Prime Representations and Extensions

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My goal is to ‘understand’ a particular category: the category of finite–dimensional representations of the quantized enveloping algebra of an affine Lie algebra.

The precise definition of the algebra is irrelevant for the moment.

There is a vast literature on the subject going back to 1990, (C, Frenkel, Hernandez, Kashiwara, Mukhin, Nakajima, Reshetikhin, to name a very few).

But many of the basic representation theoretic questions are not answered. No general character formulae, no dimension formulae and so on.

Today, I want to show you that there is a surprising connection between the tensor structure of this category and its homological properties.

Prime objects

What we have is the following: a category, say $\mathcal{F}$, of finite–dimensional modules for a nice Hopf algebra. It is an abelian tensor category: closed under finite–direct sums, tensor products, duals, has a trivial object.

Let's see what a prime object in $\mathcal{F}$ would be.

Definition

Say that an object V of $\mathcal{F}$ is prime, if it cannot be written as a tensor product of non–trivial objects of $\mathcal{F}$.

Of course, since $\mathcal{F}$ has the trivial object $\mathbf{C}$ we could always write $V \cong V \otimes \mathbf{C}$.

I want to claim that any object of $\mathcal{F}$ is isomorphic to a tensor product of non-trivial prime representations.

A proof would go as follows: if V is not prime, then $V \cong V_1 \otimes V_2$, with both V_1 and V_2 being nontrivial.

A dimension argument would give the result if I knew that $\dim V_i < \dim V$ for $i = 1, 2$.

And this is ok for me, because my Hopf algebra is nice! The only one–dimensional representation is the trivial one.

So what this does for me, is that it allows me to understand my category by understanding prime objects in it.

For the most part, my goal is to understand prime simple objects.

In my context, this problem is hard enough, but also, we will see now that this restriction is in some sense necessary.

Examples

Lets look at some examples. The first one is where neither the homological properties of the category, nor the notion of prime is going to be interesting. But it does have the virtue of being an example that everyone who's had a course in Lie theory is familiar with!

Suppose that $\mathfrak{g}$ is a complex simple finite–dimensional Lie algebra, lets take $\mathfrak{sl}_2$. Then we know the following two facts about its finite–dimensional representations:

(i) The isomorphism classes of irreducible representations are indexed by the non–negative integers: given $r \in \mathbb{Z}_+$, let $V(r)$ be an irreducible representation in the corresponding isomorphism class.

(ii) Any finite–dimensional irreducible representation is isomorphic to a direct sum of irreducible representations.

What about prime representations in this category?

Well, the Clebsch–Gordon formula for tensor products gives,

$$V(p) \otimes V(q) = V(p+q) \oplus V(p+q-2) \oplus \cdots \oplus V(p-q),$$

where $p, q \in \mathbb{Z}_+$ and $p \geq q$.

Immediate consequences.

- Any simple object is prime.

- Non–simple prime objects are trivially found.

Look at $V(p) \oplus V(p+1)$ or $V(p) \oplus V(p-4)$. The first is not prime for parity reasons and the second is not prime because $V(p-2)$ is missing.

So restricting one's attention to simple prime representations is reasonable. Similar results hold for the other simple Lie algebras.

The next example

This time we are going to work with the loop algebra $L(\mathfrak{g})$ associated to a simple Lie algebra: in other words, the quotient of the affine Kac–Moody Lie algebra by the center. $\mathcal{F}$ will continue to be the category of finite–dimensional objects of this algebra.
This time we will see that the notion of prime simple objects is more interesting. As are the homological properties of $\mathcal{F}$.

Lets recall the definition of the loop algebra: we fix an indeterminate t and let $\mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in one variable.

Then $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$, with commutator given in the obvious way:

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg, \quad x, y \in \mathfrak{g}, \quad f, g \in \mathbf{C}[t, t^{-1}].$$

Given $z \in \mathbb{C}$, $z \neq 0$, consider the evaluation map $\mathrm{ev}_z : L(\mathfrak{g}) \to \mathfrak{g}$, given by $x \otimes f \to f(z)x$. If V is any representation of $\mathfrak{g}$, one can pull back via ev_z to get a representation of $L(\mathfrak{g})$ and we denote this by $\mathrm{ev}_z V$.

An old result, due to Pressley and myself is that:

Proposition

An irreducible object of $\mathcal{F}$ is isomorphic to a finite tensor product of evaluations representations: $\mathrm{ev}_{z_1} V_1 \otimes \cdots \otimes \mathrm{ev}_{z_r} V_r$, *where z_i are distinct complex number numbers and V_i are irreducible representations of $\mathfrak{g}$. Conversely any such tensor product is irreducible.*

So lets see what this tells us about prime simple representations.

• It tells us that not all simple representations are prime.

• Combined with the example that we worked out for simple Lie algebras, it also tells us that the prime representations are exactly of the form $\mathrm{ev}_z V$ where V is a simple representation of $\mathfrak{g}$.

