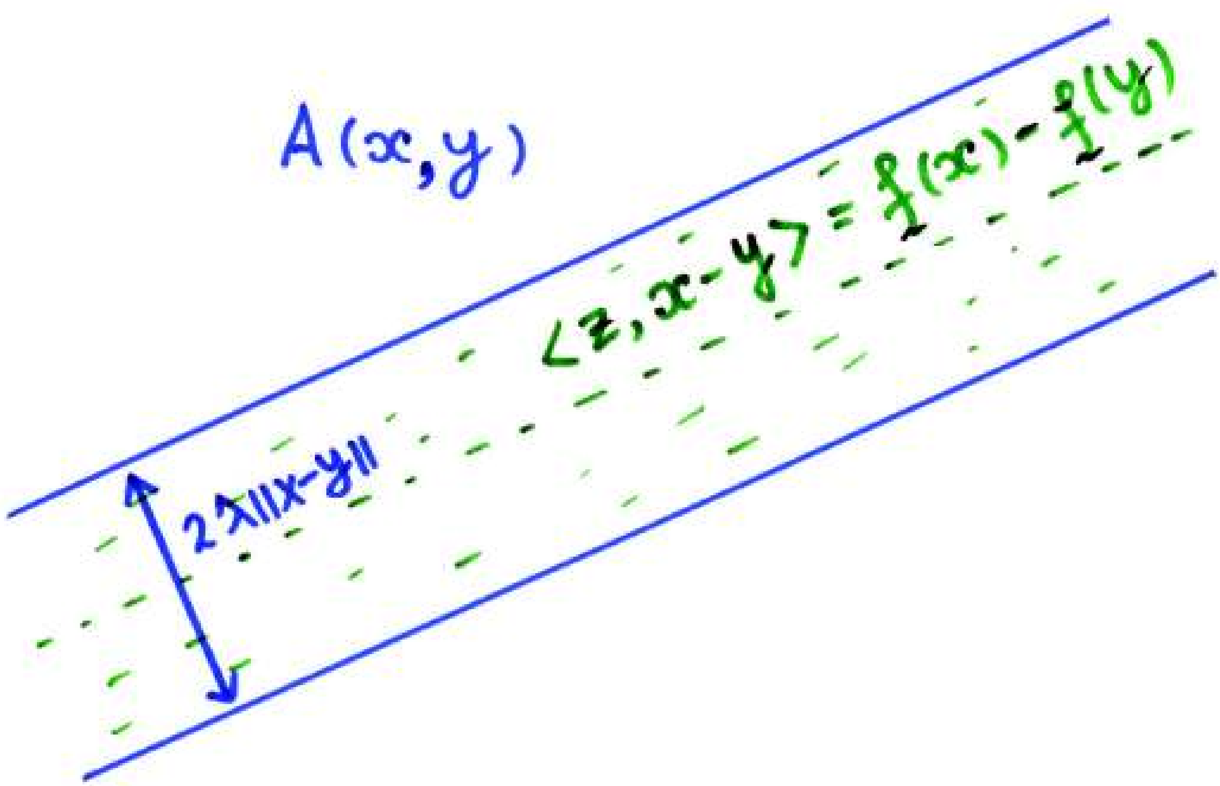
A geometrical background of the Whitney theorem.

Fix $x \in E$. For each $y \in E$ the set

$$A(x,y) := \{z \in \mathbb{R}^2 :$$

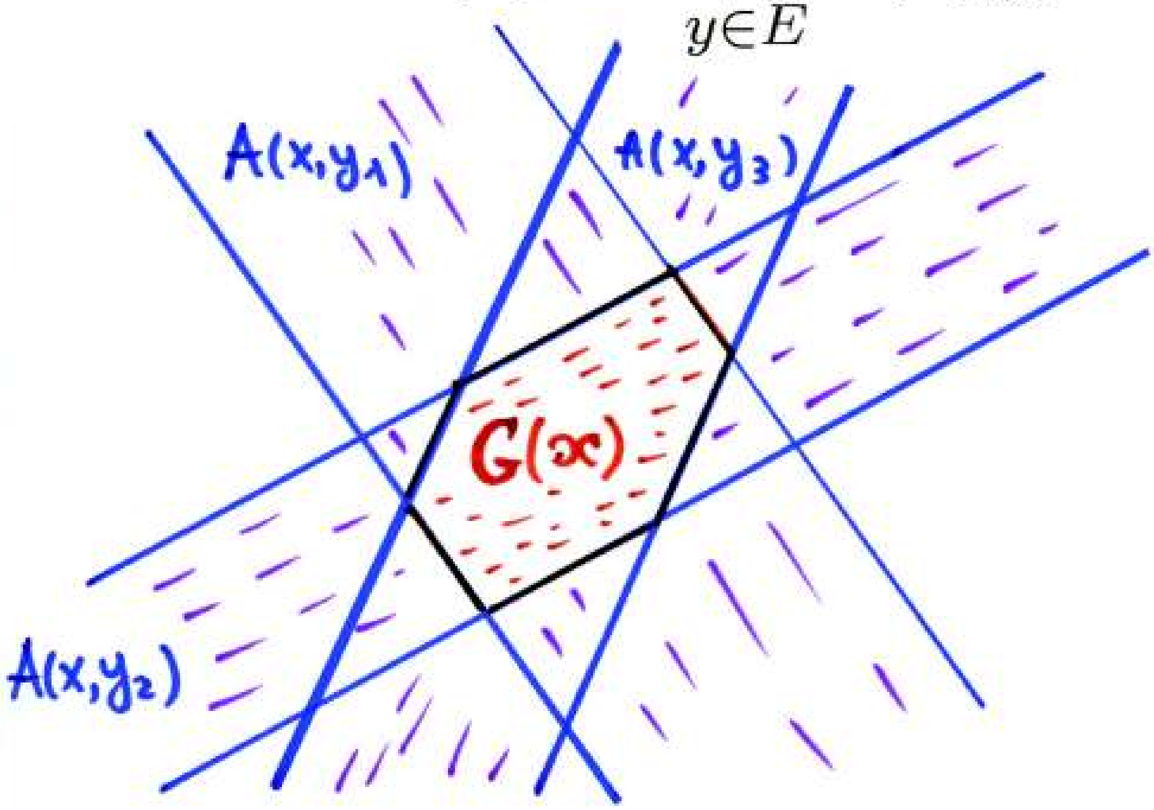
$$|f(x) - f(y) - \langle z, x-y \rangle| \leq \lambda \, \|x-y\|^2\}$$

is a strip between two parallel hyperplanes.



Put

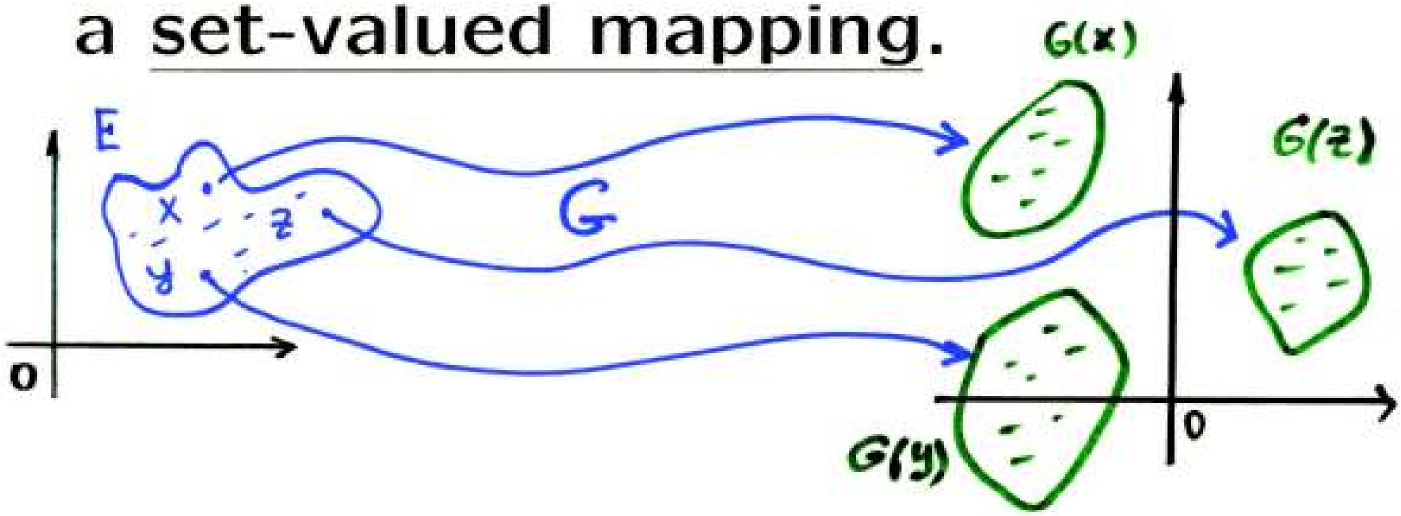
$$G(x) := \bigcap_{y \in E} A(x, y)$$



$G(x)$ is a *convex* closed subset of R^2. We may assume that $G(x)$ is compact $\Rightarrow G(x) \in \mathcal{K}(\mathrm{R}^2)$.

