Gradual Variation and Numerical Solution of Partial Differential Equations

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Outlines

- The construction method of McShane-Whitney-Kirszbraun Lipschitz function extension is Good in theory, but may not work for actual data reconstruction. (A bounded function is always Lipschitz . The Lipschitz constant may be too big for data fitting) x^2 is not Lipschitz; Lipschitz is too narrow for real data reconstruction.

- Any continuous function on the constructive-compact-metric-space will have a uniform gradually varied approximation on ε-net (Chen 1990,2005). Gradually varied functions can be applied to all local Lipschitz function.

Outlines

- Digital-Discrete Methods(Chen 2010)
  - Get gradually varied extension (digital)
  - Replace the actual values (discrete). Do finite differences
  - Get gradually varied derivatives; repeat. This is an important conceptual change.

Outlines

- Use Taylor Expansion to get C^{(k)}. (Inspirited by Whitney Workshop 2009.) This is the solution for single surface. It can be used as initial surface for Heat equations.
- Use implicit finite difference formula for heat equations.
- Application for groundwater equations
- Other work: Piecewise harmonic reconstruction for functions on manifolds.
- Other connection: Use absolute retract; gradually varied extension to trees

Whitney Construction

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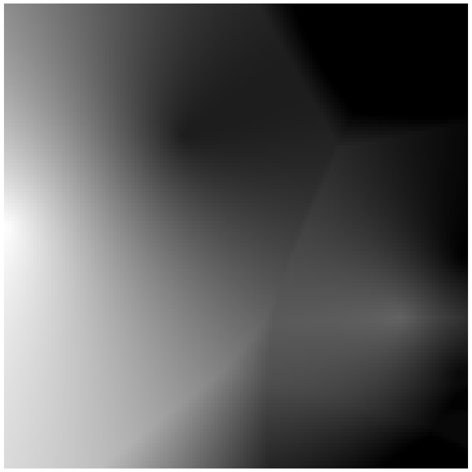
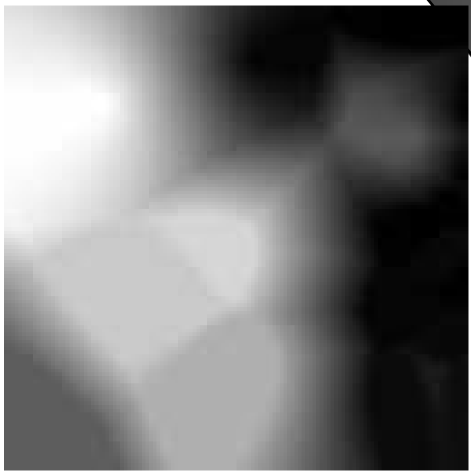




McShane-Whitney mid extensions: set one containing 10 points ; set two containing 29 points. *F=(INF+SUP)/2* where INF =minimal extension that is Lipschitz. SUP = maximum extension (SUP).

Gradually Varied Construction

Let $A_1 < A_2 < \ldots < A_n$. The Concept of Gradual Variation: Let function f: D $\{A_1, A_2, \ldots, A_n\}$. If a and b are adjacent in graph D, then it is implied that $f(a)=f(b)$, or $f(b) = A_{i-1}$ or A_{i+1} when $f(a)=A_i$. Point $(a,f(a))$ and $(b,f(b))$ are then said to be gradually varied. *A 2D function (surface) is said to be gradually varied if every adjacent pair is gradually varied.*

Discrete Surface Fitting: Given $J \subseteq D$, and f: J $\{A_1, A_2, \ldots A_n\}$, decide if there is exists an F: D $\{A_1, A_2, \ldots, A_n\}$ such that F is gradually varied where $f(x)=F(x)$, x in J.

Theorem *(Chen, 1989) The necessary and sufficient conditions for the existence of a gradually varied extension F* is: for all x,y in J, $d(x,y) \geq |i-j|$, $f(x)=A_i$ and $f(y)=A_j$, where d is the distance between x and y in D.

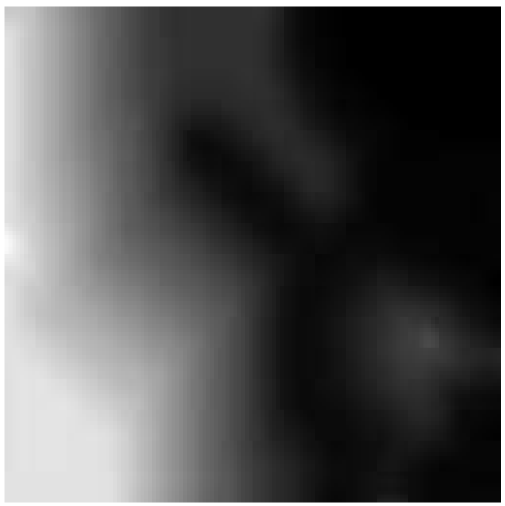
Gradually Varied Approximation in Constructive Compact Space