• Since finite–dimensional simple representations of $\mathfrak{g}$ are understood, it follows that we understand the simple finite–dimensional representations of $L(\mathfrak{g})$.

And we would be pretty happy with this, except for one fact.

The category $\mathcal{F}$ is not semisimple and understanding simple objects is not good enough. There are lost of interesting indecomposable objects in this category, such as the local Weyl modules, Kirillov–Reshetikhin modules and so on.

Self Extensions

But today, I want to look at Ext^1 in $\mathcal{F}$. More specifically at self extensions, i.e., $\mathrm{Ext}^1(V, V)$ of irreducible objects of $\mathcal{F}$.

An element of $\mathrm{Ext}^1(V, V)$ is the equivalence class of short exact sequences

$$0 \to V \to U \to V \to 0.$$

The extension is split if it is equivalent to the trivial extension,

$$0 \to V \to V \oplus V \to V.$$

From now on, I am talking about joint work with Adriano Moura and Charles Young.

Here is an easy way to construct an example of such an extension.

Consider the derivation d/dt of $\mathbf{C}[t, t^{-1}]$. This induces a derivation of $L(\mathfrak{g})$ in the obvious way, $x \otimes f \to x \otimes f'(t)$.

A canonical self extension

Let V be any representation of $L(\mathfrak{g})$. Set $U = V \oplus V$ as vector spaces. Using the product rule, its trivial to check that setting

$$(x \otimes f)(v_1, v_2) = ((x \otimes f)v_1, \ \ (x \otimes f')v_1 + (x \otimes f)v_2),$$

$x \in \mathfrak{g}$, $f \in \mathbf{C}[t, t^{-1}]$, and $v_1, v_2 \in V$ defines the structure of an $L(\mathfrak{g})$–module on V.

Let $\iota : V \to U$ be the inclusion in the second factor, $\iota(v) = (0, v)$, and let $\tau : U \to V$ be the projection on to the first factor $\tau(v_1, v_2) = v_1$.

The equivalence class

$$0 \to V \xrightarrow{\iota} U \xrightarrow{\tau} V \to 0.$$

defines an element $\mathbb{E}(V)$ of $\operatorname{Ext}^1(V, V)$.

Proposition

Suppose that V is not a sum of trivial representations. Then $\mathbb{E}(V)$ is a nontrivial element of $\operatorname{Ext}^1(V, V)$, i.e.,

$$\dim \operatorname{Ext}^1(V, V) \geq 1$$

for all objects V of $\mathcal{F}$.

Suppose that there exists an isomorphism $\varphi : \mathbb{E}(V) \to V \oplus V$ such that $\varphi.\iota = \iota_1$ and $\tau_1.\varphi = \tau$, here $\iota_1 : V \to V\oplus$ is the inclusion in the second summand and $\tau_1 : V \oplus V \to V$ is projection to the first summand. Then, we have

$$\varphi((0, v)) = (0, v), \quad \varphi((v, 0)) = (v, \psi(v)),$$

for some linear map $\psi : V \to V$. This gives,

$$(xf)\varphi(v, 0) = ((xf)v, (xf)\psi(v)) = ((xf)v, \quad (xf')v + \psi((xf)v)),$$

i.e.,

$$[xf, \psi] = xf',$$

as endomorphisms of V for all $x \in \mathfrak{g}$ and all $f \in \mathbf{C}[t, t^{-1}]$.
Since V is a non–trivial finite dimensional representation of $L(\mathfrak{g})$, there exists a proper ideal of finite codimension in $L(\mathfrak{g})$ which annihilates V. Such an ideal is necessarily of the form $\mathfrak{g} \otimes (p)$ for some $p \in \mathbf{C}[t, t^{-1}]$.

Now we have

$$0 = [xp, \psi] = xp',$$

as operators on V, i.e xp' annihilates V for all $x \in \mathfrak{g}$. But this is a contradiction. So, ψ must be the zero map, in which case φ is not a map of $L(\mathfrak{g})$–modules and the proposition is proved.

So one could ask, is there anything special about the case when $\dim \operatorname{Ext}^1(V, V) = 1$?

Tensor products and Self–Extensions

Proposition

Suppose that V_1 and V_2 are non–isomorphic irreducible, non–trivial representations of $L(\mathfrak{g})$. Then,

$$\dim \operatorname{Ext}^1(V_1 \otimes V_2, V_1 \otimes V_2) \geq 2.$$

More precisely, the extensions $\mathbb{E}(V_1) \otimes V_2$ and $V_1 \otimes \mathbb{E}(V_2)$ of $V_1 \otimes V_2$ are linearly independent.

In other words: $\dim \operatorname{Ext}^1(V, V) = 1$ and V irreducible implies that V is prime!