$\mathcal{K}(\mathrm{R}^2)$ − all convex closed subsets of R^2.

We say that $G : E \to \mathcal{K}(\mathrm{R}^2)$ is a <u>set-valued mapping</u>.

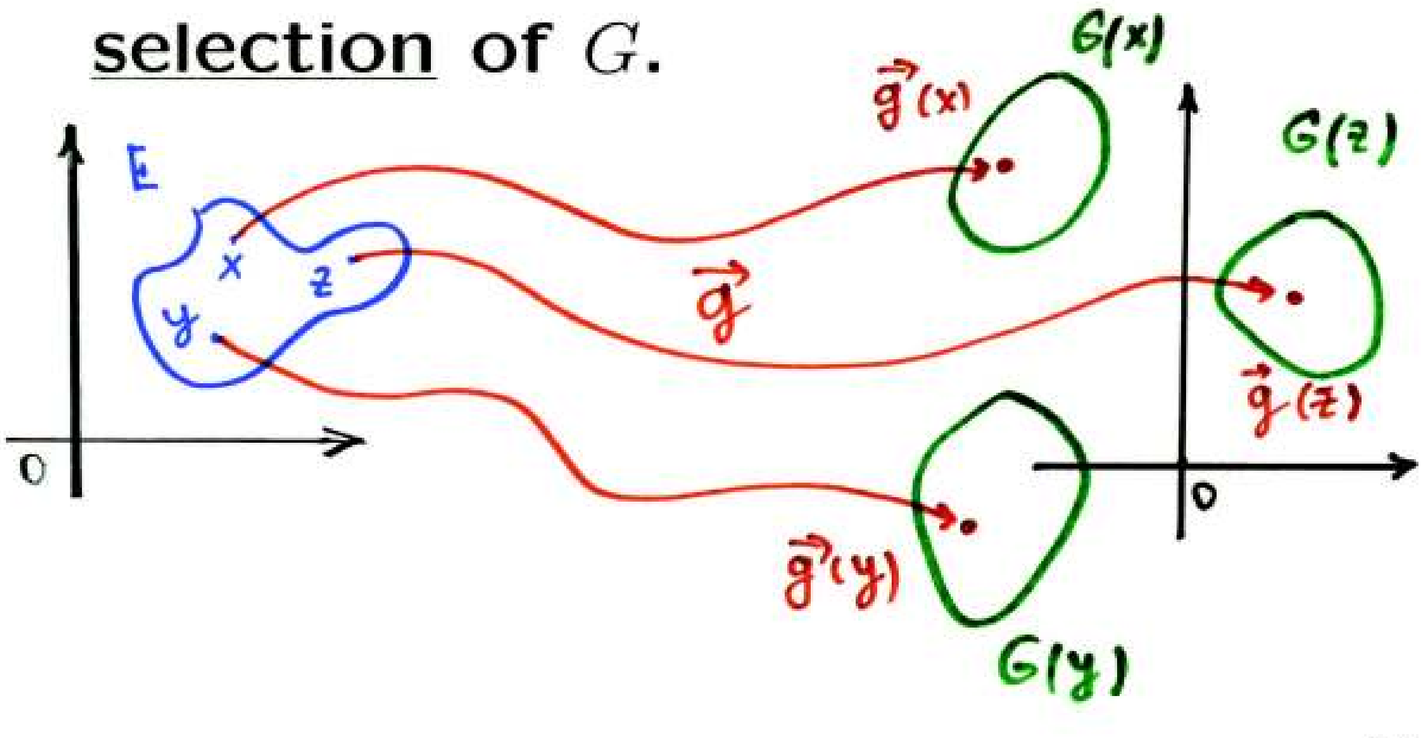


The condition:

$$|f(x)-f(y)-\langle\vec{g}(x), x-y\rangle| \le \lambda\,\|x-y\|^2$$

$\forall y \in E \Leftrightarrow \underline{\vec{g}(x) \in G(x),\ x \in E}$.

We say that $\vec{g} : E \to \mathrm{R}^2$ is a <u>selection</u> of G.



The second condition $\Leftrightarrow$

$$\vec{g} \in \mathrm{Lip}(\mathcal{M}; \mathbf{R}^2)$$

Here $\mathcal{M} := (E, \rho)$ **where**

$$\rho(x, y) := \|x - y\|$$

$\mathrm{Lip}(\mathcal{M}; \mathbf{R}^2)$ **denotes the space of all Lipschitz mappings from** $\mathcal{M}$ **into** R^2 **equipped with the semi-norm**

$$\|\vec{g}\|_{\mathrm{Lip}(\mathcal{M}; \mathbf{R}^2)} := \sup_{x,y \in \mathcal{M}} \frac{\|\vec{g}(x) - \vec{g}(y)\|}{\rho(x, y)}$$

We call $\vec{g}$ **a** **Lipschitz selection of the set-valued mapping** G**.**

4. Lipschitz selections of set-valued mappings.

- $(\mathcal{M},\rho)$ – a finite metric space;
- $\mathcal{K}(\mathbf{R}^2)$ – all *convex closed subsets* of $\mathbf{R}^2$;
- $F:\mathcal{M}\to\mathcal{K}(\mathbf{R}^2)$ – a set-valued mapping.

The Lipschitz Selection Problem.

Let f be a Lipschitz selection of F, i.e., a mapping $f:\mathcal{M}\to\mathbf{R}^2$:

(i) $f(x)\in F(x), \quad x\in\mathcal{M}$.

(ii) $f\in \mathrm{Lip}(\mathcal{M};\mathbf{R}^2)$

How small can its Lipschitz seminorm $\|f\|_{\mathrm{Lip}(\mathcal{M};\mathbf{R}^2)}$ be?

Theorem 4.1 Let $(\mathcal{M},\rho)$ be a finite metric space and let $F:\mathcal{M}\to\mathcal{K}(\mathbf{R}^2)$ be a set-valued mapping.

Suppose that for every subset $\mathcal{M}'\subset\mathcal{M}$ consisting of at most 4 elements the restriction $F|_{\mathcal{M}'}$ has a Lipschitz selection

$$f_{\mathcal{M}'}:\mathcal{M}'\to\mathbf{R}^2$$

such that

$$\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}';\mathbf{R}^2)}\le 1$$

Then F on all of the set $\mathcal{M}$ has a Lipschitz selection $f:\mathcal{M}\to\mathbf{R}^2$ with

$$\|f\|_{\mathrm{Lip}(\mathcal{M};\mathbf{R}^2)}\le 5$$

This theorem is also true for pseudometric spaces, i.e., $\rho(x,y)$ may take the value 0 for $x \neq y$.

Example 4.2 **Let $\rho \equiv 0$. Let $F : \mathcal{M} \to \mathcal{K}(\mathbf{R}^2)$ be a set-valued mapping and let $f : \mathcal{M} \to \mathbf{R}^2$ be its Lipschitz selection. Then**

$$\|f(x) - f(y)\| \leq \rho(x,y) = 0 \;\; \forall \;\; x,y \in \mathcal{M}$$

so that $f(x) = c \in \mathbf{R}^2$, $x \in \mathcal{M}$.