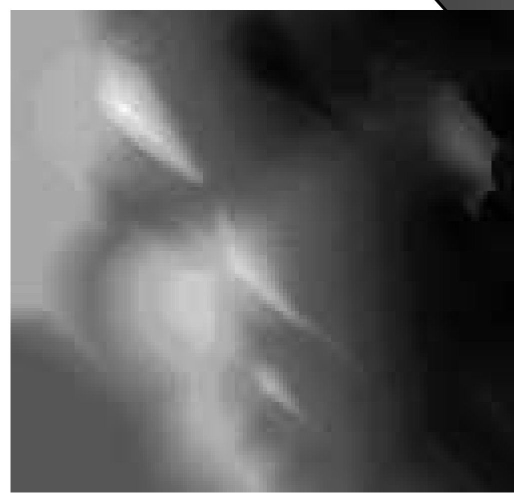
- Any continuous function on the constructive-compact-metric-space will have a uniform gradually varied approximation on ε-net (Chen 1990,2005). Gradually varied functions can be applied to all local Lipschitz function.

L. Chen, Gradually varied surfaces and gradually varied functions, in Chinese, 1990; in English 2005 CITR-TR 156, U of Auckland.

Professor Douglas Bridge, the co-author of the famous book "Constructive Analysis," replied: "I've had a look (not a detailed one) at the relevant section of your paper. It seems fine to me."

Gradually Varied Construction





Gradually varied function extensions: set one containing 10 points ; set two containing 29 points.

Gradually varied functions

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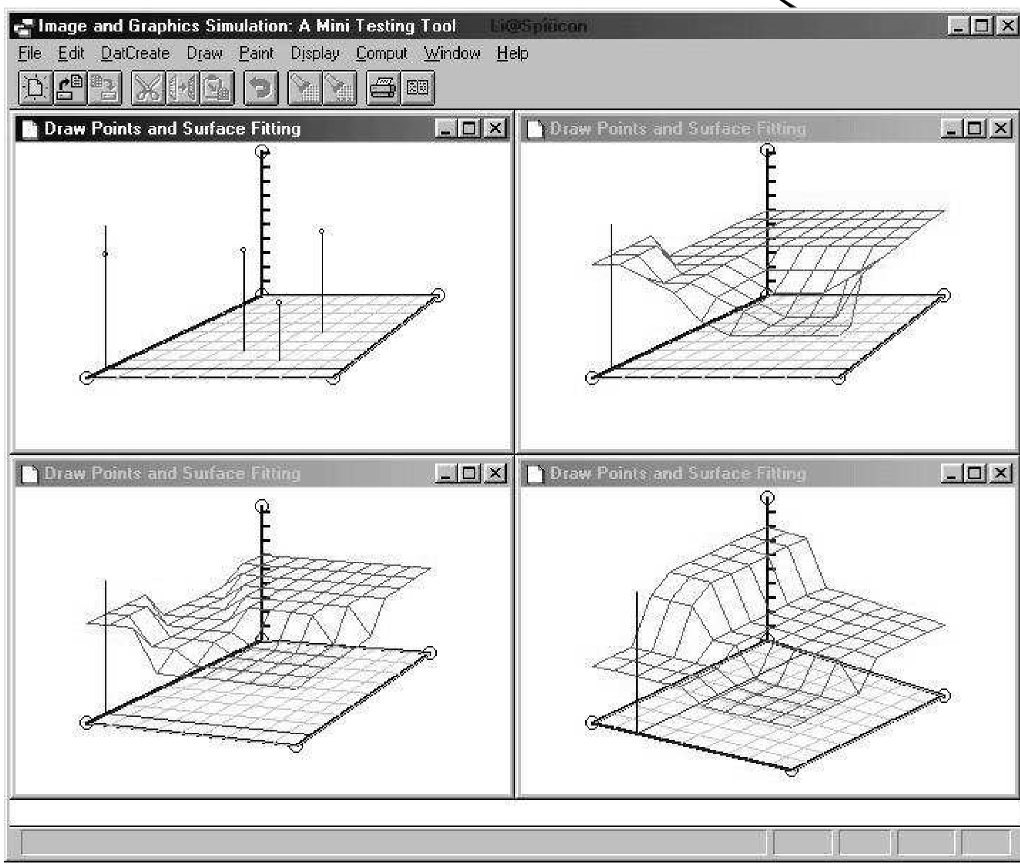


Gradually Varied Derivatives

Force result of the finite difference to be gradually varied.

