Arithmétique des groupes algébriques linéaires sur les corps de dimension deux

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Corps de nombres

Let k be a totally imaginary number field.

Quadratic forms over k are classified by the *classical invariants*.

Formes quadratiques

Let k be a field of characteristic not 2.

Let q be a quadratic form in n variables over k.

$$q = \sum_{1 \leq i \leq j \leq n} a_{ij} X_i X_j$$

Let b_q be the bilinear form associated to q.

Let A_q be the symmetric matrix associated to b_q.

We assume that q is non-degenerate, i.e. $\det(A_q) \neq 0$.

Invariants classiques

Dimension : $\dim(q) = n$.

Dimension mod 2 : $\dim_2(q) = n \pmod 2 \in \mathbb{Z}/2\mathbb{Z}$.

Discriminant : $\operatorname{disc}(q) = (-1)^{n(n-1)/2} \det(A_q) \in k^*/k^{*2}$.

Clifford invariant : This takes values in the Brauer group of k.

Groupe de Brauer

Brauer equivalence on central simple algebras over a field k :

Let A, B be central simple algebras over k. Then

$$A \sim B \Leftrightarrow M_n(A) \simeq M_m(B).$$

Brauer equivalence classes of central simple algebras over k form an abelian group under $\otimes$, denoted by $\mathrm{Br}(k)$.

Algèbre de Clifford

Let q be a quadratic form over k.

Let $C(q)$ be the Clifford algebra of q.

$$C(q) = C_0(q) \oplus C_1(q).$$

$C_0(q)$ is called the even Clifford algebra of q.

If $\dim(q)$ is even, $C(q)$ is a central simple algebra over k.

If $\dim(q)$ is odd, $C_0(q)$ is a central simple algebra over k.

The Clifford algebra (even Clifford algebra) further comes equipped with a canonical involution and defines a 2-torsion element in $\mathrm{Br}(k)$ if $\dim(q)$ is even ($\dim(q)$ is odd).

Invariants classiques

$$c(q) = \begin{cases} [C(q)] \in {}_2\mathrm{Br}(k), & \text{if } \dim(q) \text{ even} \\ [C_0(q)] \in {}_2\mathrm{Br}(k), & \text{if } \dim(q) \text{ odd.} \end{cases}$$

The dimension, the discriminant and the Clifford invariant are called the classical invariants of quadratic forms.

If k is a totally imaginary number field, the classical invariants determine the isomorphism class of a quadratic form.

These classification results lead to the fact that any principal homogeneous space under $\mathrm{Spin}(q)$ has a rational point.

Cohomologie galoisienne

Let k be a field and L/k a finite Galois extension.

Let G be a linear algebraic group defined over k.

$H^1\left(\mathrm{Gal}\left(L/k\right), G(L)\right)$ is defined to be

$$\{f : \mathrm{Gal}\left(L/k\right) \to G(L) \mid f(st) = f(s)\,{}^{s}f(t)\ \forall s, t \in \mathrm{Gal}(L/k)\}/\sim$$

where $f \sim g$ if and only if there is an $x \in G(L)$ such that

$$f(s) = x^{-1} g(s)\,{}^{s}x, \forall s \in \mathrm{Gal}(L/k)$$

Cohomologie galoisienne

Let $\overline{k}$ be a separable closure of k.

The absolute Galois group of k :

$$\begin{aligned}\Gamma_k &= \mathrm{Gal}\left(\overline{k}/k\right) \\ &= \varprojlim_{\substack{L/k \text{ finite Galois} \\ L \hookrightarrow \overline{k}}} \mathrm{Gal}\,(L/k)\end{aligned}$$

The first Galois cohomology set of G :

$$H^1(k, G) = \varinjlim_{\substack{L/k \text{ finite Galois} \\ L \hookrightarrow \overline{k}}} H^1\left(\mathrm{Gal}(L/k), G(L)\right)$$

Espaces principaux homogènes

$H^1(k, G)$ is a pointed set.

$H^1(k, G) \simeq$ Isomorphism classes of principal homogeneous spaces under G over k.

