A Hasse principle for function fields

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Hasse principle for number fields

Let k be a number field.

$\Omega_k =$ the set of places of k.

For $v \in \Omega_k$, k_v denotes the completion of k at v.

Theorem (Hasse–Minkowski)

A quadratic form q over k is isotropic if it is isotropic over k_v for all $v \in \Omega_k$.

One has more general local-global principles for homogeneous spaces under connected linear algebraic groups.

Hasse principle for number fields

Theorem (Harder)

Let X be a projective homogeneous space under a connected linear algebraic group defined over a number field. If $X(k_v) \neq \varnothing \ \forall v \in \Omega_k$, $X(k) \neq \varnothing$.

Theorem (Kneser, Harder, Chernousov)

Let G be a semisimple simply-connected linear algebraic group defined over a number field k. Let X be a principal homogeneous space under G. If $X(k_v) \neq \varnothing$ for all real places v of k, $X(k) \neq \varnothing$.

Theorem (Kneser)

If v is a finite place of k, $X(k_v) \neq \varnothing$.

Function fields over number fields

One could look for a local-global principle for the existence of rational points for homogeneous spaces under connected linear algebraic groups defined over function fields.

An obstruction to the Hasse principle for the existence of zeros of quadratic forms crops up already over $k(t)$ where k is a number field.

Function fields over number fields

Example (Riechardt and Lind (1942))

There exists a pair of quadratic forms q_1, q_2 over $\mathbb{Q}$ of dimension 4 such that the intersection X of the quadrics associated to q_1 and q_2 in $\mathbb{P}^3$ is smooth and has a local point over $\mathbb{Q}_v$ for every $v \in \Omega_{\mathbb{Q}}$, but $X(\mathbb{Q})$ is empty.

Theorem (Amer-Brumer)

Let q_1 and q_2 be two quadratic forms of dimension n over a field M. Then $q_1 + tq_2$ has a non-trivial zero over $M(t)$ if and only if q_1 and q_2 have a common non-trivial zero over M.

In particular, $q_1 + tq_2$ has a non-trivial zero over $\mathbb{Q}_v(t)$ for every $v \in \Omega_{\mathbb{Q}}$ but has no non-trivial zero over $\mathbb{Q}(t)$.

Function fields over number fields

Let $f(t) = \mathrm{disc}(q_1 + tq_2)$.

$E : y^2 = f(t)$.

E is an elliptic curve over $\mathbb{Q}$ and $(q_1 + tq_2)_{\mathbb{Q}(E)}$ is similar to a quaternion norm associated to an element $\zeta \in Br(\mathbb{Q}(E))$.

Further $q_1 + tq_2$ is hyperbolic over $\mathbb{Q}_v(E)$ for all $v \in \Omega_{\mathbb{Q}}$, but anisotropic over $\mathbb{Q}(E)$.

Even more:

$q_1 + tq_2$ over $\mathbb{Q}(E)$ is hyperbolic over $\mathbb{Q}(E)_w$ for every discrete valuation w of $\mathbb{Q}(E)$.

Function fields over number fields

The element ζ belongs to $Ш(\mathrm{Br}(E))$ which in turn is associated to an element in ${}_2Ш(E)$.

For any smooth projective curve X over a number field k, let J denote the Jacobian of X.

Let $Ш(J) = \ker(H^1(k, J) \to \prod_v H^1(k_v, J))$.

Hasse kernel for the Witt group

Theorem (Parimala-Sujatha)

Let k be a number field and X a smooth projective curve over k with a rational point. Then the kernel of the natural map

$$W(k(X)) \to \prod_{v \in \Omega_k} W(k_v(X))$$

is isomorphic to ${}_2\text{Ш}(J)$, where J denotes the Jacobian of the curve X.

In fact elements in ${}_2\text{Ш}(J)$ give rise to quadratic forms over $k(X)$ which are hyperbolic over $k(X)_w$ for every discrete valuation w of $k(X)$.

An example

Let E be the elliptic curve over $\mathbb{Q}$ defined by the affine equation

$$y^2 = x^3 + 17x$$

Let $F = \mathbb{Q}(E)$

Let $q = \langle 1, 2, -x, -2x \rangle$

Then q is anisotropic over F but hyperbolic over every completion F_v of F at its discrete valuations.

