Locally Affine Lie Algebras

at Fields Institute on March 25, 2013

Yoji Yoshii
Akita National College of Technology
Japan
Joint Work with Jun Morita

Example of LALA

$L = \mathbf{sl}_{\mathbb{N}}(F[t^{\pm}]) \oplus Fc \oplus D$

c = central, D = derivations

$d := t\frac{d}{dt}$ (degree derivation)

If $D = Fd$, then L is called a minimal standard LALA

F = a field of characteristic 0

L = a Lie algebra over F

H = a subalgebra of L

$B : L \times L \longrightarrow F$ (a bilinear form)

(L, H, B) is called a Locally Extended Affine Lie Algebra, LEALA for short if it satisfies the following (LE1) – (LE4).

(LE1) Root Space Decomposition

$L = \oplus_{\xi \in H^*} L_\xi$

$H^* = Hom_F(H, F)$

$L_\xi := \{x \in L \mid [h, x] = \xi(h)x \text{ for all } h \in H)\}$

★ $H = L_0$

H becomes abelian.

(a Cartan subalgebra!)

(LE2) Existence of Form

$B(x, y) = B(y, x)$: symmetric

$B([x, y], z) = B(x, [y, z])$: invariant

B : nondegenerate

By (LE1),

$B(L_\xi, L_\eta) = 0$ if $\xi + \eta \neq 0$

$B|_{L_\xi \times L_{-\xi}}$ is also nondegenerate.

Properties of LEALA

The set of roots

$R = \{\xi \in H^* \mid L_\xi \neq 0\}$

Define $t_\xi \in H$:

$\forall \xi \in R, \ \exists! \ t_\xi \in H$ s.t. $B(t_\xi, h) = \xi(h)$

Then

$[x, y] = B(x, y)t_\xi$ if $x \in L_\xi, y \in L_{-\xi}$

$R^\times = \{\xi \in R \mid B(t_\xi, t_\xi) \neq 0\}$

anisotropic roots

$R^0 = R \setminus R^\times$: isotropic roots

(LE3) Integrability

$\mathrm{ad}\,x \in \mathrm{End}(L)$ is locally nilpotent for all $x \in L_\alpha$ and $\alpha \in R^\times$.

(LE4) Irreducibility

$R^\times$: irreducible

$R^\times = R_1 \cup R_2$

$B(t_{\xi_1}, t_{\xi_2}) = 0$ for $\xi_1 \in R_1$ and $\xi_2 \in R_2$

$\Longrightarrow R_1 = \emptyset$ or $R_2 = \emptyset$

$$\dim L_\xi \begin{cases} = 1 & \text{if } \xi \in R^\times \\ \geq 1 & \text{if } \xi \in R^0 \end{cases}$$

$\xi \in R^\times$

$$L_\xi \oplus F t_\xi \oplus L_{-\xi} \simeq \mathbf{sl}_2(F)$$

$\xi \in R^0$

$$x \in L_\xi, y \in L_{-\xi} \quad \text{s.t. } B(x, y) \neq 0$$

$$Fx \oplus F t_\xi \oplus F y \simeq \mathcal{H}$$

(3-dim. Heisenberg)

Under a suitable scalar multiple of B, one can assume $B(t_\alpha, t_\alpha) \in \mathbb{Q}_{>0}$ for all $\alpha \in R^\times$.

Kac-Conjecture for LEALA

(Morita-Y, J. Algebra, 2006)

$(\cdot, \cdot)|_{V\times V}$ is positive semidefinite.

For EALA, AABGP, Mem. AMS, 1997

Thus

$\overline{R^\times}$ is a locally finite irreducible root system, and $R^\times$ is a natural generalization of an extended affine root system, introduced by K. Saito in 1985. In fact, $R^\times$ is an extended affine root system if L is an EALA, and an irreducible affine root system by Macdonald in 1974 if the nullity is 1.

Core of (L, H, B)

$$L_c = \langle L_\alpha \mid \alpha \in R^\times \rangle < L$$

$L_c \lhd L$: ideal

$L_c = [L_c, L_c]$ (L_c is perfect.)

Let

$$C_L(L_c) := \{x \in L \mid [x, y] = 0, \ \forall y \in L_c\}$$

be the centralizer.

Tame Condition

$C_L(L_c) < L_c$

$L = L_c \oplus D$

$D \hookrightarrow \mathrm{Der}(L_c)/\mathrm{ad}(L_c)$

Thus D can be viewed as outer derivations of L_c. (In fact, they should be skew outer derivations relative to B.)

Nullity

$\Gamma = \langle R^0 \rangle < H^*$: additive subgroup

"null group" of (L, H, B)

nullity $n \iff \Gamma \simeq \mathbb{Z}^n$

Note that the null group Γ is not necessarily a free group.