What about the converse? Well, for sl_2 it holds, but for sl_3 it does not. Consider the adjoint representation of $\mathfrak{sl}_3$. This is prime but one can prove that the space of self extensions is two dimensional. Too bad.

The Quantum Case

In fact, it was the precise example, the adjoint representation of $\mathfrak{sl}_3$, which made us realize that the quantum case might be interesting! So, in the remaining time, I will quickly discuss the case of quantum loop algebra $\mathbf{U}_q(L(\mathfrak{g})$ and I shall let $\mathcal{F}_q$ be the category of its finite–dimensional representations.

The classification of the simple objects was given by Pressley and myself in 1990, 1994. What we proved was, that the isomorphism classes of simple objects is given by an n–tuple of polynomials with constant term one where n is the rank of $\mathfrak{g}$. Although the classification I gave today of the irreducible representations of affine algebras, looks a bit different, it can be made to look the same.

And there the similarity ends.

The reason is that outside $\mathfrak{sl}_{n+1}$ there do not exist analogs of the evaluation map, i.e, we do not have maps $\mathbf{U}_q(L(\mathfrak{g})) \to \mathbf{U}_q(\mathfrak{g})$. So, finding examples of prime representations is non–trivial, leave alone classifying them.

Even for $\mathfrak{sl}_{r+1}$, $r \geq 2$ it is not hard to show that there are irreducible representations which are not isomorphic to the tensor product of evaluations.

For $\mathfrak{sl}_2$, however, we proved in joint work with Pressley in 1990, that: every irreducible representation is isomorphic to a tensor product of evaluation representations. The condition for irreducibility is more subtle. $\mathrm{ev}_z\, V(n) \otimes \mathrm{ev}_{z'}\, V(m)$ is reducible whenever $z/z' \text{or} \;\; z'/z \in \{q^{m+n}, q^{m+n-2} \cdots, q^{m-n+2}\}$ (compare with the Clebsch Gordon formula). In particular, the tensor square of an irreducible representation is irreducible because 1 is not in our set.

No such result is still known in any other case. Instead people have focused on understanding particular families of representations and perhaps proving such theorems for those special families: standard modules of Nakajima, Kirillov–Reshethikhin modules, their generalizations, called minimal affinizations. These latter modules are known to be prime, but this is kind of easy to see from the sl_2 case. Recently, Hernandez and Leclerc using cluster algebras identified some new families of prime representations which are not at all obvious. So, finding prime representations in this category is hard, on the other hand finding them is obviously useful in understanding the simple objects.

A theorem,a conjecture and some wishful thinking

The algebra $\mathbf{U}_q(L(\mathfrak{g}))$ is $\mathbb{Z}$–graded, in an obvious way, roughly, it corresponds to the derivation td/dt of $L(\mathfrak{g})$.

So, one can construct self extensions as before, if $x \in \mathbf{U}_q(L(\mathfrak{g}))$ has grade r, then x acts on $V \oplus V$ by

$$x(v,0) = (xv, rxv).$$

Call this $\mathbb{E}(V)$ as before. Then, the theorem we prove (C,Moura, Young) is: suppose that V is a nontrivial irreducible object of $\mathcal{F}_q$.

- $\mathbb{E}(V)$ is a nontrivial extension and so $\dim \operatorname{Ext}^1(V,V) \geq 1$.
- If $\dim Ext^1(V,V) = 1$ and V is simple, then V is either prime or a prime power.
- For sl_2 an irreducible representation is prime iff $\dim \operatorname{Ext}^1(V,V) = 1$.

• Many of the well–known families of prime representations satisfy the condition that $\dim \mathrm{Ext}^1(V, V) = 1$.

For instance, consider the adjoint representation of $\mathfrak{sl}_3$ which we recall had a two dimensional space of self extensions.

Well, we can consider the analogous representation of $\mathbf{U}_q(\mathfrak{sl}_3)$. Recall that here we have an evaluation map $\mathbf{U}_q(L(\mathfrak{g})) \to \mathbf{U}_q(\mathfrak{g})$ and so we can pull back the adjoint to get a representation of $\mathbf{U}_q(L(\mathfrak{g}))$. Well, this representation is covered by our theorem and so has a one–dimensional space of self extensions.

Conjecture A simple non–trivial object V of $\mathcal{F}_q$ is prime iff $\dim \mathrm{Ext}^1(V, V) = 1$.

The theorem does give evidence for this conjecture, at least enough evidence to suggest that this approach might be useful.

Wishful Thinking, (aka a thesis problem?)

An irreducible representation is a tensor product of r prime representations iff $\dim \operatorname{Ext}^1(V, V) = r$.

This is not known even for $\mathfrak{sl}_2$.

A result of R.Kodera shows that (for sl_2) that if V is a tensor product of r prime representations then, $\dim \operatorname{Ext}^1(V, V) \leq r$. With Moura and Lunde, we think we can do the sl_2 case by establishing the other inequality. Maybe.

Thank you!