Unramified Brauer group

Key observation

Let k be a number field and $\mathcal{O}$ the ring of integers in k

Let X be a smooth projective curve over k.

Let $\mathscr{X} \to \mathcal{O}$ be a regular proper model for X.

Then $\mathrm{Br}(\mathscr{X}) \neq 0$ in general.

Unramified Brauer group

This phenomenon of nontrivial unramified Brauer group does not occur for the following classes of fields over which one could look for Hasse principle results.

Type I fields

Type I

Let K be a complete discrete valued field with residue field κ algebraically closed or a finite field. Let X be a smooth projective curve over K and $F = K(X)$.

Let $\mathcal{O}$ be the valuation ring of K

Let $\mathscr{X} \to \mathcal{O}$ be a regular proper model of X.

Then $\mathrm{Br}(\mathscr{X}) = 0$ (Grothendieck).

Type II fields

Type II

Let A be a 2-dimensional henselian local domain with residue field κ algebraically closed or a finite field. Let F be the field of fractions of A.

Let $\mathscr{X} \to A$ be a regular proper desingularisation of A.

${}_n\mathrm{Br}(\mathscr{X}) = 0$ for $(n, \mathrm{char}\,\kappa) = 1$.

For fields of type I and II, there are a host of conjectures and theorems concerning Hasse principle for homogeneous spaces under connected linear algebraic groups.

Type II fields

We look at type II fields with residue fields algebraically closed of characteristic 0.

Let A be a 2-dimensional henselian local domain with residue field κ algebraically closed of characteristic 0. Let F be the field of fractions of A.

Theorem (Artin)

$cd(F) \leq 2$.

Type II fields

Theorem (Colliot-Thélène–Ojanguren–Parimala)

The field F has the following properties:

1. *Hasse principle holds for quadratic forms of dimension at least 3 over F with respect to Ω_F.*
2. *Every 5-dimensional quadratic form over F is isotropic.*
3. *Index is equal to exponent for finite-dimensional central simple algebras over F; further, division algebras over F are cyclic.*
4. $\mathrm{cd}(F^{ab}) \leq 1$.

Type II fields

These theorems lead to the following Hasse-Principle results.

Theorem (Colliot-Thélène–Gille–Parimala)
Hasse principle holds for projective homogeneous spaces under connected linear algebraic groups over F, with respect to Ω_F.

(analogue of Harder's theorem for number fields)

Theorem (Colliot-Thélène–Ojanguren–Parimala)
Every principal homogeneous space under a semisimple simply-connected linear algebraic group over F has a rational point.

(i.e. Serre's Conjecture II holds for F.)

Type II fields

Let F be the field of fractions of a 2-dimensional henselian local domain with residue field κ finite. We have the following

Theorem (Hu)

1. *Every rank nine quadratic form over F is isotropic (char$(\kappa) \neq 2$).*
2. *Every 5-dimensional quadratic form over F satisfies Hasse principle with respect to Ω_F*
3. *Locally cyclic algebras of prime exponent ℓ are cyclic (char$(\kappa) \neq \ell$).*

Theorem (Hu, Preeti)

The Hasse principle holds for principal homogeneous spaces under certain classes of classical simple simply connected groups defined over F.

Function fields of p-adic curves

We discuss some conjectures proposed by *Colliot-Thélène–Parimala–Suresh* concerning homogeneous spaces over fields of type I, more specifically function fields of p-adic curves.

Conjecture (A)

Let F be a function field in one variable over a p-adic field. Let Y be a projective homogeneous space under a connected linear algebraic group defined over F. If $Y(F_v) \neq \emptyset$ for all $v \in \Omega_F$, then $Y(F) \neq \emptyset$.

Conjecture (B)

Let F be a function field in one variable over a p-adic field. Let Y be a principal homogeneous space under a semisimple simply-connected linear algebraic group defined over F. If $Y(F_v) \neq \varnothing$ for all $v \in \Omega_F$, then $Y(F) \neq \varnothing$.

Function fields of p-adic curves

The connectedness assumption cannot be dispensed with.

