A note on the simplicity and the universal covering of some Kac-Moody group

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Contents

§ Recent Topic - Simplicity -

2009, P.E. Caprace - B. Rémy

Theorem

Let A be an $n \times n$ indecomposable GCM, and $\mathbb{F}_q$ a finite field with $q = p^\ell$ elements. Let $G_u(A, \mathbb{F}_q)$ be the universal Kac-Moody group over $\mathbb{F}_q$ of type A, and $G'_u(A, \mathbb{F}_q) = [G_u(A, \mathbb{F}_q), G_u(A, \mathbb{F}_q)]$ its derived subgroup. We suppose that A is not of affine type, and $q \geq n > 2$. Then $G'_u(A, \mathbb{F}_q)$ is simple modulo its center.

2012, P.E. Caprace - B. Rémy

Theorem

Let $A = \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$ *be a* 2×2 *hyperbolic GCM, that is,* $a > 4$*, and* $\mathbb{F}_q$ *a finite field with* $q > 3$*. Let* $G_u(A, \mathbb{F}_q)$ *be the universal Kac-Moody group over* $\mathbb{F}_q$ *of type A. Then* $G_u(A, \mathbb{F}_q)$ *is simple modulo its center.*

Remaining Case (with B. Rémy)

Theorem

Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ *be a* 2×2 *hyperbolic GCM satisfying* $ab > 4$ *with* $a > 1$ *and* $b > 1$*, and* F *the algebraic closure of a finite field* $\mathbb{F}_p$*. Let* $G_u(A, F)$ *be the universal Kac-Moody group over* F *of type* A*. Then* $G_u(A, F)$ *is simple modulo its center.*

Uniformization

Theorem

Let A be an indecomposable GCM, and F the algebraic closure of a finite field $\mathbb{F}_p$. Let $G_u(A, F)$ be the universal Kac-Moody group over F of type A, and $G'_u(A, F)$ its derived subgroup. We suppose that A is not of affine type. Then $G'_u(A, F)$ is simple modulo its center.

Simple Group with Trivial Schur Multiplier

Theorem

Let A be an indecomposable GCM, and F the algebraic closure of a finite field $\mathbb{F}_p$. Then the following two conditions are equivalent.

(1) $\det(A) = \pm p^c$ for some $c \geq 0$.

(2) $G_u(A, F)$ is a simple group with trivial Schur multiplier.

Rank 2 Case

Example

Let F be the algebraic closure of a finite field $\mathbb{F}_p$. Then the following groups are simple groups with trivial Schur multipliers.

(1) $G_u(\begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}, F)$, $p = 2$

(2) $G_u(\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}, F)$, $p = 5$

(3) $G_u(\begin{pmatrix} 2 & -5 \\ -17 & 2 \end{pmatrix}, F)$, $p = 3$

Anther Infinite Field

Question

Let A be an indecomposable non-finite & non-affine GCM, and F another infinite field.

(1) Is $G_u'(A, F)$ simple modulo its center ?

(2) Especially how about $G_u'(A, \mathbb{C})$?

§ Notation

Set Up I

Let A be an $n \times n$ GCM, and put $n' = corank(A)$. We let $G_u(A, -)$ denote the so-called Tits group functor associated with A. Let $G_u(A, F)$ be the universal Kac-Moody group over F of type A, and $G'_u(A, F)$ the derived subgroup of $G_u(A, F)$. There is an embedding : $T = Hom(\mathbb{Z}^{n+n'}, F^{\times}) \overset{\exists}{\hookrightarrow} G_u(A, F)$.

Set Up II

Let $\mathfrak{g}$ be the Kac-Moody algebra over $\mathbb{C}$ of type A, and Δ^{re} the set of real roots. For each $\alpha \in \Delta^{\mathrm{re}}$, there is a group homomorphism $x_\alpha : F \hookrightarrow G_u(A, F)$. Put $U_\alpha = \mathrm{Im}(x_\alpha) = \{x_\alpha(t) \mid t \in F\}$. Then,

$$G_u(A, F) = \langle T,\ U_\alpha \mid \alpha \in \Delta^{\mathrm{re}} \rangle,$$
$$G_u'(A, F) = \langle U_\alpha \mid \alpha \in \Delta^{\mathrm{re}} \rangle,$$
$$G_{ad}'(A, F) = G_u'(A, F)/Z(G_u'(A, F)),$$
$$G_u(A, F) = G_u'(A, F) \text{ if } det(A) \neq 0,$$
$$G_{ad}(A, F) = G_{ad}'(A, F) \text{ if } det(A) \neq 0.$$

§ Presentation

1986, J. Tits

Theorem (G'_u-version)

The group $G'_u(A, F)$ is presented by the generators $x_\alpha(t)$ with $\alpha \in \Delta^{\mathrm{re}}$ and $t \in F$, and the following defining relations:

(A) $x_\alpha(s)x_\alpha(t) = x_\alpha(s+t)$,
(B) $[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}s^i t^j)$,
(B′) $w_\alpha(u)x_\beta(t)w_\alpha(-u) = x_{\beta'}(t')$,
(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$.