Since $f(x) \in F(x)$, $x \in \mathcal{M}$, $\Longrightarrow$

$$c \in F(x), \quad \forall x \in \mathcal{M}$$

Thus F has a Lipschitz selection with respect to $\rho \equiv 0 \iff$

$$\bigcap \{F(x) : x \in \mathcal{M}\} \neq \emptyset$$

By Helly's Theorem

$$\bigcap \{F(x) : x \in \mathcal{M}\} \neq \emptyset$$

$\iff$

$$\bigcap \{F(x) : x \in \mathcal{M}'\} \neq \emptyset$$

for every $\mathcal{M}' \subset \mathcal{M}$, $\operatorname{card} \mathcal{M} \leq 3$,

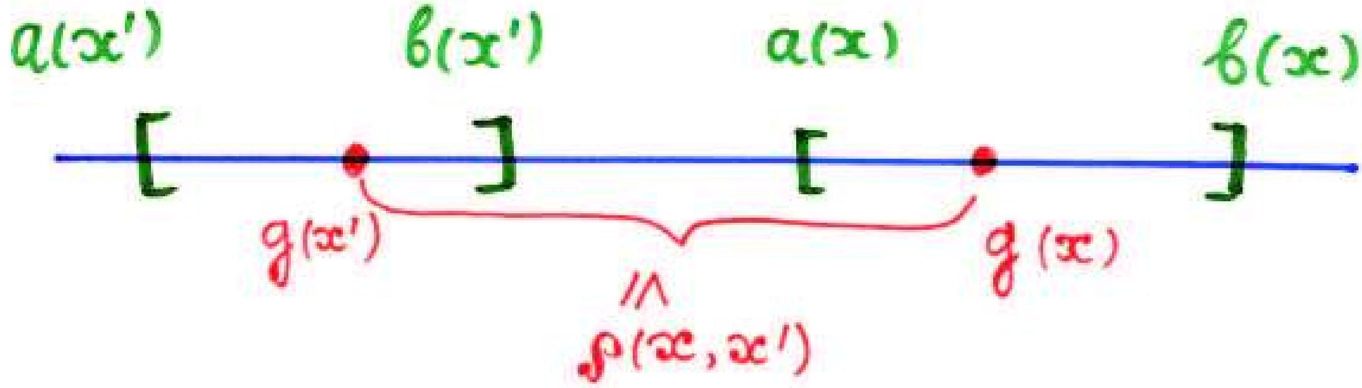
$\iff$ $F|_{\mathcal{M}'}$ **has a Lipschitz selection for every subset**

$$\mathcal{M}' \subset \mathcal{M}, \ \operatorname{card} \mathcal{M} \leq 3$$

Let $F : \mathcal{M} \to \mathcal{K}(\mathrm{R})$ be a set-valued mapping, i.e., $F(x) := [a(x), b(x)],\ x \in \mathcal{M}$.

Assume that for every $F(x), F(x')$ there exist $g(x) \in F(x), g(x') \in F(x')$ such that

$$|g(x) - g(x')| \le \rho(x, x')$$



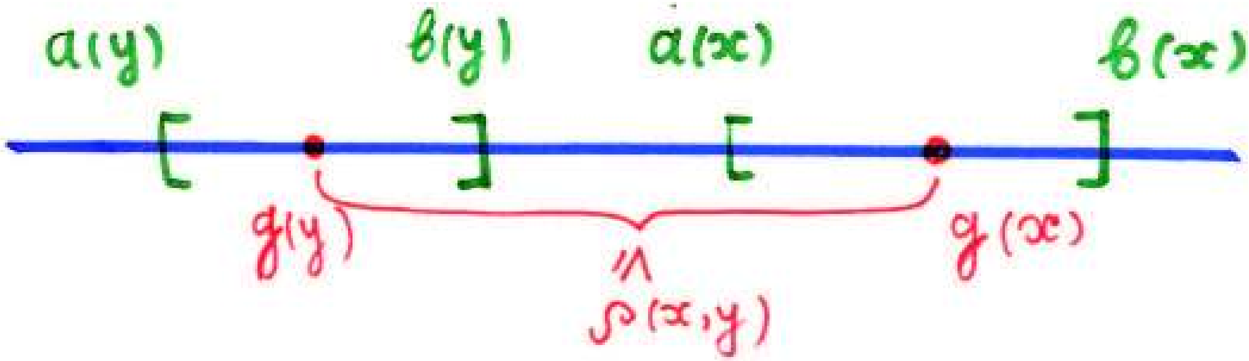
Given $x \in \mathcal{M}$ we define

$$f(x) := \inf_{y \in \mathcal{M}} \{b(y) + \rho(x, y)\}$$

Then $f(x) \le b(x)$ (put $y = x$).

For every $y \in \mathcal{M}$ there are points $g(x) \in [a(x), b(x)]$, $g(y) \in [a(y), b(y)]$ such that