$E\colon y^2 = x(1-x)(x-p)$ defined over $\mathbb{Q}_p$, p odd.

$F = \mathbb{Q}_p(E)$.

$1 - x \in F^\times$ is a square in $F_v^\times$ at every completion F_v of F.

$1 - x \notin F^{\times 2}$.

$(1 - x) \in H^1(F, \mu_2)$ is not zero, but locally everywhere zero.

Projective homogeneous spaces

There are two cases where Conjecture A is settled, namely projective homogeneous spaces under special orthogonal groups or the projective linear groups.

Theorem (Colliot-Thélène–Parimala–Suresh .)

Let F be a function field in one variable over a complete discrete-valued field with residue field characteristic not 2. let X be a quadric of dimension at least 1 over F. Then $X(F_v) \neq \varnothing \ \forall v \in \Omega_F \Longrightarrow X(F) \neq \varnothing$.

In particular, Conjecture A holds for quadrics.

Projective homogeneous spaces

Theorem (Surendranath Reddy, V. Suresh.)

Let F be a function field in one variable over a complete discrete-valued field K. Let A be a central simple algebra of exponent coprime to the characteristic of the residue field of K. Let X be a generalized Brauer–Severi variety associated to A. Then $X(F_v) \neq \varnothing \; \forall v \in \Omega_F \implies X(F) \neq \varnothing$.

More specifically,

$$\mathrm{index}(A) = \operatorname*{lcm}_{v \in \Omega_F} \mathrm{index}(A \underset{F}{\otimes} F_v).$$

In particular, if A is a p-primary index division algebra over F, $A \otimes_F F_v$ is division over some completion v of F.

Projective homogeneous spaces

Remark

To discuss the Hasse principle, one can confine oneself to discrete valuations centered on a suitable regular proper model for the curve, chosen with reference to the given projective homogeneous space.

Conjecture A remains open for a general projective homogeneous space under a connected linear algebraic group.

The proofs of the above theorems use certain patching theorems of Harbater-Hartmann-Krashen.

HHK patching

Let K be a complete discrete-valued field with residue field κ.

Let $\mathcal{O}$ the valuation ring of K.

Let X be a smooth projective geometrically integral curve over K.

Let $\mathscr{X} \to \mathcal{O}$ be a regular proper model for X.

Let $\mathscr{X}_0 \to \kappa$ be the special fiber.

We assume that $\mathscr{X}_0$ has regular components with normal crossings.

HHK patching

For $x \in \mathscr{X}_0$, let $\hat{\mathcal{O}}_{\mathscr{X},x}$ be the completion of the local ring $\mathcal{O}_{\mathscr{X},x}$.

Let F_x be the field of fractions of $\hat{\mathcal{O}}_{\mathscr{X},x}$.

For $x \in \mathscr{X}_0$ corresponding to a component of $\mathscr{X}_0$, F_x is the completion at the discrete valuation of F associated to x.

For closed point $x \in \mathscr{X}_0$, F_x is the field of fractions of a 2-dimensional complete regular local ring.

HHK patching

Theorem (Harbater-Hartmann-Krashen)

Let K be a complete discrete-valued field and F the function field of a smooth projective geometrically integral curve X over K. Let G be a connected linear algebraic group over F. Suppose G is F-**rational**. *Let Y be a principal homogeneous space or a projective homogeneous space under G. Then there is a regular model $\mathscr{X}$ of X over $\mathcal{O}$ such that if $Y(F_x) \neq \varnothing\ \forall x \in \mathscr{X}_0$ then $Y(F) \neq \varnothing$.*

HHK patching

The Conjectures A and B would be true for F-rational groups G if the following is true :

($\star$) Given a homogeneous space Y under G, there exists a model $\mathscr{X}$ such that $Y(F_v) \neq \varnothing$ for all $v \in \Omega_F \Rightarrow Y(F_x) \neq \varnothing$ for every $x \in \mathscr{X}_0$

The proof of Conjecture A in the two known cases is via proving that ($\star$) is true.

Conjecture B

Conjecture B could be thought of as a 2-dimensional analogue of Kneser's conjecture for number fields.