The point corresponds to the isomorphism class of a principal homogeneous space with a rational point.

Let q be a quadratic form in at least three variables.

The Spin group $\mathrm{Spin}(q)$ is a double cover of the special orthogonal group $\mathrm{SO}(q)$ and we have an exact sequence of algebraic groups

$$1 \to \mu_2 \xrightarrow{i} \mathrm{Spin}(q) \xrightarrow{\eta} \mathrm{SO}(q) \to 1$$

This gives rise to an exact sequence of pointed sets

$$1 \to k^*/k^{*2}\,\mathrm{sn}(q) \xrightarrow{i} H^1(k, \mathrm{Spin}(q)) \xrightarrow{\eta} H^1(k, \mathrm{SO}(q))$$

where sn denotes the spinor norm.

Groupes Spin

$$1 \to k^*/k^{*2}\,\mathrm{sn}(q) \xrightarrow{i} H^1(k, \mathrm{Spin}(q)) \xrightarrow{\eta} H^1(k, \mathrm{SO}(q))$$

The image of η consists of isomorphism classes of quadratic forms with the same dimension, discriminant and Clifford invariant as q.

Classification theorems over totally imaginary number fields imply that the image of η is the point.

The spinor norms contain norms from certain quaternion algebras. Any quarternion norm form is universal. (Any 5-dimensional quadratic form has a non-trivial zero by the Hasse-Minkowski theorem)

Thus $H^1(k, \mathrm{Spin}(q)) = 1$.

Conjecture de Kneser

Conjecture (Kneser)

Let k be a totally imaginary number field and G a semisimple simply connected linear algebraic group defined over k. Then $H^1(k, G) = \{1\}$.

Kneser's conjecture was proved for classical groups by Kneser himself, for exceptional groups other than E_8 by Harder and for groups of type E_8 by Chernousov.

Serre made a conjecture in the 60's which placed Kneser's conjecture in a far more general context.

Dimension cohomologique

For any discrete Γ_k module M, let $H^n(k, M)$ be the Galois cohomology group with coefficients in M.

Definition

Cohomological dimension of k is at most n ($\mathrm{cd}(k) \leq n$) if $H^N(k, M) = 0$ for every discrete torsion Γ_k module M, for all $N \geq n + 1$.

Totally imaginary number fields, p-adic fields and function fields of surfaces over algebraically closed fields are examples of fields of cohomological dimension ≤ 2.

Conjecture II de Serre

Conjecture II (Serre)

Let k be a perfect field of cohomological dimension ≤ 2 and G a semisimple simply connected linear algebraic group defined over k. Then $H^1(k, G) = \{1\}$.

Conjecture II

We break up tracing the progress on Conjecture II into two streams, namely special classes of groups and special classes of fields for which the conjecture is resolved.

$$\begin{tikzcd}[column sep=small] & \text{Classical groups} \\ \text{Restriction on Groups} \arrow[ur] \arrow[dr] & \\ & \text{Special exceptional groups} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \text{fields of arithmetic type} \\ \text{Restriction on fields} \arrow[ur] \arrow[dr] & \\ & \text{fields of geometric type} \end{tikzcd}$$

Absolutely simple simply connected classical groups

- $\mathrm{SL}_1(A)$, where A is a central simple algebra over k.
- $\mathrm{SU}(A, \sigma)$, where σ is an involution of the second kind on A.
- $\mathrm{Spin}(A, \sigma)$, where σ is an orthogonal involution on A.
- $\mathrm{Sp}(A, \sigma)$, where σ is a symplectic involution on A.

Un théorème de Merkurjev-Suslin

The first major breakthrough towards Conjecture II is a theorem of Merkurjev-Suslin for groups of inner type A_n.

Let A be a central simple algebra over k.

$\mathrm{SL}_1(A)$ is the group of elements of reduced norm 1 in A.

There is an exact sequence of algebraic groups

$$1 \to \mathrm{SL}_1(A) \to \mathrm{GL}_1(A) \overset{\mathrm{Nrd}}{\to} \mathbb{G}_m \to 1,$$

which yields a natural bijection.