$$|g(x) - g(y)| \leq \rho(x, y)$$



$$a(x) \leq g(x) \leq g(y) + \rho(x, y) \leq b(y) + \rho(x, y)$$

$$a(x) \leq \inf_{y \in \mathcal{M}} \{b(y) + \rho(x, y)\} = f(x).$$

Hence, $a(x) \leq f(x) \leq b(x) \Leftrightarrow$

$$f(x) \in F(x)$$

Clearly, $\|f\|_{\mathrm{Lip}(\mathcal{M};\mathbf{R})} \leq 1.$

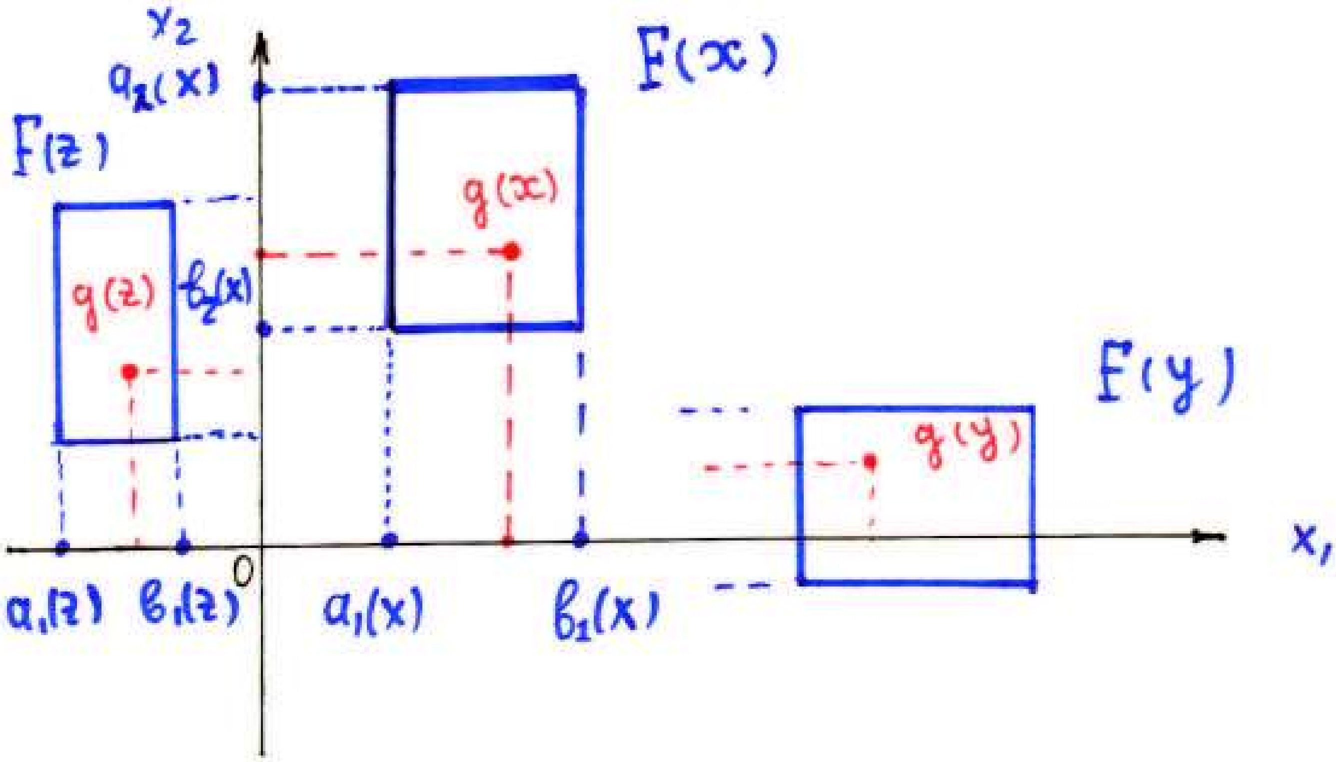
Let

$$\|a\|_\infty := \max_{i=1,2} |a_i|$$

Consider

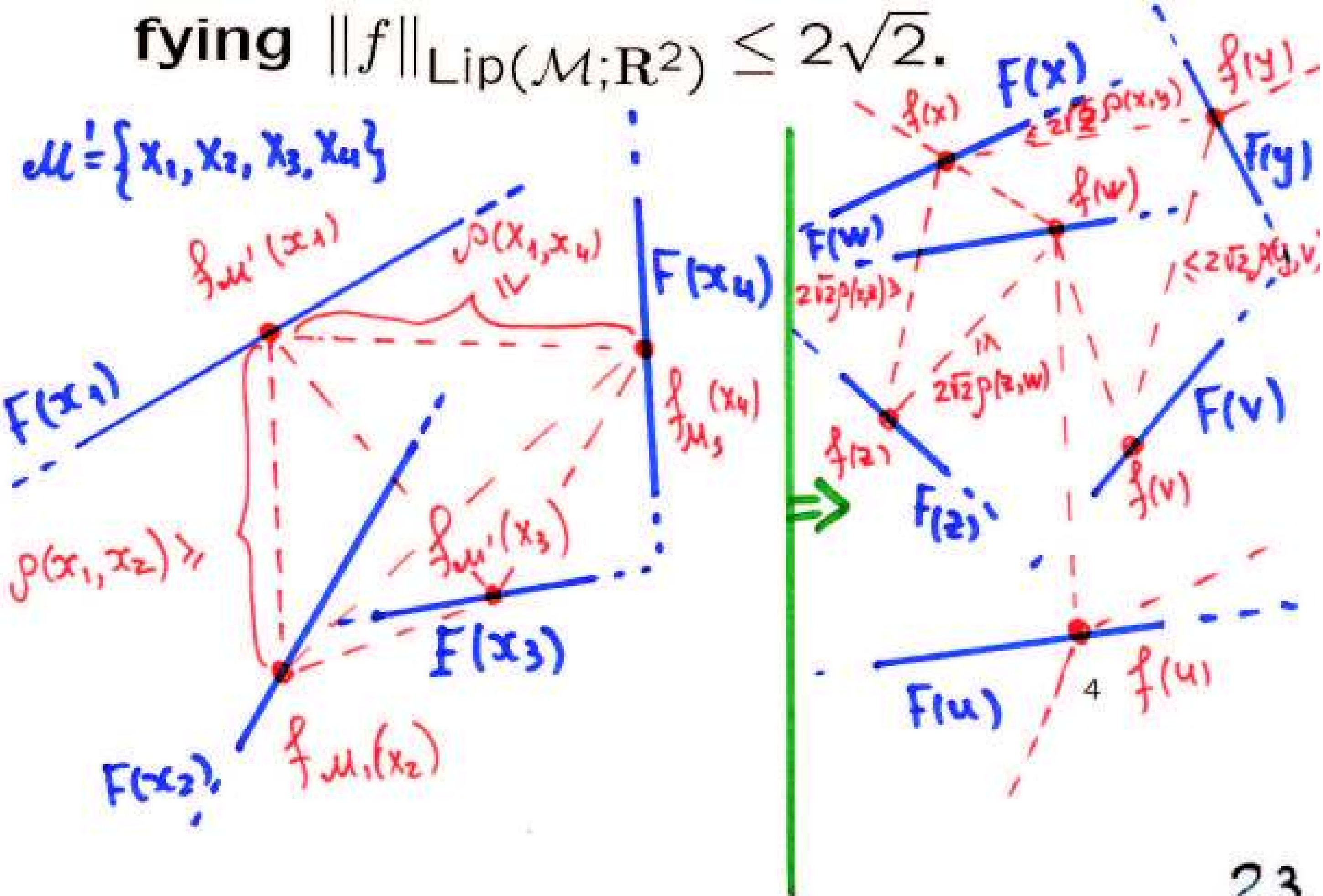
$$F(x) := \prod_{k=1}^{2} [a_k(x), b_k(x)], \ x \in \mathcal{M}$$

Then F **has a selection** f **with** $\|f\|_{\text{Lip}(\mathcal{M})} \leq 1 \quad \Leftrightarrow \forall \mathcal{M}' \subset \mathcal{M}$, card $\mathcal{M}' = 2$, **the restriction** $F|_{\mathcal{M}'}$ **has such a selection.**



Prove that the theorem is true for $X = \mathrm{R}^2$ (equipped with the Euclidean norm) with $N = 4$ and $\gamma = 2\sqrt{2}$.

We know that $\forall\ \mathcal{M}' \subset \mathcal{M}$ with $\operatorname{card} \mathcal{M}' \le 4$, the restriction $F|_{\mathcal{M}'}$ has a Lipschitz selection $f_{\mathcal{M}'}$ with $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}';\mathrm{R}^2)} \le 1$. We have to prove that F on $\mathcal{M}$ has a Lipschitz selection $f : \mathcal{M} \to \mathrm{R}^2$ satisfying $\|f\|_{\mathrm{Lip}(\mathcal{M};\mathrm{R}^2)} \le 2\sqrt{2}$.



Step 1. Let $x_1, x_2 \in \mathcal{M},\ x_1 \neq x_2$ and let $F(x_1) \nparallel F(x_2)$.