One has Galois cohomological invariants in degree 3 for principal homogeneous spaces under simply connected groups, the vanishing of which is a necessary condition for the triviality of torsors.

The Rost Invariant

Let G be a simple simply connected linear algebraic group defined over a field k with $\mathrm{char}(k) = 0$

There is an invariant

$$R_G : H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

defined by Rost.

The Rost Invariant

Let $G = \mathrm{SL}_1(A)$, where A is a central simple algebra over F; $\mathrm{index}(A) = n$. The invariant R_G is the Suslin invariant:

$$R_G : H^1(F, \mathrm{SL}_1(A)) \to H^3(F, \mu_n^{\otimes 2})$$

$R_G([\lambda]) = (\lambda) \cdot [A]$.

Theorem (Merkurjev–Suslin)

R_G has trivial kernel if $\mathrm{index}(A)$ *is square-free.*

The Rost invariant

Let F be a field of cohomological dimension 3.

A natural (naive) question is whether the Rost invariant has trivial kernel over F.

There is some evidence to the triviality of the Rost kernel.

Since $\mathrm{cd}(F)$ is 3, $H^4(F, \mathbb{Z}/2\mathbb{Z}) = 0$ and quadratic forms over F are classified up to isomorphism by the dimension, discriminant, Clifford and Arason invariants.

Thus if G is $\mathrm{Spin}(q)$ with q isotropic, R_G has trivial kernel.

The Rost Invariant

One has the following theorem for quasi-split groups.

Theorem (Colliot-Thélène–Parimala–Suresh)

Let F be a field of cohomological dimension 3 and G a quasi-split simple simply-connected group defined F. Then R_G has trivial kernel.

The Rost Invariant

Case-by-case discussion and classification of hermitian forms by Galois cohomology invariants lead to the theorem for classical groups and groups of type G_2.

If G is a quasi-split of type ${}^{3,4}D_4, E_6$, or E_7, $\ker(R_G)$ is trivial in view of a theorem of Rost-Garibaldi.

For split groups of type F_4, the theorem follows from Springer's classification results for Albert algebras.

The proof for split groups of type E_8 is more delicate. It is a consequence of certain Hasse principle results.

Conjecture B for classical groups

The first word of caution to greater expectations on the triviality of the Rost kernel for fields of cohomological dimension three came from Merkurjev.

Merkurjev's example: Let k be a field of cohomological dimension 2 which admits a biquaternion division algebra A.

Let $F = k(t)$. Then $\operatorname{cd} F = 3$.

$[t^2] \in H^1(F, \mathrm{SL}_1(A))$

$R_{SL_1(A)}([t^2]) = (t^2).[A] = 0.$

However, t^2 is not a reduced norm from A and $[t^2] \neq 1$ in $H^1(F, \mathrm{SL}_1(A))$.

Conjecture B for classical groups

Thus R_G could have non-trivial kernel for fields of cohomological dimension 3 in general.

However fields K of cohomological dimension 2 admitting a biquaternion division algebra are “non-existent" for number theorists.

Over all "good fields" of cohomological dimension 2, biquaternion division algebras do not exist.

Conjecture B on Hasse principle only concerns function fields of p-adic curves.

Kato's theorem

Let K be a p-adic field and F a function field in one variable over K. Let Ω_F denote the set of all discrete valuations of F.

We have the following injectivity result of Kato for degree three Galois cohomology of F

Theorem (Kato.)

The map

$$H^3(F, \mu_\ell^{\otimes 2}) \to \prod_{v \in \Omega_F} H^3(F_v, \mu_\ell^{\otimes 2})$$

has trivial kernel.

Kato's theorem

We also have a commutative diagram

$$
\begin{tikzcd}
H^1(F,G) \arrow[r, "R_G"] \arrow[d, "\mathrm{res}"'] & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "\mathrm{res}"] \\
\prod\limits_{v\in\Omega(F)} H^1(F_v,G) \arrow[r] & \prod\limits_{v\in\Omega(F)} H^3(F_v,\mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}
$$

Thus $\ker(H^1(F,G) \to \prod H^1(F_v,G))$ is contained in the Rost kernel.

In particular, for all groups G for which R_G has trivial kernel, Conjecture B is true.