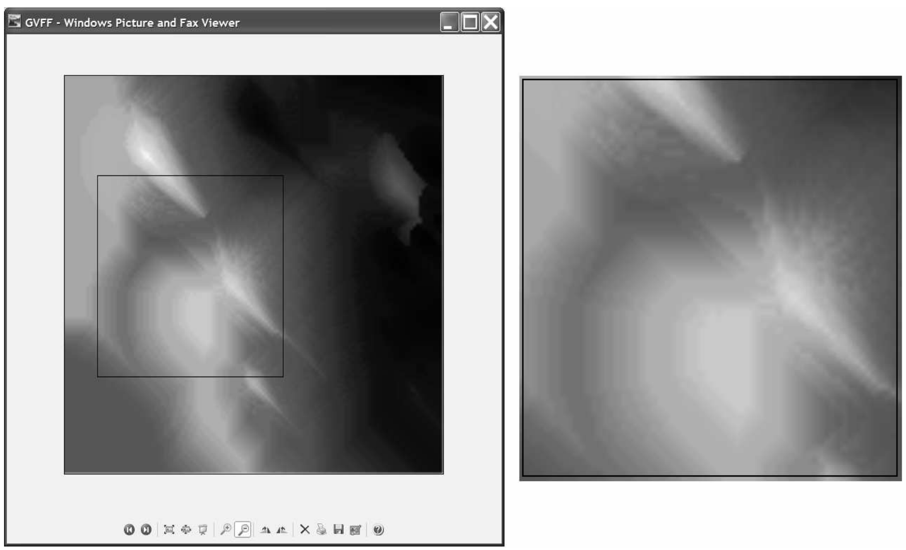
- selecte derivatives on at least sample points
- do gradually vaired interpolation

Taylor Extension Based on Gradually Varied Derivatives

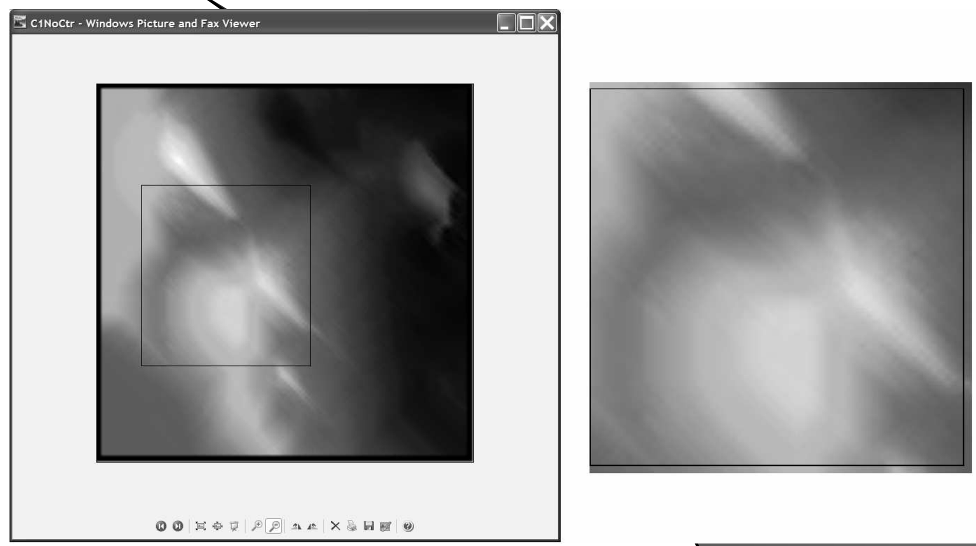
$$f(x_1, \cdots, x_d) =$$
$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1+\cdots+n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \right) (a_1, \ldots, a_d).$$

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)\, f_x(x_0, y_0) + (y - y_0)\, f_y(x_0, y_0)$$

Taylor Extension Based on Gradually Varied Derivatives (Examples)

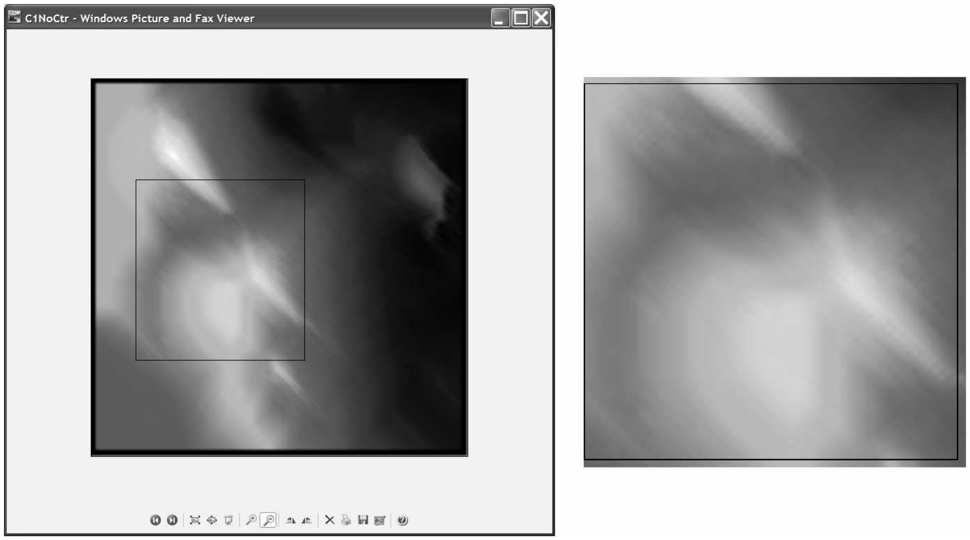


(a) The "continuous" function.