$k^* / \mathrm{Nrd}(A^*) \simeq H^1(k, \mathrm{SL}_1(A))$

Un théorème de Merkurjev-Suslin

Theorem (Merkurjev-Suslin)

Let k be a perfect field. The following are equivalent :

1. $\mathrm{cd}(k) \leq 2$
2. For every finite extension ℓ/k and any central simple algebra A over ℓ, $Nrd : A \to \ell$ is surjective.

L'invariant de Suslin

Given a central simple algebra A over k, there is a finite dimensional division algebra D over k such that $A \simeq M_n(D)$.

index : $\mathrm{ind}(A) = \sqrt{[D:k]}$

exponent : $\exp(A) =$ order of $[A]$ in $\mathrm{Br}(k)$.

Let A be a central simple algebra over k of index n.

Assume $(n, char(k)) = 1$.

Let $\lambda \in k^*$.

$[\lambda]$ denotes the class of λ in $k^*/Nrd(A^*) \simeq H^1(k, \mathrm{SL}_1(A))$

(λ) denotes its class in $k^*/k^{*n} \simeq H^1(k, \mu_n)$.

L'invariant de Suslin

The Suslin invariant is defined as follows :

$$R : H^1\left(k, \mathrm{SL}_1(A)\right) \to H^3\left(k, \mu_n^{\otimes 2}\right)$$
$$[\lambda] \rightsquigarrow (\lambda) \cdot [A]$$

where $[A] \in H^2(k, \mu_n) \simeq {}_n\mathrm{Br}(k)$.

The injectivity of R for algebras A of square-free index is critical to the proof of Conjecture II for groups of inner type A_n.

Groupes classiques

Theorem (E.Bayer-Fluckiger–Parimala)

Conjecture II is true for all classical groups.

The proof is via classification of hermitian forms over division algebras with involutions in terms of classical invariants, namely, the dimension, the discriminant and the Clifford invariant.

The triviality of the central cocycle classes is achieved via certain norm principle results of Gille-Merkurjev.

Principe de norme

Suppose $G = \mathrm{Spin}\,(A, \sigma)$, A being a central simple algebra over k with an orthogonal involution σ. We have an exact sequence :

$$1 \to \mu_2 \xrightarrow{i} \mathrm{Spin}\,(A, \sigma) \xrightarrow{\eta} \mathrm{SU}\,(A, \sigma) \to 1$$

The image of $H^1\,(k, \mu_2) \to H^1\,(k, \mathrm{Spin}\,(A, \sigma))$ is trivial provided the spinor norm group $\mathrm{sn}\,(A, \sigma)$ is k^*/k^{*2}.

Surjectivity of the spinor norm is proved via the following description of the spinor norm group using the norm principle.

$$\mathrm{sn}(A, \sigma) = \langle N_{L/k}(k^*),\ L/k \textit{ finite with } A_L \textit{ split and } \sigma_L \textit{ isotropic} \rangle \cdot k^{*2}$$

Groupes de type G_2

Let G be a simple simply connected group of type G_2.

$G = \mathrm{Aut}(\mathbb{O})$, $\mathbb{O}$ octonion algebra over k.

Let $n_{\mathbb{O}}$ be the norm form of $\mathbb{O}$.

$n_{\mathbb{O}} \simeq \langle 1, a\rangle \otimes \langle 1, b\rangle \otimes \langle 1, c\rangle$, $a, b, c \in k^*$.

$e_3(n_{\mathbb{O}}) = (-a) \cdot (-b) \cdot (-c) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$.

e_3 denotes the Arason invariant.

$e_3(n_{\mathbb{O}})$ determines the isomorphism class of the octonion algebra $\mathbb{O}$.

Corollary

Conjecture II holds for groups of type G_2.

Groupes de type F_4

Let G be a simple simply connected group of type F_4.

There exists an Albert algebra A such that $G = \mathrm{Aut}(A)$.

There is an invariant for G in $H^3(k, \mathbb{Z}/3\mathbb{Z})$ due to Rost, the vanishing of which ensures that A is a reduced Albert algebra.