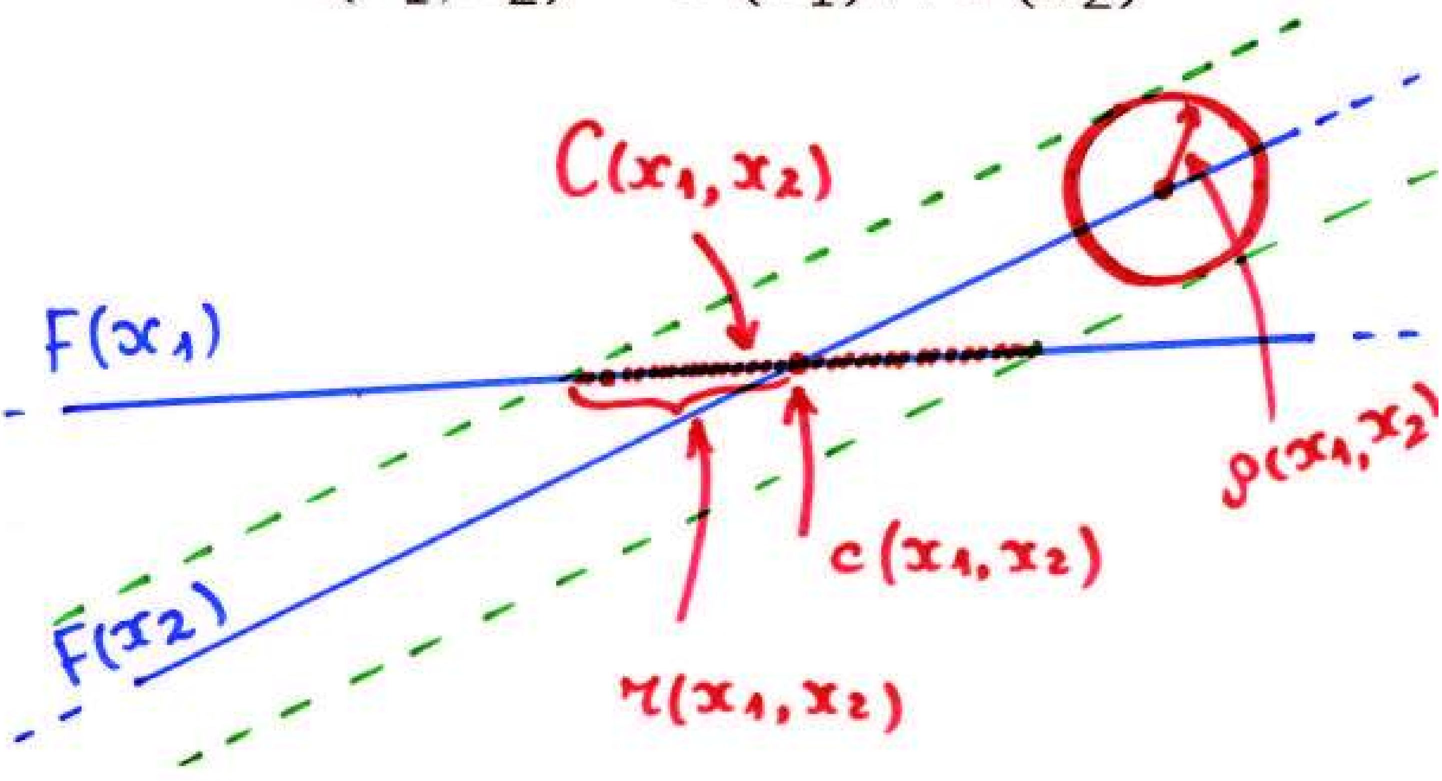
Put

$$C(x_1, x_2) :=$$

$$F(x_1) \cap \{F(x_2) + B(0, \rho(x_1, x_2))\}$$

$C(x_1, x_2)$ is a line segment on $F(x_1)$ with center

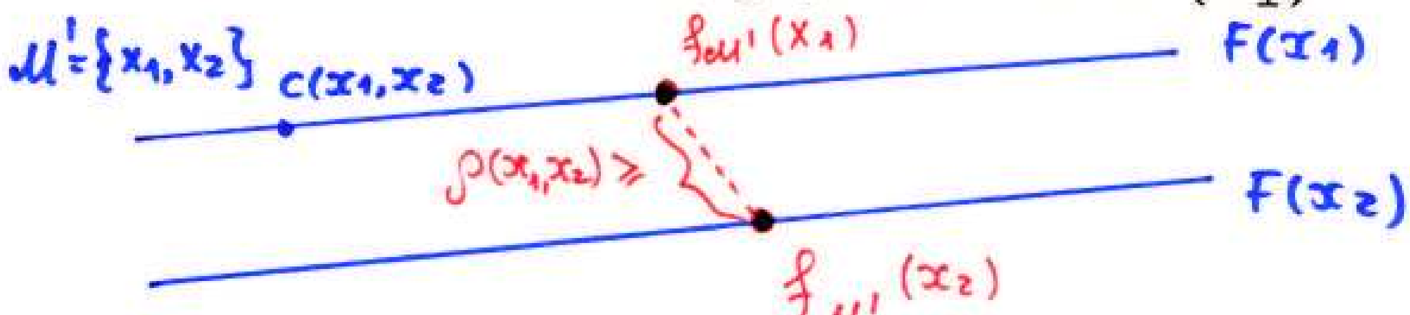
$$c(x_1, x_2) = F(x_1) \cap F(x_2)$$



Put

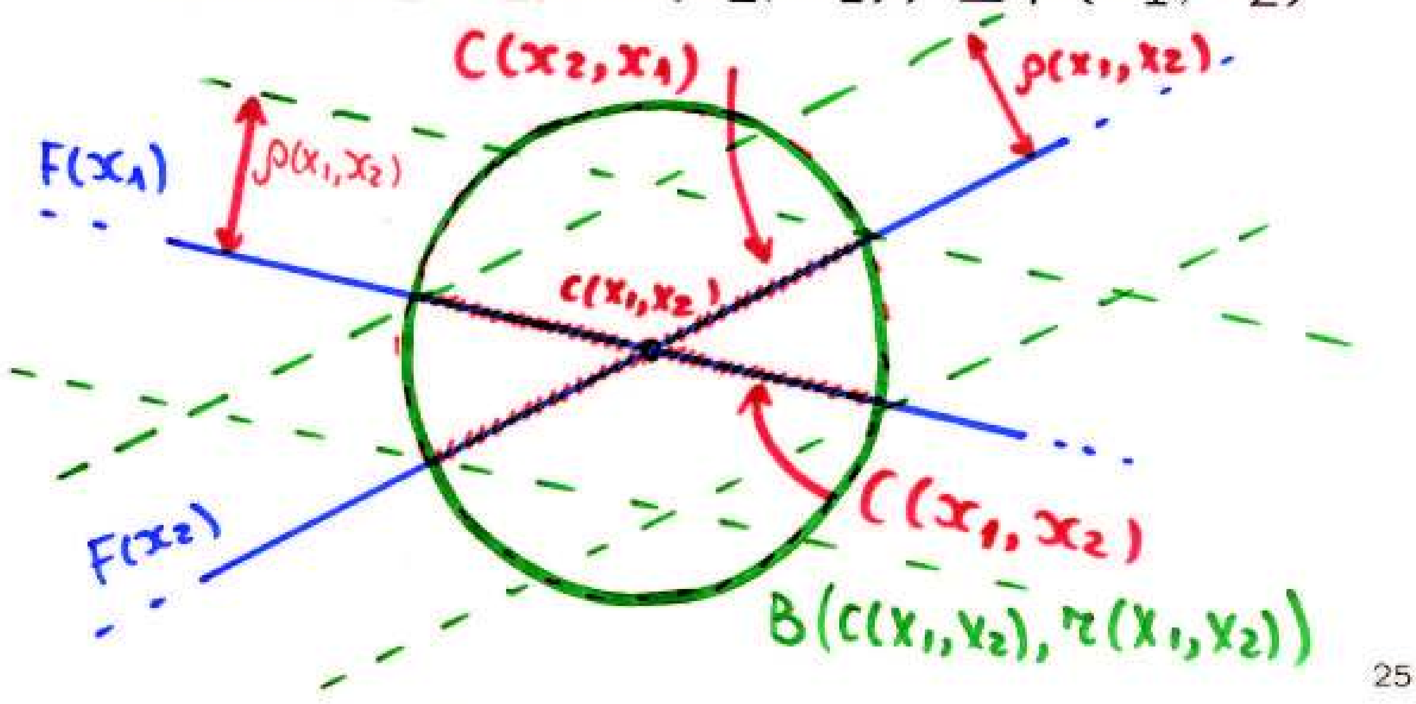
$$r(x_1, x_2) = \frac{1}{2} \operatorname{diam} C(x_1, x_2)$$

If $F(x_1) \parallel F(x_2) \Rightarrow C(x_1, x_2) := F(x_1)$**.** **We put** $r(x_1, x_2) = +\infty$ **and** $c(x_1, x_2)$ **to be an arbitrary point on** $F(x_1)$**.**



We have

$$C(x_1, x_2) = F(x_1) \cap B(c(x_1, x_2), r(x_1, x_2))$$
$$C(x_2, x_1) = F(x_2) \cap B(c(x_1, x_2), r(x_1, x_2))$$
$$\mathrm{d_H}(C(x_1, x_2), C(x_2, x_1)) \leq \rho(x_1, x_2)$$



Here $\mathsf{d_H}$ stands for the Hausdorff distance:

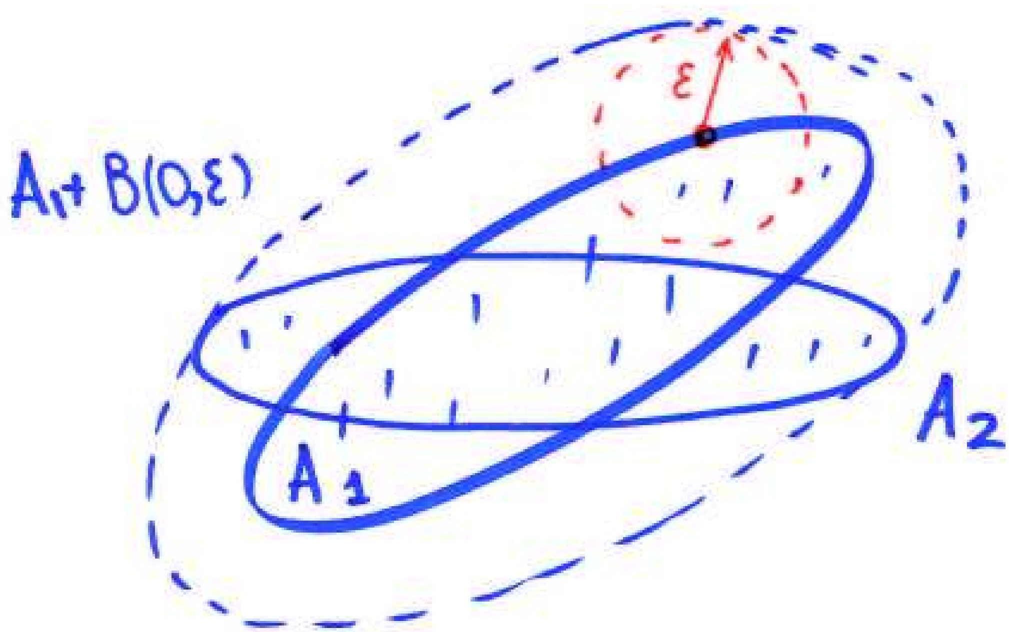
$$\mathsf{d_H}(A_1, A_2) := \inf\{\varepsilon > 0 :$$

$$A_1 + B(0,\varepsilon) \supset A_2, A_2 + B(0,\varepsilon) \supset A_1\}$$

or, equivalently,

$$\mathsf{d_H}(A_1, A_2) :=$$

$$\max\{\sup_{x\in A_1} \mathrm{dist}(x, A_2), \sup_{x\in A_1} \mathrm{dist}(x, A_2)\}$$