Condition for (B)

Let $\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ (\forall h \in \mathfrak{h})\},$$
$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}, \quad \mathfrak{g}_0 = \mathfrak{h}.$$

Put $Q_{\alpha,\beta} = \{i\alpha + j\beta \mid i, j \in \mathbb{Z}_{>0}\} \cap \Delta$.

Then, we have

$$\text{(B)} \ [x_\alpha(s), x_\beta(t)] = \prod_{Q_{\alpha,\beta}} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j} s^i t^j)$$

whenever $Q_{\alpha,\beta} \subset \Delta^{\mathrm{re}}$.

Relation (B), 1987, J. M.

Theorem

There are essentially five type relations in (B).

$$[x_\alpha(s), x_\beta(t)] = 1$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm(r+1)st)$$

$$r = max\{i \in \mathbb{Z} \mid \beta - i\alpha \in \Delta^{\mathrm{re}}\}$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2 t)\cdot x_{\alpha+2\beta}(\pm 3st^2)$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)\cdot x_{3\alpha+\beta}(\pm s^3 t)x_{3\alpha+2\beta}(\pm 2s^3 t^2)$$

About (B′), (C)

For $u, v \in F^\times$, we put

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u),$$
$$h_\alpha(u) = w_\alpha(u)w_\alpha(-1).$$

Then,

(B′) $w_\alpha(u)x_\beta(t)w_\alpha(-u) = x_{\beta'}(t'),$

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv),$

where h_α is the coroot of α and

$$\beta' = \beta - \beta(h_\alpha)\alpha, \qquad t' = \pm u^{-\beta(h_\alpha)}t.$$

$SL_2(F)$

For each $\alpha \in \Delta^{\mathrm{re}}$, there is a group isomorphism $\varphi_\alpha : \langle U_\alpha,\ U_{-\alpha}\rangle \xrightarrow{\simeq} SL_2(F)$ satisfying

$$x_\alpha(t) \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$w_\alpha(u) \mapsto \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix},$$

$$h_\alpha(u) \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

§ Universal Covering

Central Extension

A group epimorphism $E \longrightarrow G$ is called an extension, and an extension $E \longrightarrow G$ is called a central extension if $Ker\,[E \longrightarrow G] \subset Z(E)$.

Universal Covering

A central extension $E \longrightarrow G$ is called a universal covering (or a universal central extension) if for any central extension $E' \longrightarrow G$, there uniquely exists a group homomorphism $E \longrightarrow E'$ such that the following diagram is commutative.

$$\begin{array}{ccc} E & \rightarrow & G \\ \downarrow & \nearrow & \\ E' & & \end{array}$$

Steinberg Group

The Steinberg group $St(A, F)$ over a field F of type A is defined to be the group generated by $\hat{x}_\alpha(t)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $t \in F$ with the defining relations corresponding to (A), (B), (B′).

1990, J. M. - U. Rehmann

Theorem

Let A be a GCM, and F an infinite field. Then, $St(A, F)$ is a universal covering of $G'_u(A, F)$, which is induced by $\hat{x}_\alpha(t) \mapsto x_\alpha(t)$.

Application

Theorem

Let A be an indecomposable GCM, and F the algebraic closure of a finite field $\mathbb{F}_p$. We suppose that A is not of affine type. Then, $G'_u(A, F)$ is a universal covering of a simple group $G'_{ad}(A, F)$.