Springer has a classification of reduced Albert algebras by their trace forms.

Springer's classification leads to the fact that Albert algebras are split over fields of cohomological dimension 2 yielding Conjecture II for groups of type F_4.

L'invariant de Rost

Let G be a simple simply connected linear algebraic group defined over a field k with $\mathrm{char}(k) = 0$.

There is an invariant

$$R_G : H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

defined by Rost.

If $G = \mathrm{SL}_1(A)$, R_G is the Suslin invariant (Gille-Quéguiner).

Groupes exceptionnels spéciaux

There are classes of simply connected groups G for which R_G has trivial kernel over any field.

If $G = \mathrm{SL}_1(A)$ with A central simple algebra of square free index or $G = \mathrm{Aut}(\mathbb{O})$ where $\mathbb{O}$ is an octonian algebra, then R_G has trivial kernel.

Theorem (Rost-Garibaldi)

If G is quasi split of type trialitarian D_4, E_6 or E_7, R_G has trivial kernel.

Corollary

Conjecture II is true for quasi split groups of type trialitarian D_4, E_6 or E_7.

Corps non-parfait

Serre posed a strengthening of Conjecture II to include fields which are not necessarily perfect.

Conjecture II in this more general setting has been proved in several cases.

> Gille : Groups of type ${}^1A_n$.
>
> Berhuy-Frings-Tignol : Groups of classical type.

Corps de type arithmétique

If k is a number field, then index is equal to exponent for all central simple algebras over k.

We say that a field k is of arithmetic type if $\mathrm{cd}(k) \leq 2$ and index is equal to exponent for all central simple algebras of exponent 2 or 3.

Besides number fields, C_2-fields are examples of fields of arithmetic type.

Corps C_2

A field k is a C_2-field if every homogenous form of degree d in at-least $d^2 + 1$ variables has a non-trivial zero.

Any C_2-field has cohomological dimension at most 2.

This is a consequence of the Merkurjev-Suslin theorem.

Theorem (Artin)

If k is a C_2-field, then index is equal to exponent for central simple algebras of 2 or 3 primary exponent.

Question (Artin)

Is index equal to exponent for all central simple algebras over C_2-fields ?

Corps de type arithmétique

Theorem (Gille)

Let k be a field of arithmetic type. Then Conjecture II holds for all groups of type trialitarian D_4, E_6 or E_7 over k.

If G is of type trialitarian D_4 or E_7 over a field of arithmetic type, the Tits algebra of G can be split by an extension L/k of degree at most 2.

Gille proves that $H^1(L/k, G) = 1$.

Corps de type arithmétique

Let k^{ab} denote the maximal abelian extension of k.

We look at the following condition on k :

$$\mathrm{cd}\left(k^{ab}\right) \leq 1 \qquad (\star)$$

Number fields satisfy the condition $(\star)$.(Every central simple algebra over a number field has an abelian splitting field.)

If k is the function field of a surface over $\mathbb{C}$, it is open whether condition $(\star)$ holds for k.

Chernousov's proof of Kneser's conjecture for groups of type E_8 over number fields can be extended to fields k satisfying $\mathrm{cd}(k) \leq 2$ and $\mathrm{cd}(k^{ab}) \leq 1$.

Groupes de type E_8

Let G be the split group of type E_8 and $\zeta \in H^1(k, G)$.

There is a maximal torus T in G and an element η in $H^1(k, T)$ such that η maps to ζ under the natural map :

$$i : H^1(k, T) \to H^1(k, G)$$

The order of η is $2^p 3^q 5^r$ for some p, q, r. Since $cd(k^{ab}) \leq 1$, Steinberg's theorem given $H^1(k^{ab}, T) = 1$.

There exists a finite abelian extension L/k such that $\eta_L = 0$.

Groupes de type E_8

There is a filtration $L \supset L_1 \supset k$ with $([L : L_1], 60) = 1$ and $[L_1 : k] = 2^a 3^b 5^c$

Since the order of η is coprime to $[L : L_1]$, $\eta_L = 0$ implies $\eta_{L_1} = 0$. Thus $\zeta_{L_1} = 0$.