We introduce a family of ordered pairs of points from $\mathcal{M}$:

$$\widetilde{\mathcal{M}} := \{\tilde{x} = (x_1, x_2) : x_1, x_2 \in \mathcal{M}, x_1 \neq x_2\}$$

Given $\tilde{x} = (x_1, x_2) \in \widetilde{\mathcal{M}}$ **we put**

$C(\tilde{x}) := C(x_1, x_2)$, $c(\tilde{x}) := c(x_1, x_2)$, $r(\tilde{x}) := r(x_1, x_2)$ **and**

$$\mathbb{B}(\tilde{x}) := B(c(x_1, x_2), r(x_1, x_2))$$

We know that

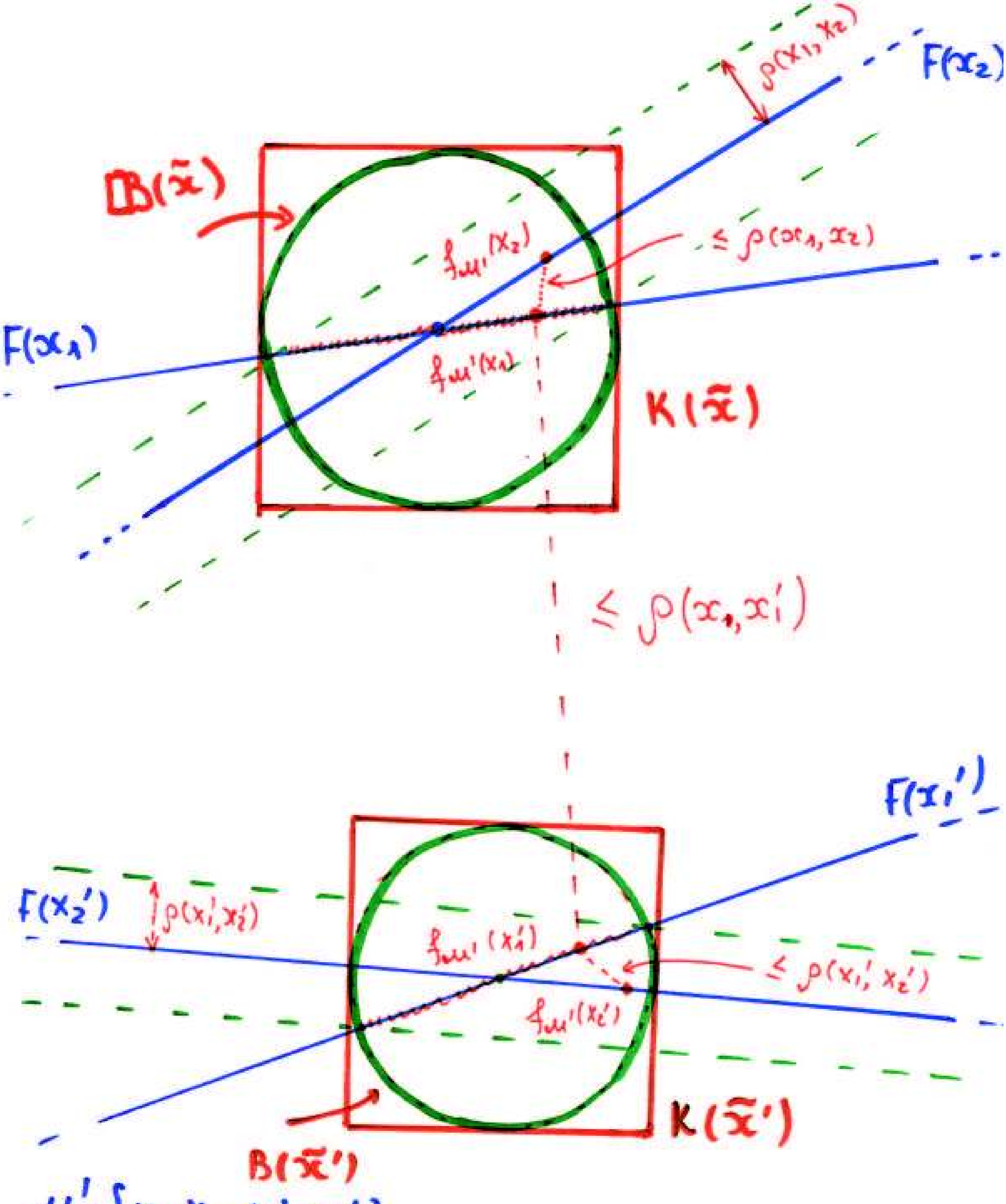
$$\mathrm{d_H}(F(x_1) \cap \mathbb{B}(\tilde{x}), F(x_2) \cap \mathbb{B}(\tilde{x})) \le \rho(x_1, x_2)$$

Prove that

$$\mathrm{dist}(\mathbb{B}(\tilde{x}), \mathbb{B}(\tilde{x}')) \le \rho(x_1, x_1')$$

for every

$$\tilde{x} = (x_1, x_2), \ \tilde{x}' = (x_1', x_2') \in \widetilde{\mathcal{M}}$$



$\mathcal{M}' = \{x_1, x_2, x_1', x_2'\}$

$A = f_{\mathcal{M}'}(x_1) \in K(\tilde{x})$, $A' = f_{\mathcal{M}'}(x_1') \in K(\tilde{x}')$

$\|A - A'\| \leq \rho(x_1, x_1')$

In fact, let

$$A := f_{\mathcal{M}'}(x_1), \quad A' := f_{\mathcal{M}'}(x_1')$$

Then $A \in \mathbb{B}(\tilde{x})$ **and** $A' \in \mathbb{B}(\tilde{x}')$**. Furthermore,**

$$\|A - A'\| \leq \rho(x_1, x_1')$$

Hence $A \in K(\tilde{x})$ **and** $A' \in K(\tilde{x}')$**, and**

$$\|A - A'\|_\infty \leq \rho(x_1, x_1')$$

Step 2. Given

$$\tilde{x} = (x_1, x_2), \quad \tilde{x}' = (x_1', x_2') \in \widetilde{\mathcal{M}}$$