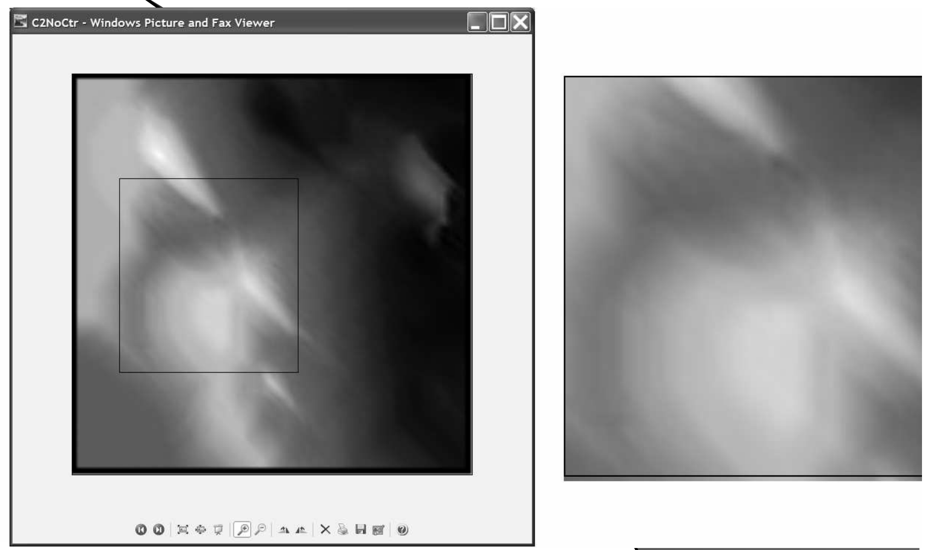


(b) The first order derivative
function.

Taylor Extension Based on Gradually Varied Derivatives (Examples)



(b) The first order derivative function.



(c) The second order derivative function.

(c) Is smoother than (b) ; (c) still contains detailed information not just look like an average of surrounding points.

PDE Research

- Find connection between flow equations and gradually varied functions
- Data Input format and database (T. Branham)
- Algorithm Design gradually varied function alone combined with difference form of flow equations
- Real data processing
- Testing

Groundwater flow equations

- Darcy's Law

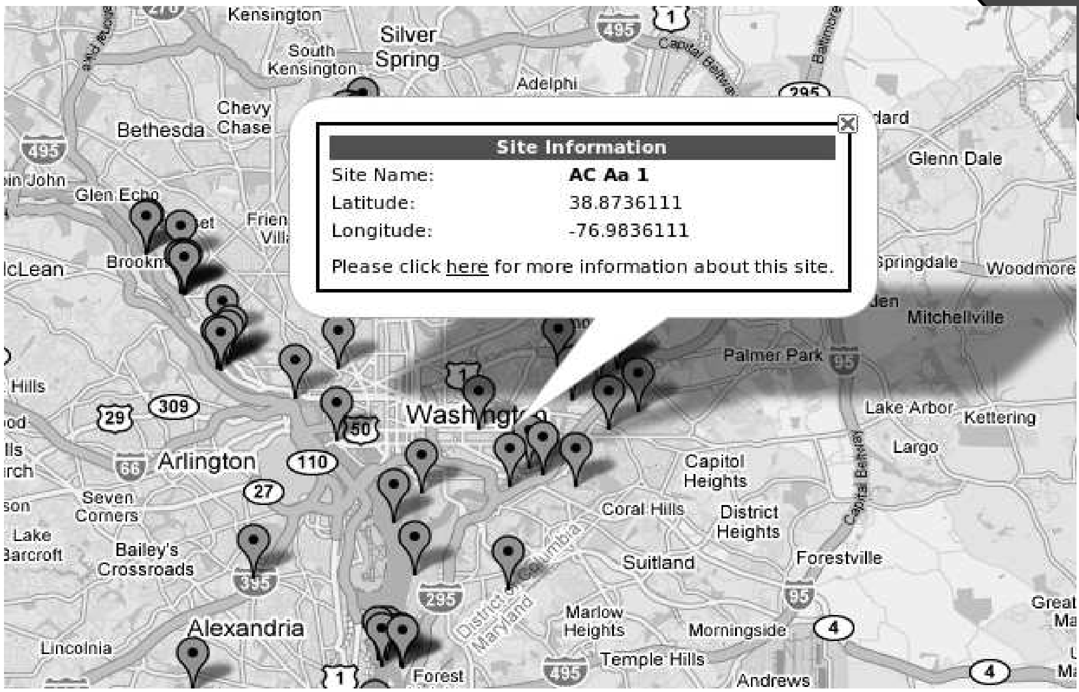
$$\frac{\Delta M_{stor}}{\Delta t} = \frac{M_{in}}{\Delta t} - \frac{M_{out}}{\Delta t} - \frac{M_{gen}}{\Delta t}$$

- Differential form

$$\frac{\partial h}{\partial t} = \alpha \left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right] - G$$

Data Preparation

Use PHP build a web application to access groundwater log data in VA and MD. Data is stored in MySQL databases.