§ Remark

Remark I

Let A be an $n \times n$ GCM, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots. The principal divisors of A is denoted by $\pi(A) = (d_1, \cdots, d_n)$, and we put $\Gamma = \oplus_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$. Then, for a field F, we have

$$
\begin{aligned}
Z(G'_u(A, F)) \\
&= \{h_{\alpha_1}(u_1) \cdots h_{\alpha_n}(u_n) \mid u_1^{a_{1j}} \cdots u_n^{a_{nj}} = 1, \forall j\} \\
&\simeq Hom(\Gamma, F^{\times}).
\end{aligned}
$$

Remark II

Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a 2×2 hyperbolic GCM, and we suppose $ab > 4, \; a > 1, \; b > 1$. Then, in many cases, we see that $G_{ad}(A, \mathbb{F}_q)$ is not simple.

Non-Simple Case

Example

Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a 2×2 hyperbolic GCM satisfying $ab > 4$ with $a > 1$ and $b > 1$. We suppose $a \equiv b \equiv 2 \pmod{q-1}$. Then, we have $G_{ad}(A, \mathbb{F}_q) \simeq PSL_2(\mathbb{F}_q[X, X^{-1}])$, which is not simple.

$a = 8,\ b = 14,\ q = 7$

Example

We see

$$G_{ad}\left(\begin{pmatrix} 2 & -8 \\ -14 & 2 \end{pmatrix}, \mathbb{F}_7\right) \simeq PSL_2(\mathbb{F}_7[X, X^{-1}]).$$

§ Schur Multiplier

Schur Multiplier

If $E \longrightarrow G$ is a universal covering, then

$$M(G) = \mathrm{Ker}[E \longrightarrow G]$$

is called the Schur multiplier of G, in the sense that every projective representation of G can be lifted to an ordinary representation of E.

Fact

Let A be a GCM, and F the algebraic closure of a finite field $\mathbb{F}_p$. Let $\pi(A) = (d_1, \cdots, d_n)$ be the principal divisors of A, where we write $d_i = p^{c_i} m_i$ with $p \nmid m_i$ if $d_i \neq 0$. Then, we have

$$M(G'_{ad}(A, F)) \simeq \begin{cases} Z_{m_1} \times \cdots \times Z_{m_n} & \text{if } d_n \neq 0, \\ Z_{m_1} \times \cdots \times Z_{m_k} \times (F^\times)^{n-k} & \\ \qquad \text{if } d_k \neq 0, d_{k+1} = 0. & \end{cases}$$

Rank 2 Case

Example

Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ with $d = ab - 4 > 0$, and

$$\pi(A) = \begin{cases} (2, d') \text{ if } a \equiv b \equiv 0 \pmod{2}, d = 2d', \\ (1, d) \text{ otherwise.} \end{cases}$$

Let F be the algebraic closure of a finite field $\mathbb{F}_p$.

Then,

$M(G_{ad}(A, F))$	$(1, d)$	$(2, d')$
$p = 2$	Z_m	$Z_{m'}$
$p > 2$	Z_m	$Z_2 \times Z_{m'}$

if $d = p^c m,\ d' = p^{c'} m',\ p \nmid m,\ p \nmid m'$.

As Before

Example

Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a 2×2 hyperbolic GCM satisfying $ab = p^c + 4$ for some prime number p and some integer $c \geq 0$, and F the algebraic closure of a finite field $\mathbb{F}_p$. Then, $G_u(A, F)$ is a simple group with trivial Schur multiplier.

§ Conclusion

Conclusion

Summary

(1) A : Indecomposable non-affine GCM
$\mapsto \exists$ *Simple Groups*

(2) A : Indecomposable GCM with $\det(A) = \pm p^c$
$\mapsto \exists$ *Simple Group with Trivial Schur Multiplier*

§ Appendix

Remark III

We can construct some completion of a Kac-Moody group. Suppose that A is indecomposable and F is any field. Then, its derived subgroup is always simple modulo its center. In characteristic 0 case, this is done by R. Moody (as a unpublished paper) for a non-affine GCM, and this is known to many specialists for an affine GCM (as a folk result, cf. $\exists$ an explicit description by J. M.). In general case, this is due to J. Tits.

Matsumoto-Type Presentation

$K_2(A, F)$

Let A be a GCM, and F a field. We define $K_2(A, F)$ by

$$1 \to K_2(A, F) \to St(A, F) \to G'_u(A, F) \to 1 \ .$$

Then, $K_2(A, F)$ has a Matsumoto-type presentation (cf. J. M. - U. Rehmann). This gives a lot of information on $K_2(A, F)$. In this case, $K_2(A, F)$ is just the Schur multiplier of $G'_u(A, F)$ for an infinite field F.