let

$$\tilde{\rho}(\tilde{x}, \tilde{x}') := \rho(x_1, x_1')$$

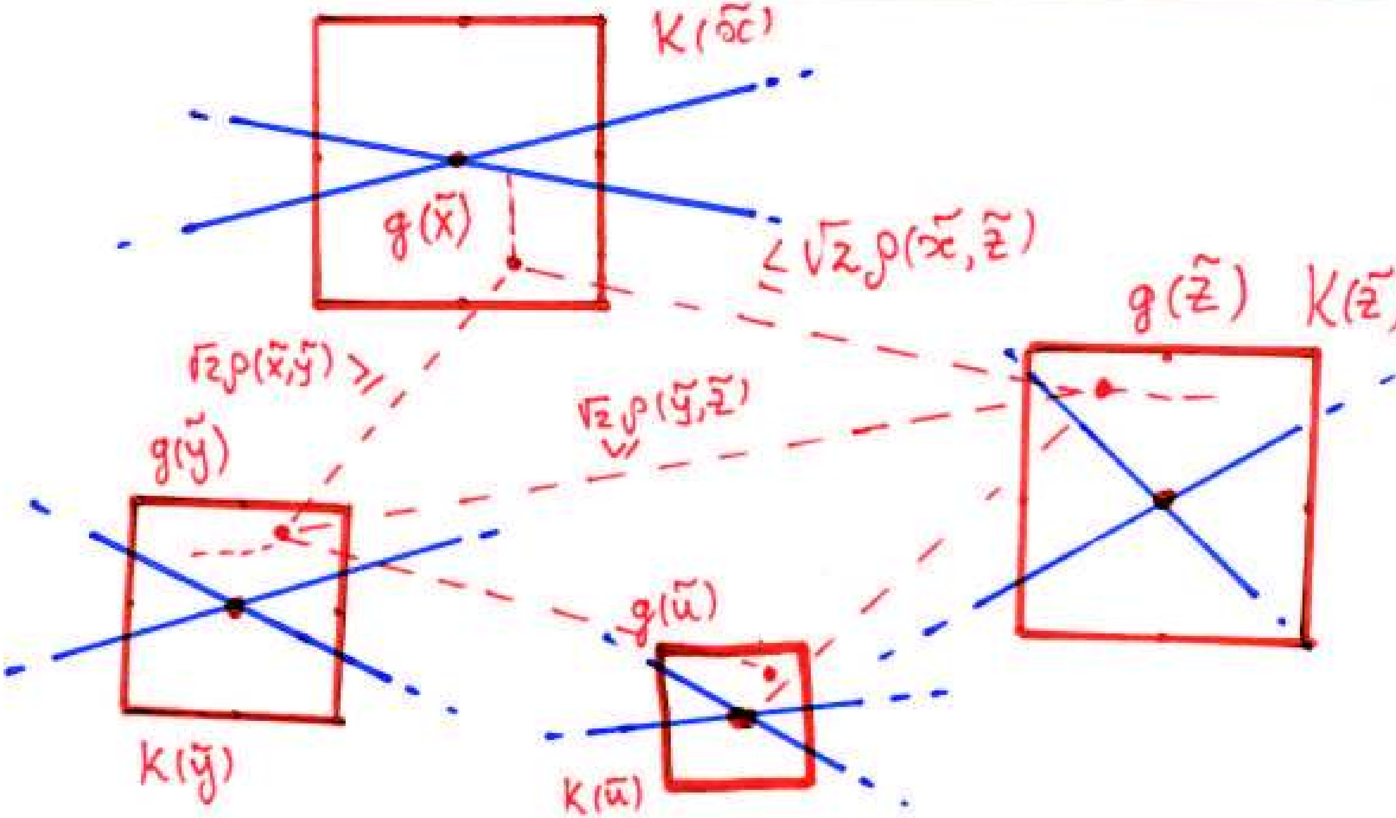
Let

$$K := K(\tilde{x}), \quad \tilde{x} \in \widetilde{\mathcal{M}}$$

be a set-valued mapping from $\widetilde{\mathcal{M}}$ into the family of all squares in $\mathbf{R}^2$.

We have proved that the restriction $K|_{\{\tilde{x},\tilde{x}'\}}$ to every subset $\{\tilde{x},\tilde{x}'\}$ of $\widetilde{\mathcal{M}}$ has a Lipschitz selection (with respect to $\tilde{\rho}$) with the Lipschitz constant (in ℓ^2_∞) at most 1.

Then K on all of $\widetilde{\mathcal{M}}$ has a Lipschitz selection $g : \widetilde{\mathcal{M}} \to \mathbf{R}^2$ with $\|g\|_{\mathrm{Lip}(\widetilde{\mathcal{M}},\ell^2_\infty)} \le 1$.



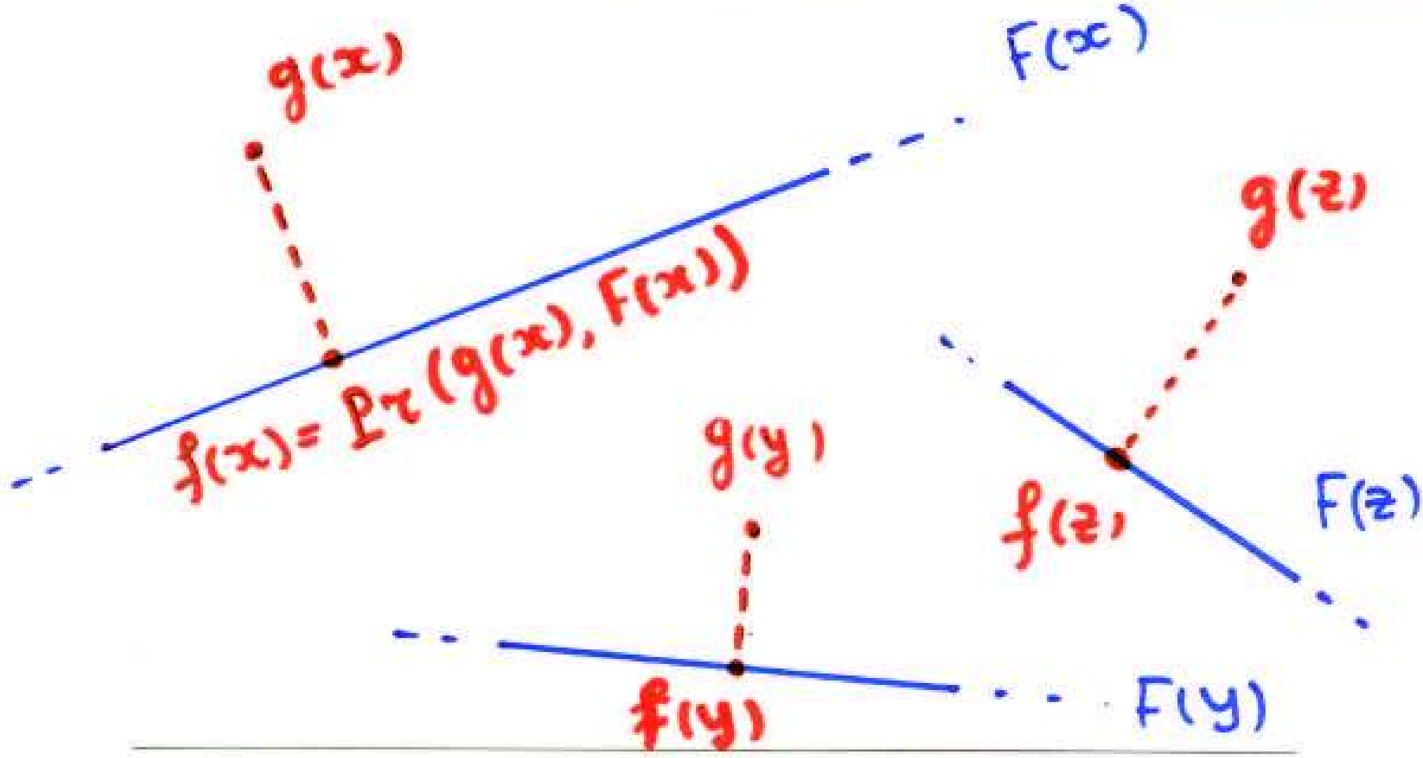
Compare $g(x, x_2)$ and $g(x, x_2')$:

$$\|g(\tilde{x}) - g(\tilde{x}')\|_\infty \leq \tilde{\rho}(\tilde{x}, \tilde{x}') =: \rho(x, x) = 0$$

$\Rightarrow g(\tilde{x}) = g(\tilde{x}')$ if

$$\tilde{x} = (x, x_2), \quad \tilde{x}' = (x, x_2')$$

$g(\tilde{x}) = g(x_1, x_2)$ depends only on $x_1 \Rightarrow g$ defines a <u>mapping on $\mathcal{M}$</u> which we denote by the same symbol g.



Put

$$f(x) := \mathsf{Pr}(g(x), F(x))$$

where $\mathsf{Pr}(\cdot, L)$ stands for the orthogonal projection on a straight line $L \subset \mathrm{R}^2$.