Data Preparation

- *Travis L. Branham,* ***Development of a Web-based Application to Geographically Plot Water Quality Data, UDC, 2008***

Algorithm Design

Individual surface fitting

- Use original algorithm
- Problem: real data does not satisfy the condition of fitting
- New algorithm based on the sample points contribution to the fitting point.

Algorithm Design

```
for (k=0;k<nGuildPoints;k++){
        ii=(latIndex[k]-latMin)/latDet;
        jj=(longtIndex[k]-longtMin)/longtDet;

        distance=sqrt((ii -i)*(ii-i)+(jj -j)*(jj-j));
         temp_j=abs((array[i][j] - dat[k][time]))/Ratio-distance;
          if(temp_j>0){ // not satisfy gvs condition

        if( array[i][j] > dat[k][time])
                temp=-temp_j *Ratio ;
        else
                temp= temp_j *Ratio;
                array[i][j]=array[i][j]+temp ;
    }
```

Algorithm Design

- Sequential surface fitting
  - Involve flow equation
  - Iteration
  - Error control

Algorithm Design

Sequential surface fitting

Involve flow equation (implicit form)

h2-h1 = alpha (h2(x-1,y)+h2(x+1,y)-2h2(x,y) +
h2(x,y-1)+h2(x,y+1)-2h2(x,y)) -G

f4= (h2(x,y)-h1(x,y)+G)/alpha + 4*h2(x,y)
- h2(x-1,y)+h2(x+1,y)+ h2(x,y-1)+h2(x,y+1)

Algorithm Design

Sequential surface fitting

Iteration

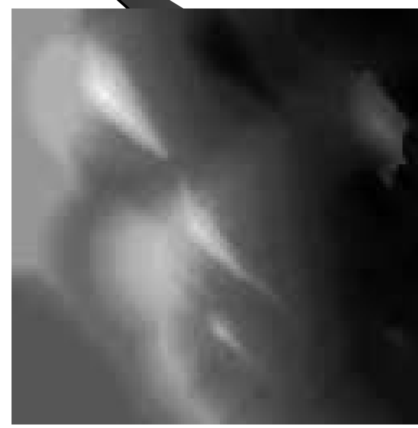
– use gradually varied function to get initial surface

Real data processing and application

Individual surface fitting

Day 1

Day 30

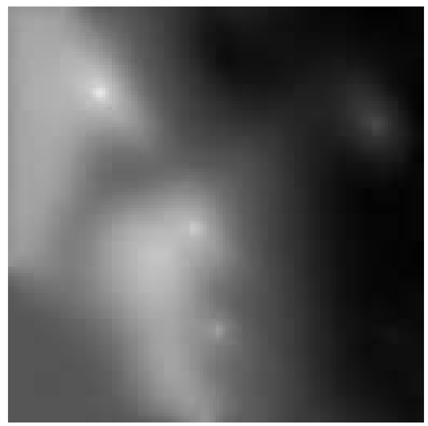


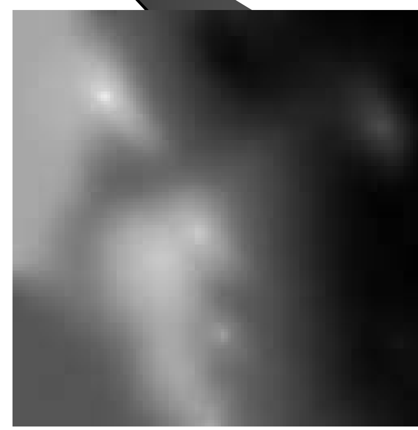
Day50

(Starting with fitted surface at each time the process will be faster to converge. It will not affect to the final result if there are enough iterations)