The Rost Kernel

There are certain non-split classical groups for which the triviality of the Rost kernel has been proved.

Theorem (Hu, Preeti)

Let F be a function field in one variable over a p-adic field. Let G be simple simply-connected linear algebraic group of classical type B_n, C_n or D_n, or of type ${}^2A_n$ with $G = \mathrm{SU}(A, \sigma)$, $\mathrm{index}(A) = 2m$, m odd. Then R_G has trivial kernel.

The method of proof of Hu and Preeti involves classifying hermitian forms over division algebras with involution by classical invariants together with the Rost invariant.

$SL_1(A)$ revisited

Let F be a function field in one variable over a p-adic field.

Let $G = SL_1(A)$, with A a biquaternion division algebra over F.

Theorem

R_G has trivial kernel.

$SL_1(A)$ revisited

The theorem follows from the following facts:

• (Merkurjev) $\ker(R_G) = F^{\times 2}.\,\mathrm{Nrd}(A^\times)/\,\mathrm{Nrd}(A^\times)$

• $\mathrm{sn}(\phi) = F^*$ for an Albert form ϕ, since the 12 dimensional form $\phi \perp -\lambda\phi$ is isotropic.

• $F^{*2} \subset Nrd(A^*)$, since squares of spinor norms are precisely reduced norms.

Corollary

Let A be a central simple algebra over F of index dividing $4m$, m odd, squarefree. Then Conjecture B holds for $\mathrm{SL}_1(A)$.

Group schemes over $\mathcal{O}$

We have a Hasse principle for a principal homogeneous spaces under a class of connected reductive groups which leads to the Conjecture B for split groups of type E_8.

Theorem (Colliot-Thélène–Parimala–Suresh.)

Let F be the function field of a p-adic curve. Let G be a connected reductive group scheme over the discrete valuation ring $\mathcal{O}$. Then the Hasse principle is true for principal homogeneous spaces under G over F with respect to its discrete valuations.

The group G in the theorem is *stably F-rational*. One uses HHK patching results to prove the theorem.

The group E_8

Corollary

Let G be a split simple simply-connected group of type E_8 over F. Then the Hasse principle holds for G.

The Rost kernel for E_8

Curiously the Hasse principle leads to triviality of the Rost kernel for groups of type E_8.

Corollary (Corollary)

For G be a split simple simply-connected group of type E_8 over F. Then R_G has trivial kernel.

The Rost kernel for E_8

We have a commutative diagram

$$\begin{tikzcd}
H^1(F,G) \arrow[r, "R_G"] \arrow[d, "\mathrm{res}"'] & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "\mathrm{res}"] \\
\prod_{v\in\Omega(F)} H^1(F_v,G) \arrow[r] & \prod_{v\in\Omega(F)} H^3(F_v,\mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}$$

The two restriction maps have trivial kernel. Further, $R_G\colon H^1(F_v,G) \to H^3(F_v,\mathbb{Q}/\mathbb{Z}(2))$ has trivial kernel (Bruhat–Tits). Thus R_G has trivial kernel.

Conjecture B

Here is the list of groups for which Conjecture B holds :

- Type ${}^1A_n$, $G = \mathrm{SL}_1(A)$, index$(A) = 4m$, m odd, squarefree.
- Type ${}^2A_n$, $G = \mathrm{SU}(A, \sigma)$, index$(A) = 2m$, m odd.
- Type B_n, C_n, D_n (nontrialitarian D_4), G_2.
- All quasi-split groups.

Non-rational groups

The Hasse principle for more general connected linear algebraic groups fails to be true!

Theorem (Colliot-Thélène–Parimala–Suresh)

There are examples of non-rational tori G over F and principal homogeneous spaces Y under G such that $Y(F_v) \neq \varnothing\ \forall v$, but $Y(F) = \varnothing$.

These also give examples of non-rational groups for which HHK patching fails.

Une dernière question

The following question still remains open:

Question

Let G be a connected linear algebraic group over F with G F-rational. If Y is a principal homogeneous space under G which has a rational point over F_v for all completions at discrete valuations of F, does Y admit rational points locally for the HHK patch?