Clearly, $f(x) \in F(x)$, i.e., f is a <u>selection of F.</u> Prove

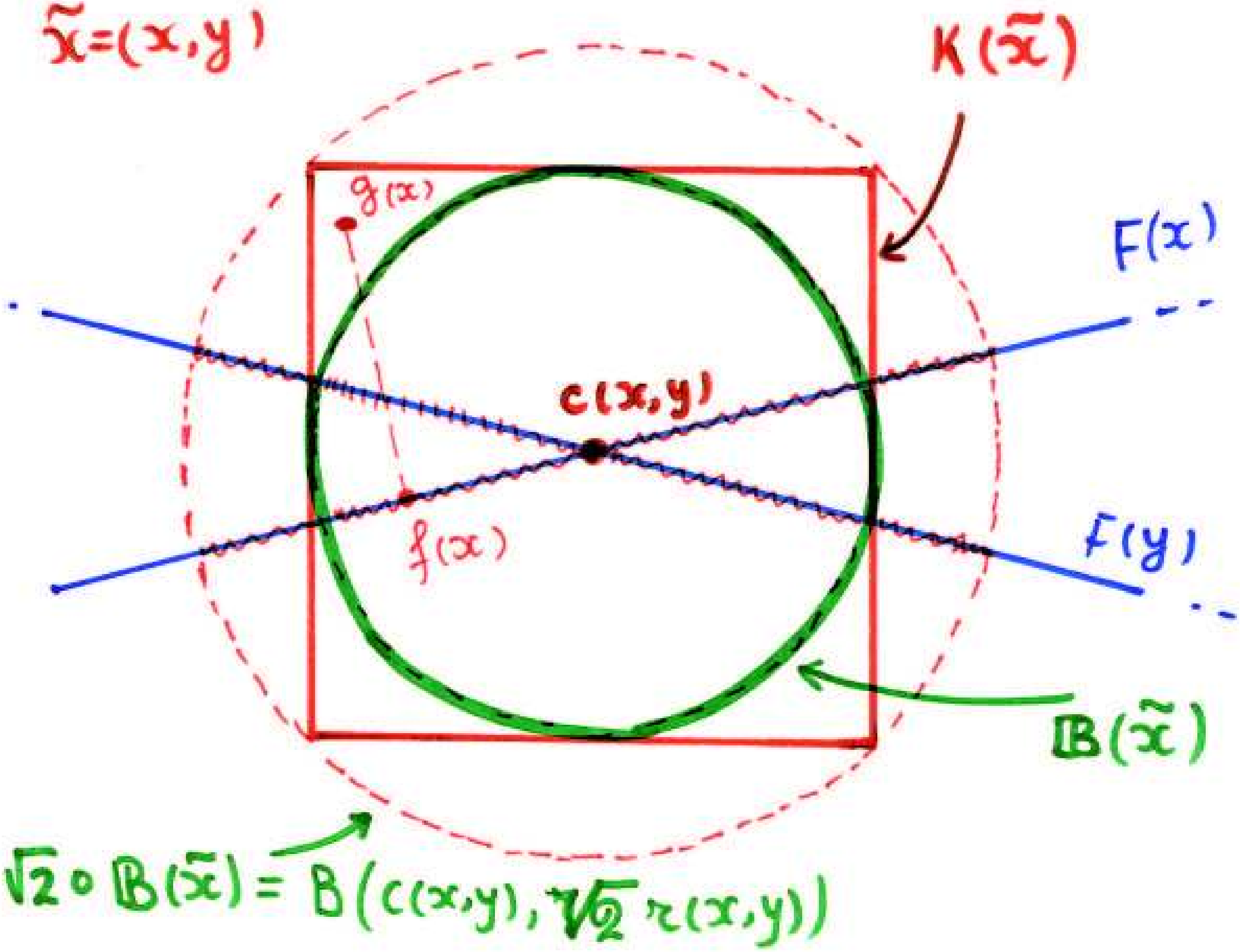
$$\|f(x) - f(y)\| \le 2\sqrt{2}\rho(x, y), \quad x, y \in \mathcal{M}.$$

For every $x, y \in \mathcal{M}$ we have

$$g(x) = g(x,y),\ g(y) = g(y,x) \Rightarrow$$

$$\|g(x) - g(y)\| \le \sqrt{2}\|g(x) - g(y)\|_\infty$$

$$= \sqrt{2}\|g(x,y) - g(y,x)\|_\infty \le \sqrt{2}\rho(x,y)$$



Since g is a selection of K, for every $\tilde{x} = (x, y) \in \widetilde{\mathcal{M}}$ we have

$$g(x) = g(\tilde{x}) \in K(\tilde{x}), \qquad y \in \mathcal{M}.$$

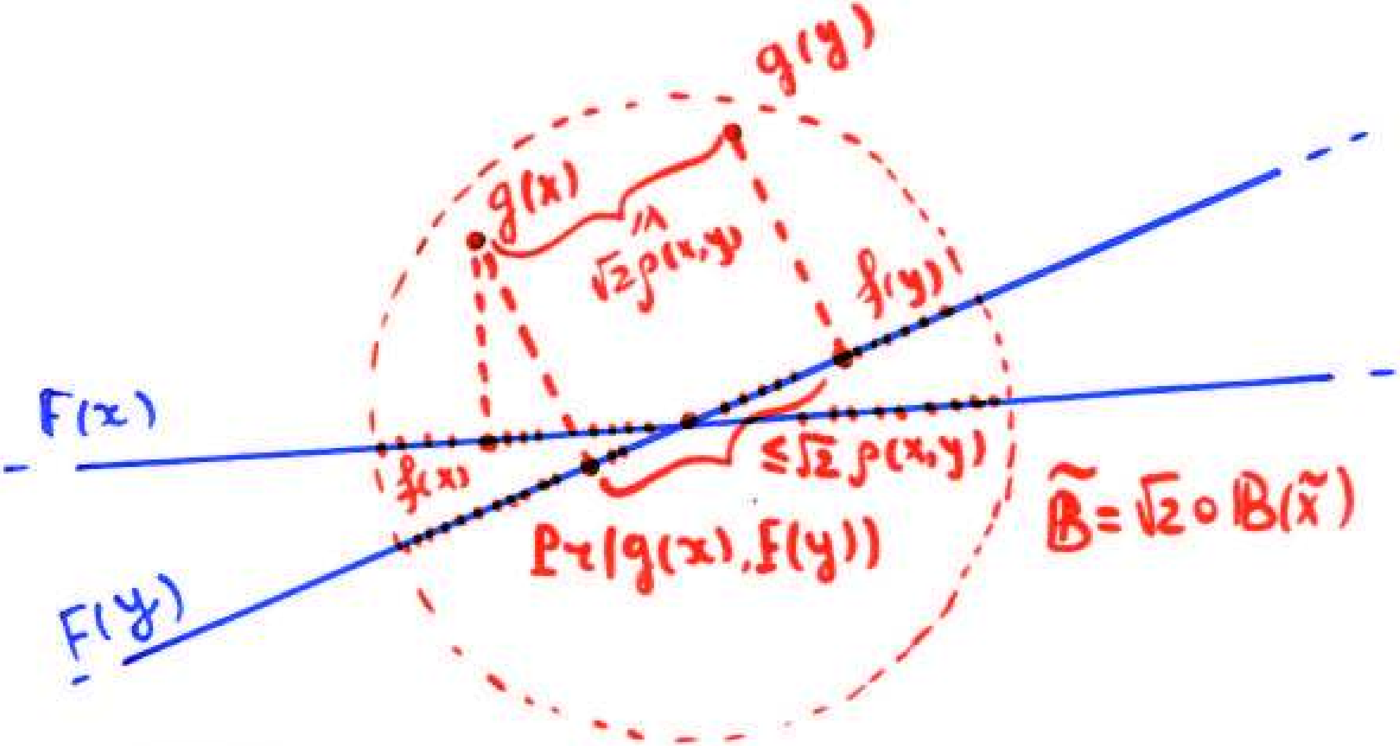
But $K(\tilde{x}) \subset \sqrt{2} \circ \mathsf{B}(\tilde{x})$ so that

$$g(x) \in \sqrt{2} \circ \mathsf{B}(\tilde{x}) = B(c(x, y), \sqrt{2} r(x, y)).$$

By dilation with respect to $c(x, y)$

$$\mathsf{d}_{\mathsf{H}}(F(x) \cap \sqrt{2} \circ \mathsf{B}(\tilde{x}), F(y) \cap \sqrt{2} \circ \mathsf{B}(\tilde{x}))$$

$$\leq \sqrt{2} \rho(x, y)$$



Lemma. L_1, L_2- subspaces of R^2, $\dim L_1 = \dim L_2 = 1$. Let $B = B(0,r)$, and let $a \in B$. Then

$$\|\operatorname{Pr}(a,L_1)-\operatorname{Pr}(a,L_2))\| \leq \mathrm{d_H}(L_1 \cap B, L_2 \cap B)$$