Real data processing and application

Water equation solution

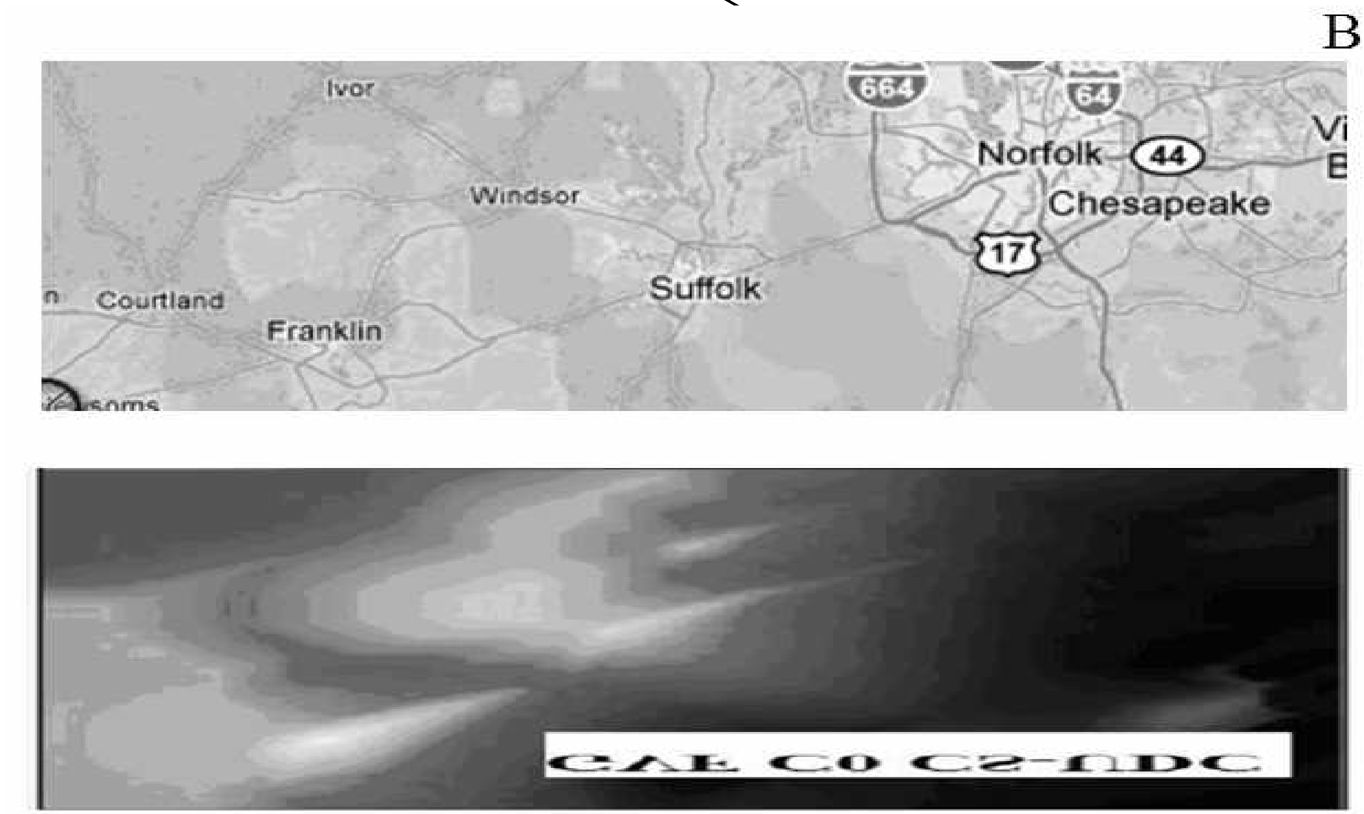




Day 3 Day 30 Day50

(The little bright dots indicates that the errors in the reconstruction)

Testing



A

Dimensions (Latitude, Longitude)

A = (36.62074879, -77.17746768)

B = (36.92515020, -76.00948530)