Root String I

$\alpha \in \Delta^{\mathrm{re}},\ \beta \in \Delta$

$S^{\alpha}(\beta) = \{\beta + i\alpha \mid i \in \mathbb{Z}\} \cap \Delta$: <u>Root String</u>

$\bullet$: Real Root, $\circ$: Imaginary Root

$$S^{\alpha}(\beta) \ : \ \left\{ \begin{array}{l} \bullet \ \bullet \ \circ \ \cdots \ \circ \ \bullet \ \bullet \\ \hline \bullet \ \bullet \ \bullet \\ \hline \bullet \ \circ \ \cdots \ \circ \ \bullet \\ \hline \bullet \\ \hline \circ \ \cdots \ \circ \end{array} \right.$$

Proposition I

$$\alpha, \beta \in \Delta^{\mathrm{re}}, \;\; \alpha + \beta \neq 0$$
$$\beta(h_\alpha) \geq 0 \quad (\Leftrightarrow \;\; \alpha(h_\beta) \geq 0)$$
$$\Longrightarrow$$
$$Q_{\alpha,\beta} \subset \{\alpha + \beta\} \cap \Delta^{\mathrm{re}},$$
where $Q_{\alpha,\beta} = (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Delta$

Root String II

$\alpha, \beta \in \Delta^{\mathrm{re}}, \;\; \alpha + \beta \neq 0, \;\; S^\alpha(\beta) \subset \Delta^{\mathrm{re}}$

$\beta(h_\alpha) < 0 \quad (\Leftrightarrow \;\; \alpha(h_\beta) < 0)$

$\Longrightarrow$

$$S^\alpha(\beta) \;\; : \;\; \left\{ \begin{array}{llll} \bullet_{\beta} & \bullet_{\beta+\alpha} & & \\ \hline \bullet_{\beta} & \bullet_{\beta+\alpha} & \bullet_{\beta+2\alpha} & \\ \hline \bullet_{\beta-\alpha} & \bullet_{\beta} & \bullet_{\beta+\alpha} & \bullet_{\beta+2\alpha} \\ \hline \bullet_{\beta} & \bullet_{\beta+\alpha} & \bullet_{\beta+2\alpha} & \bullet_{\beta+3\alpha} \end{array} \right.$$

Proposition II

$\alpha, \beta \in \Delta^{\mathrm{re}}, \ \alpha + \beta \neq 0, \ Q_{\alpha,\beta} \subset \Delta^{\mathrm{re}}$

$\Longrightarrow$

$$\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = \begin{cases} \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \\ \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \\ \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \\ \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_{\alpha+2\beta} \\ \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta} \end{cases}$$

(modulo exchanging α and β)

Chevalley Pair

$\mathfrak{g} = \langle \mathfrak{h}, e_1, f_1, \ldots, e_n, f_n \rangle$

e_i, f_i : Chevalley Generators

$\omega \in Aut(\mathfrak{g})$: Chevalley Involution

$\omega(e_i) = -f_i, \ \omega(f_i) = -e_i, \ \omega(h) = -h \ (\forall h \in \mathfrak{h})$

$(e_\alpha, e_{-\alpha}) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$: a Chevalley Pair for $\alpha \in \Delta^{\mathrm{re}}$

$[e_\alpha, e_{-\alpha}] = h_\alpha, \ \omega(e_\alpha) + e_{-\alpha} = 0$

$x_\alpha(t) = exp(te_\alpha)$

Proposition III

$\alpha, \beta, \alpha + \beta \in \Delta^{\mathrm{re}}$

$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$

$S_\alpha(\beta) = \{\beta - r\alpha, \ldots, \beta, \ldots, \beta + r'\alpha\}$

$r = \max \{i \in \mathbb{Z} \mid \beta - i\alpha \in \Delta^{\mathrm{re}}\}$

$r' = \max \{i \in \mathbb{Z} \mid \beta + i\alpha \in \Delta^{\mathrm{re}}\}$

$\Longrightarrow$

$$N_{\alpha,\beta} = \pm(r + 1)$$

Relation (B)

There are essentially five type relations in (B).

$$[x_\alpha(s), x_\beta(t)] = 1$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm(r+1)st)$$

$$r = \max\{i \in \mathbb{Z} \mid \beta - i\alpha \in \Delta^{\mathrm{re}}\}$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2 t)\cdot x_{\alpha+2\beta}(\pm 3st^2)$$

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)\cdot x_{3\alpha+\beta}(\pm s^3 t)x_{3\alpha+2\beta}(\pm 2s^3 t^2)$$

- END -

Thank you !