Extended Affine Lie Algebra

EALA (AABGP, Neher)

(EA1) (L,H,B) : LEALA

(EA2) $\dim H < \infty$

(EA3) Tame

(EA4) $\Gamma \simeq \mathbb{Z}^n$: nullity n

Induced form on $V_F = \mathrm{span}_F R$

$$\lambda = \sum a_\xi \xi \Longrightarrow t_\lambda := \sum a_\xi t_\xi$$

$$(\cdot\,,\,\cdot) : V_F \times V_F \longrightarrow F$$
$$(\lambda, \mu) := B(t_\lambda, t_\mu)$$

In particular, $V := \mathrm{span}_{\mathbb{Q}} R$

$$\bar{\ } : V \longrightarrow \overline{V} = V/\,\mathrm{rad}(\cdot, \cdot)$$

① tame EALA of nullity 0
= {f. dim'l split simple Lie algebras}

② tame EALA of nullity 1
= {affine Lie algebras} by [ABGP]

③ tame LEALA of nullity 0
= {locally finite split simple Lie algebras} ∪ {a little more}, e.g.
$\mathbf{sl}_{\mathbb{N}}(F)$, etc., and $\mathbf{sl}_{\mathbb{N}}(F) \oplus D$, etc.

Tame LEALA of nullity 1 is called an Locally Affine Lie Algebra, LALA for short.

(L, H, B): a LALA
$\Rightarrow$
$R^{\times}$ is a locally affine root system.
$L = L_c \oplus D \Rightarrow L_c = ?, \quad D = ?$

$L_c = \Omega \oplus Fc$, where Ω is a locally loop algebra, for example,

$\Omega = \mathbf{sl}_{\mathbb{N}}(F) \otimes_F F[t^{\pm 1}]$.

$$D = \mathrm{span}_F(d + \mathfrak{d}_0) \oplus \bigoplus_{m \in \mathbb{Z}^\times} \mathrm{span}_F\, \mathfrak{d}_m \otimes t^m,$$

where

$\mathfrak{d}_m = \{$certain diagonal matrices$\}$.

(recall $d := t\frac{d}{dt}$)

f. dim'l split simple Lie alg. $\mathfrak{g}(\mathrm{X}_n)$

$\mathrm{X}_n = \mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$

Each type X_n corresponds to a finite irreducible root system.

Locally finite split simple Lie alg.

I : an infinite index set

$\mathfrak{g}(X_I)$

$= \mathfrak{sl}_I(F), \mathfrak{o}_{2I+1}(F), \mathfrak{sp}_{2I}(F), \mathfrak{o}_{2I}(F)$

$X_I = A_I, B_I, C_I, D_I$

Each type X_n corresponds to a locally finite irreducible root system.

Affine Lie algebra $\mathfrak{g}(X_n^{(r)})$

untwisted type:

$A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}$

twisted type:

$A_{2n-1}^{(2)} = C_n^{(2)}, \ A_{2n}^{(2)} = BC_n^{(2)}, \ D_{n+1}^{(2)} = B_n^{(2)},$

$E_6^{(2)} = F_4^{(2)}, \ D_4^{(3)} = G_2^{(3)}$

Loop Algebra

$$\Omega(X_n^{(r)}) = \left(\mathfrak{g}(X_n) \otimes F[t^{\pm 1}]\right)^{\sigma}$$

$\sigma = \sigma_1 \otimes \sigma_2, \ |\sigma_1| = |\sigma_2| = r$

σ_1 : Dynkin diag. auto. of $\mathfrak{g}(X_n)$

$\langle \sigma_2 \rangle = \mathrm{Gal}(F[t^{\pm 1}]/F[t^{\pm r}])$

$\omega \in F$ if $r = 3$ or using $\tilde{F} := F(\omega)$

Note: $r = 1, 2, 3$.

$r = 1 \Longrightarrow$ untwisted loop algebra

$r = 2, 3 \Longrightarrow$ twisted loop algebra

Realization of Affine Lie Algebra

$$\mathfrak{g}(X_n^{(r)}) = \Omega(X_n^{(r)}) \oplus Fc \oplus Fd$$

$L_c = \Omega(X_n^{(r)}) \oplus Fc,$

c : central, d : degree derivation

Untwisted Locally Loop Algebra

$$\Omega(A_l^{(1)}) = \mathfrak{sl}_l(F) \otimes F[t^{\pm 1}]$$
$$\Omega(B_l^{(1)}) = \mathfrak{o}_{2l+1}(F) \otimes F[t^{\pm 1}]$$
$$\Omega(C_l^{(1)}) = \mathfrak{sp}_{2l}(F) \otimes F[t^{\pm 1}]$$
$$\Omega(D_l^{(1)}) = \mathfrak{o}_{2l}(F) \otimes F[t^{\pm 1}]$$