There is a filtration L_1/k by cyclic extensions E_i/E_{i+1} of degree 2, 3, or 5. Gille shows that $H^1(E_i/E_{i+1}, G) = 0$.

Un exemple

Let A be a 2-dimensional henselian local domain, with field of fractions K and residue field algebraically closed of characteristic 0.

Then $\mathrm{cd}(K) \leq 2$.

Theorem (Colliot-Thélène–Ojanguren–Parimala)

The field K is of arithmetic type with $\mathrm{cd}(K^{ab}) \leq 1$.

Thus Conjecture II holds for K.

Corps de type géométrique

Let k be the function field of a surface over an algebraically closed field.

Theorem (de Jong - He - Starr)

Let G be a split simple simply connected linear algebraic group over k. Let $\zeta \in H^1(k, G)$. If B is a k-Borel subgroup of G and $V = {}_{\zeta}(G/B)$. Then $V(k) \neq \emptyset$.

$$V(k) \neq \emptyset \implies \zeta \in Image(H^1(k, B) \to H^1(k, G))$$

$$H^1(k, B) = 1 \implies \zeta = 1.$$

Corps de type géométrique

Function fields of surfaces over algebraically closed fields are C_2 and hence of arithmetic type. Conjecture II is already established for all groups other than of type E_8.

The theorem of de Jong, He and Starr completes the proof of Conjecture II for function fields of surfaces over algebraically closed fields.

Corps de type géométrique

The theorem of de Jong, He and Starr gives a uniform proof of Conjecture II for split simply connected groups over function fields of surfaces over algebraically closed fields.

A similar classification free proof for Kneser's conjecture over number fields is still a goal to be achieved.

Une question de Serre

Question (Serre)

[QS]. Let k be a field and G a connected linear algebraic group over k. Let X be a principal homogeneous space under G over k. If X has a zero-cycle of degree 1, does X have a k-rational point?

Equivalently if ℓ_i/k, $1 \leq i \leq m$, are field extensions of k of degree n_i with $\mathrm{gcd}_i(n_i) = 1$, does the map

$$\mathrm{Res} : H^1(k, G) \to \prod_i H^1(\ell_i, G)$$

have trivial kernel?

Corps p-spéciaux

A field k is *p-special*, if Γ_k is a pro-p-group.

If [QS] has an affirmative answer, the proof of Conjecture II is reduced to p-special fields.

This can be seen as follows.

Corps p-spéciaux

Let k be a field with cohomological dimension less than or equal to 2 and G a simple simply connected linear algebraic group defined over k.

Let $\xi \in H^1(k, G)$ and L/k a finite Galois extension such that $\xi_L = 1$.

Let $\{p_1, p_2, \cdots, p_r\}$ be the finite set of primes dividing $[L : k]$.

Let L_i be the fixed field of the pro-p_i-Sylow subgroup of Γ_k.

Corps -p-spèciaux

If Conjecture II is true for p-special fields, $\xi_{L_i} = 1$.

There exists finite extensions ℓ_i over k such that $\xi_{\ell_i} = 1$ and $([\ell_i : k], p_i) = 1$.

If [QS] is true, the restriction map

$$H^1(k, G) \to \left(\prod_i H^1(\ell_i, G)\right) \times H^1(L, G)$$

has trivial kernel. Thus $\xi = 1$.

Corps p-spéciaux

For groups of type trialitarian D_4 or E_8, Conjecture II holds for p-special fields.

For instance suppose G is a group of type trialitarian D_4 over a p-special field of cohomological dimension 2.

• If $p \neq 3$, G is of classical type and hence the Conjecture II holds.

• If $p = 3$, then G is quasi split and Conjecture II holds for G.

Thus if [QS] has an affirmative answer for groups of type trialitarian D_4 or E_8, then Conjecture II holds for these groups.

Groupes classiques

Theorem (Jodi Black)

[QS] has a positive answer if G is simple group which is simply connected or adjoint of classical type.