Reducing the structural group by using stabilizers in general position

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Motivation: Essential dimension

How many independent parameters do you need to describe a generic object of the following types?

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Octonion algebra	$3 \leq \mathrm{ed}(G_2) \leq 3$
Albert algebra	$5 \leq \mathrm{ed}(F_4) \leq 7$
Freudenthal triple system	$7 \leq \mathrm{ed}(E_7) \leq 17$

Key techniques for finding upper bounds

Let G be a smooth linear algebraic group over a field F, and V a linear representation.

Theorem 1

If V is generically free that $\mathrm{ed}(G) \leq \dim V - \dim G$.

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"Theorem 2"

If the stabilizers of points in general position (in V) are all conjugate to the same $H \subset G$ subgroup, then

$$H^1(L, N_G(H)) \to H^1(L, G)$$

is surjective for all fields L/F. Hence, $\mathrm{ed}(G) \leq \mathrm{ed}(N_G(H))$.

Classifying torsors

Given a G-torsor π and an L-rational point α of the base we can form the pullback torsor over Spec(L):

$$\begin{array}{ccc} X_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(L) & \xrightarrow{\alpha} & X/G \end{array}$$

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Definition

A G-torsor is **classifying** if for all infinite fields L/F and all G-torsors T over L, the following set is *dense* in X/G:

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Example

If $G \subset \mathrm{GL}_n$, then the G-torsor $\mathrm{GL}_n \to \mathrm{GL}_n / G$ is classifying.

Sketch proof of Theorem 2

Assume $W \to W/G$ is a classifying G-torsor induced from a generically free representation, and $U \subset V$ is an open dense on which the stabilizers are conjugate to H.

If T is any G-torsor over L, then we have an N-torsor T' over L:

$$\begin{array}{ccccccc}
T' & \longrightarrow & U^H \times W & \longrightarrow & U \times W & \longleftarrow & T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} L & \xrightarrow{\alpha} & (U^H \times W)/N & \xrightarrow{\cong} & (U \times W)/G & \xleftarrow{\alpha} & \operatorname{Spec} L
\end{array}$$

Then $[T'] \mapsto [(G \times T')/N] = [T]$.

F_4 and E_7^{sc}

$$
\begin{aligned}
5 \leq \mathrm{ed}(F_4) \quad & \leq \mathrm{ed}(N_{F_4}(\mathrm{Spin}_8)) && \text{Theorem 2} \\
& \leq \mathrm{ed}(N_{N_{F_4}(\mathrm{Spin}_8)}(\mathrm{SL}_3)) && \text{Theorem 2} \\
& \leq 17 - 10 = 7 && \text{Theorem 1}
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E_n will denote the split *simply connected* group

$$
\begin{aligned}
7 \le ed(E_7) &\le \mathrm{ed}(N_{E_7}(E_6)) && \text{Theorem 2}\\
&\le \mathrm{ed}(N_{N_{E_7}(E_6)}(\mathrm{Spin}_8)) && \text{Theorem 2}\\
&\le 48 - 31 = 17 && \text{Theorem 1}
\end{aligned}
$$

Thank you

Thanks!