Twisted Locally Loop Algebra

type $D_{2l+2}^{(2)} = B_l^{(2)}$:

$$\Omega(B_l^{(2)}) = (\mathfrak{o}_{2l+2} \otimes F[t^{\pm 1}])^{\sigma}$$

type $A_{2l}^{(2)} = C_l^{(2)}$:

$$\Omega(C_l^{(2)}) = (\mathfrak{sl}_{2l} \otimes F[t^{\pm 1}])^{\sigma}$$

type $A_{2l+1}^{(2)} = BC_l^{(2)}$:

$$\Omega(BC_l^{(2)}) = (\mathfrak{sl}_{2l+1} \otimes F[t^{\pm 1}])^{\sigma}$$

Locally Affine Lie Algebra

$$\Omega(\mathbf{X}_I^{(r)}) \oplus Fc \oplus D \quad (r = 1, 2)$$

$$\mathbf{X}_I^{(1)} = \mathbf{A}_I^{(1)}, \mathbf{B}_I^{(1)}, \mathbf{C}_I^{(1)}, \mathbf{D}_I^{(1)}$$
$$\mathbf{X}_I^{(2)} = \mathbf{B}_I^{(2)}, \mathbf{C}_I^{(2)}, \mathbf{BC}_I^{(2)}$$

The core, $L_c = \Omega(\mathbf{X}_I^{(r)}) \oplus Fc$, is a universal central extension!

A LALA is called minimal if $\dim D = 1$. For example,

$$\Omega(\mathbf{X}_I^{(r)}) \oplus Fc \oplus Fd,$$

which is called standard.

Is there exist a non-standard minimal LALA?

Yes if I is an infinite set!

For example, let

$L(p) := \mathfrak{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus F(d + p)$

be a minimal LALA, where

$p = \mathrm{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$.

Then $L(p)$ is not isomorphic to a minimal standard LALA $L(0)$ as Lie algebras.

An isomorphism between two LEALAs, (L, H, B) and (L', H', B'): Write $L \cong L'$ as LEALAs if there exists a Lie algebra isomorphism $\varphi : L \longrightarrow L'$ so that $\varphi(H) = H'$ and $B(x, y) = B'(\varphi(x), \varphi(y))$ for all $x, y \in L$.

Remark. The value for $B(d, d)$ does not much matter for an affine Lie algebra, and it is usually defined to be 0 so that $Fc \oplus Fd$ is a hyperbolic plane relative to B. Following this convention, we always assume that $Fc \oplus F(d + p)$ is a hyperbolic plane for a minimal LALA $L(p)$. Concretely, we simply assume that $B(d + p, d + p) = 0$. (Then it is automatically a hyperbolic plane by the choice of d.)

Thm 1

For $L(p) = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus F(d + p)$,

$p = \mathrm{diag}(m_1, m_2, m_3, \ldots)$, $m_i \in \mathbb{Z}$

$\Longrightarrow L(p) \cong L(0)$ (min. stan. LALA)

Proof) Use the diagonal matrix $g := \mathrm{diag}(t^{m_1}, t^{m_2}, t^{m_3}, \ldots)$, and take the conjugate by g.

How about the case p has the trace?

Such an $L(p)$ is called traceable.

Thm 2 For a traceable min. LALA $L(p)$,

$$L(p) \cong L(0) \iff \mathrm{tr}(p) \in \mathbb{Z}$$

For example,

$L(\frac{1}{2}e_{11}) \not\cong L(0)$ (e_{11} is the matrix unit)

Moreover:

Thm 3

For traceable min. LALAs $L(p)$ and $L(p')$,

$$L(p) \cong L(p') \iff \mathrm{tr}(p) \pm \mathrm{tr}(p') \in \mathbb{Z}$$

For example,

$$L(\frac{1}{5}e_{11}) \not\cong L(\frac{2}{5}e_{11}), \quad L(\frac{2}{5}e_{11}) \cong L(\frac{3}{5}e_{11})$$

When $F = \mathbb{R}$, isomorphic classes of traceable min. LALAs of type $\mathbf{A}_{\mathbb{N}}^{(1)}$
$\longleftrightarrow$ the closed interval $[0, \frac{1}{2}]$ in $\mathbb{R}$
(In fact, $\mathbb{N}$ can be any infinite set.)
For example,

$$L(\sqrt{2}e_{11}) \cong L((\sqrt{2}-1)e_{11})$$
$$L(\sqrt{2}e_{11}) \ncong L(2\sqrt{2}e_{11})$$

However:

Problem

Are

$L(\frac{1}{2}e_{11})$ and $L(0)$

isomorphic as Lie algebras?

MAX LALA:

$$(\mathbf{sl}_{\mathbb{N}}(F) + T') \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd$$

is a maximal LALA of typte $A_{\mathbb{N}}^{(1)}$,

where

T ={all $\mathbb{N} \times \mathbb{N}$ diagonal matrices}

$\iota \in T$: the identity matrix

$T = T' \oplus F\iota$

(Take a complement T' of $F\iota$.)