Testing

B


Ivor
664
64
Norfolk
44
Chesapeake
Windsor
17
Suffolk
Courtland
Franklin

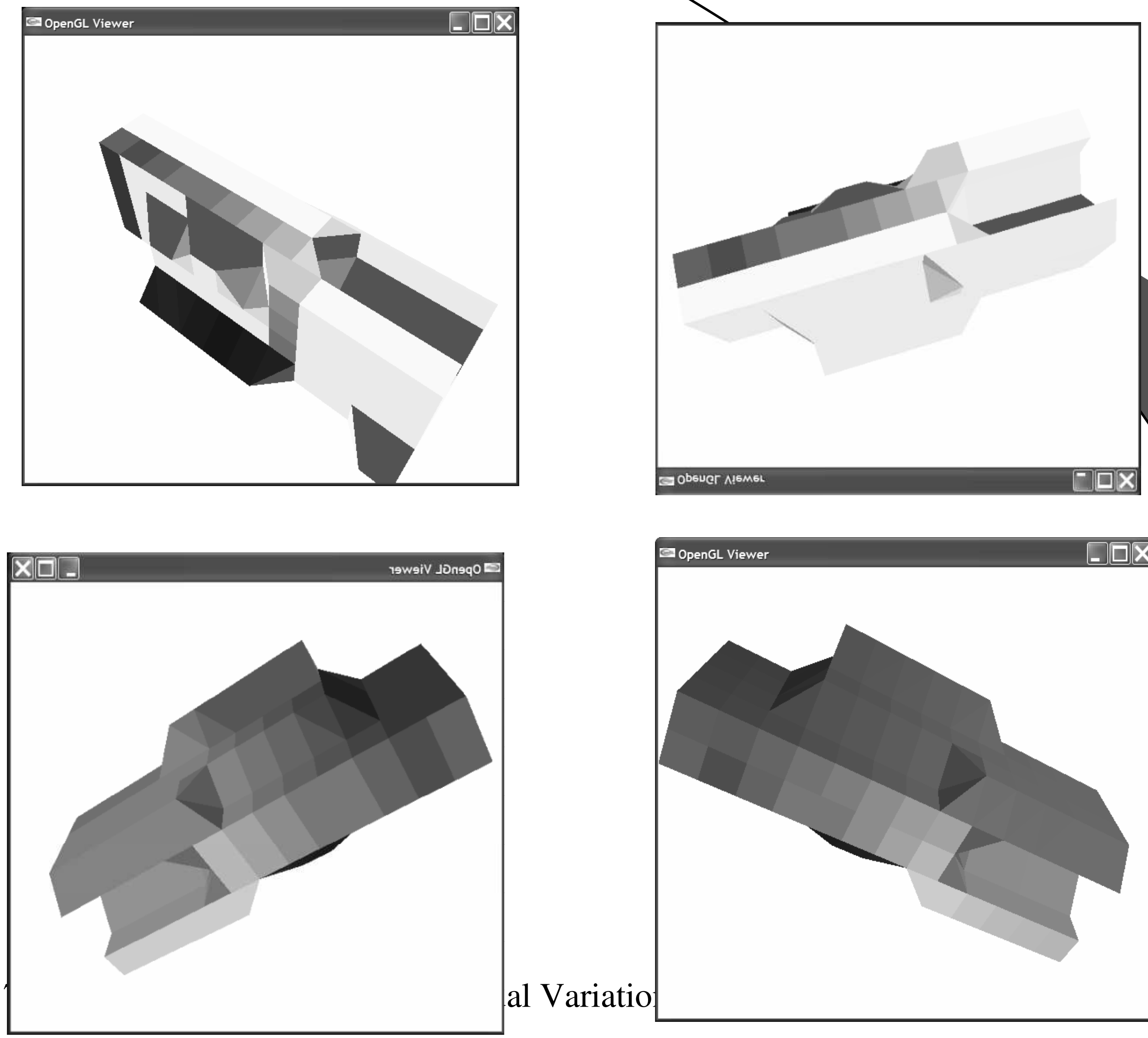
A
D

Testing

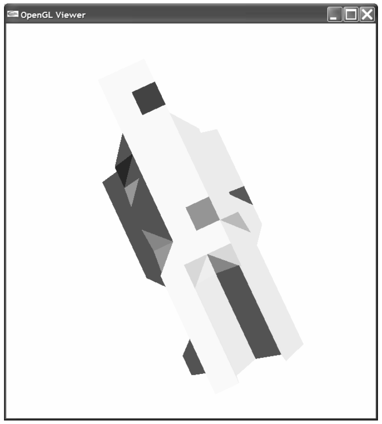
Selected Points used in reconstruction
One can find the location at
http://findlatitudeandlongitude.com/

4.65	36.62074879	-76.10938540
75.37	36.92515020	-77.17746768
6.00	36.69104276	-76.00948530
175.80	36.78431615	-76.64328700
168.33	36.80403855	-76.73495750
157.71	36.85931567	-76.58634110
208.26	36.68320624	-76.91329390
7.26	36.78737704	-76.05153760

Function extension on Manifolds



Piecewise Harmonic Function extension on Manifolds



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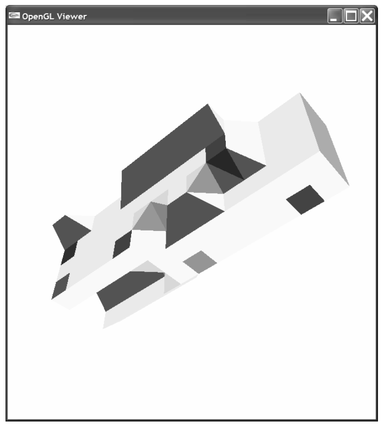
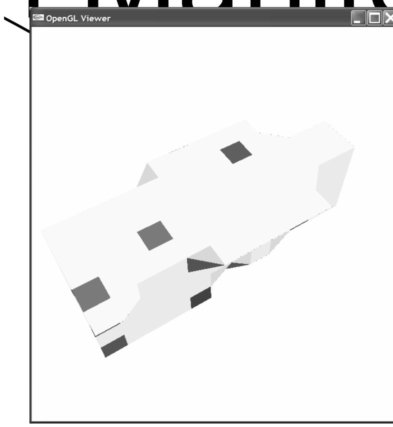




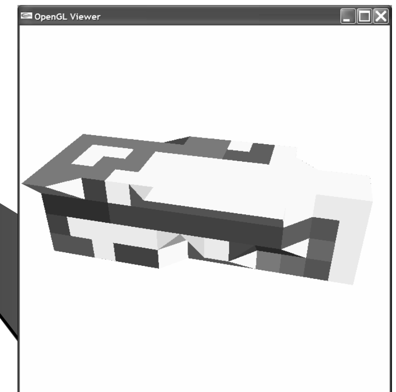
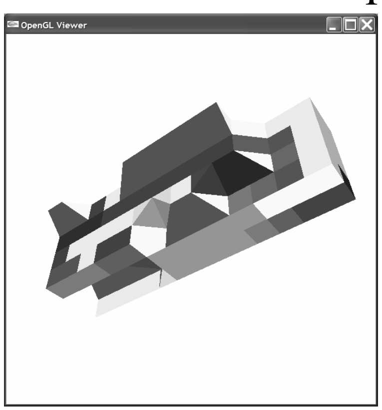
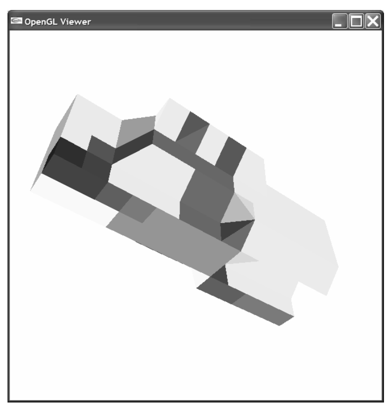
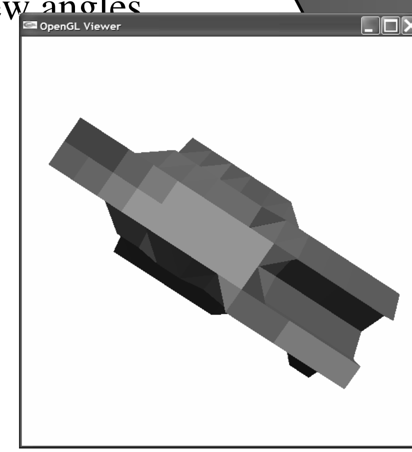


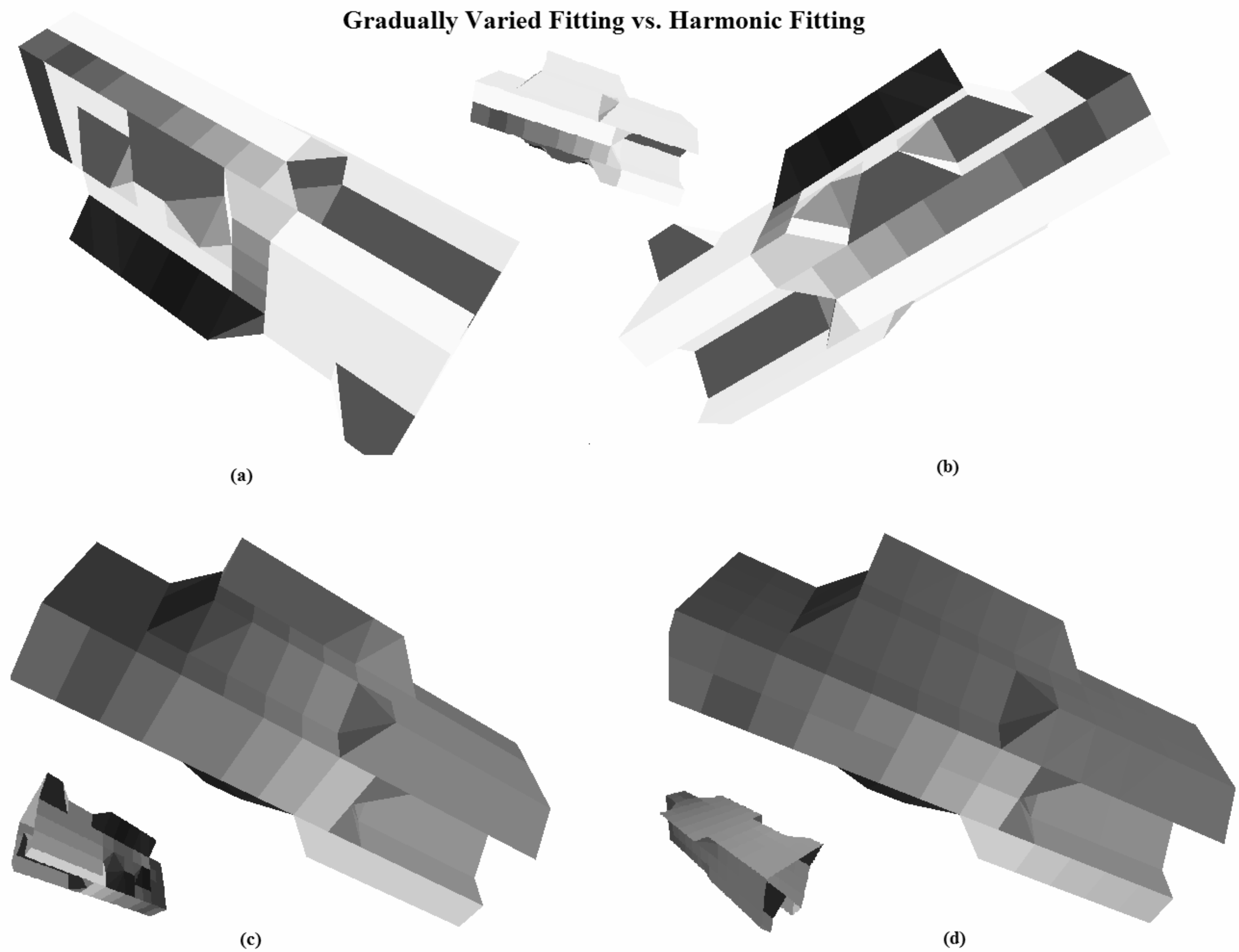
Fig. 4.5. Twelve guiding points from difference view angles







Gradually Varied vs Harmonic on Manifolds



Gradually Varied Fitting vs. Harmonic Fitting: (a) The selected cells form a boundary curve that is gradually varied. (b) Another view of the guiding points. (c) The gradually varied fitting (GVF) result. (d) The

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Acknowledgements

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