Entropy, Determinants, and ℓ^2-Torsion

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Conference in honor of Marc Rieffel

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2. Volumes and covolumes

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3. Entropy of algebraic actions

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4. ℓ^2-Torsion for amenable groups

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Almost all results are obtained in joint work with Hanfeng Li.

Determinants

Gábor Szegő (1895-1985)

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If $f(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$ is a polynomial, then

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The number $M(f)$ is called the Mahler measure of the function f.

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On $\ell^2\mathbb{Z}$, the operator M_f has matrix coefficients

$$M_f = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_1 & a_2 & \ddots \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ \ddots & a_{-2} & a_{-1} & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

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Determinants of operators on Hilbert space

We consider the matrix

$$D_f^{(n+1)} := \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n \\ a_{-1} & a_0 & a_1 & \ddots & a_{n-1} \\ \vdots & a_{-1} & a_0 & \ddots & \vdots \\ a_{-n+1} & \ddots & \ddots & \ddots & a_1 \\ a_{-n} & a_{-n+1} & \dots & a_{-1} & a_0 \end{pmatrix}$$

If $f \geq 0$, then $D_f^{(n)}$ is positive semi-definite.

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Amenable groups

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For a function φ defined on all finite subsets of Γ, we write

$$\lim_{F \to \infty} \varphi(F)$$

to denote the limit of φ as F becomes more and more invariant.

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defines a unital, positive, faithful trace on $L\Gamma$.

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We can think about μ_a as the distribution of eigenvalues of the operator $a \in B(\ell^2\Gamma)$.

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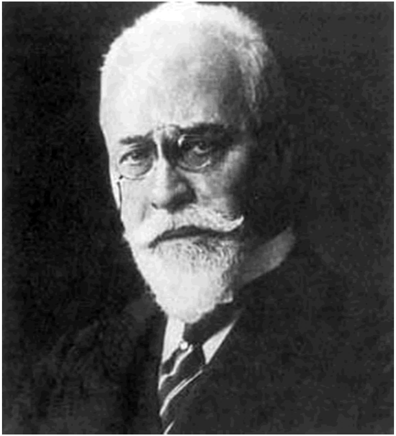
Volumes and covolumes



Hermann Minkowski (1864-1909)

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3. *For $f \in \mathbb{Z}\Gamma$, how is the covolume of $\ell^\infty(\Gamma, \mathbb{Z}) \cdot f$ related to the "size" of $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$?*

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Let $f \in \mathbb{Z}\Gamma$ *be arbitrary. Every weakly closed, symmetric, convex subset of* $\ell^\infty(\Gamma)$ *with*

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contains some non-zero element of $\ell^\infty(\Gamma, \mathbb{Z}) \cdot f$*.*

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$$\det_\Gamma(a) = \lim_{F \to \infty} \det(a_F)^{\frac{1}{|F|}}.$$

This was conjectured by Deninger and only known in special cases and for strictly positive elements in $L\Gamma$.

Ingredients of the proof

Lemma (Gantmacher-Kreĭn)

Let X and Y be finite sets.

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Let X *and* Y *be finite sets. Let* $g \in B(\ell^2(X \cup Y))$ *be positive and invertible.*

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1. $\varphi(\emptyset) = 0$ *and* $\varphi(Fs) = \varphi(F)$ *for all* F *and* $s \in \Gamma$,

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Let φ be a $\mathbb{R}$-valued function defined on finite subsets of Γ, such that

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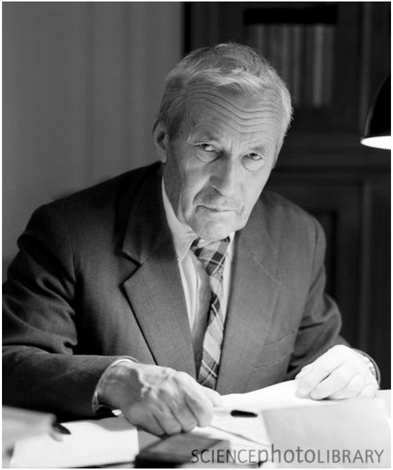
Then

$$\lim_F \frac{\varphi(F)}{|F|} = \inf_F \frac{\varphi(F)}{|F|}.$$

Entropy



Andrei Kolmogorov (1903-1987)

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Consider the partition $[0, 1] = [0, 1/4) \cup [1/4, 1/2) \cup [1/2, 1]$.

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Consider the partition $[0,1] = [0,1/4) \cup [1/4,1/2) \cup [1/2,1]$. For points in $[1/2,1]$ one bit is revealed, whereas for points in $[0,1/2)$, two bits are revealed. Hence, $H = 3/2$; using $\log = \log_2$.

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Kolmogorov showed that one P is enough if P is *generating*.

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A programme to study the question above for principal algebraic actions in the non-commutative case was started by Deninger in 2005.

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- by Li in general if f is invertible in $L\Gamma$.

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Let $g \in L\Gamma$ *be positive such that* $\det_\Gamma g > 0$. *Let* $\lambda > 1$. *Then there exists* $0 < \kappa < \min(1, \|g\|)$ *such that*

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Refined techniques from:

H. Li. *Compact group automorphisms, addition formulas and Fuglede-Kadison determinants.* Ann. of Math. **176** (2012), no. 1, 303-347.

ℓ^2-Torsion

Michael Atiyah (1920-)

Classification of lens spaces

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Definition (Tietze (1908))

The lens spaces are the closed oriented 3-dimensional manifolds

$$L(m, n) = \left\{(a, b) \in \mathbb{C}^2 \middle| |a|^2 + |b|^2 = 1\right\} / (a, b) \sim (\zeta a, \zeta n b),$$

with $\zeta = \exp(\frac{2\pi i}{m})$ a primitive m-th root of unity, and m, n coprime.

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Theorem (Franz, Rueff and Whitehead (1940))

1. *$L(m,n)$ is homotopy equivalent to $L(m,n')$ iff $n \equiv \pm n' r^2 \mod m$ for some $r \in \mathbb{Z}/m\mathbb{Z}$.*
2. *$L(m,n)$ is homeomorphic to $L(m,n')$ iff $n \equiv \pm n' r^2 \mod m$ for $r \equiv 1$ or $r \equiv n \mod m$.*

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Remark

For $G = \{e\}$ or $G = \mathbb{Z}^d$, this a consequence of the classical Milnor-Turaev formula; and related to formulas for the Alexander polynomial.

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Lemma (Chung-T.)

If $\rho(M) < \infty$, then $\rho_p(M) \geq 0$.

The torsion of general $\mathbb{Z}\Gamma$-modules

Theorem (Chung-T.)

Let M be a $\mathbb{Z}\Gamma$-module with finite torsion. Then, we have

$$\rho(M) = \rho_\infty(M) + \sum_p \rho_p(M). \tag{1}$$

Moreover, for any exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathbb{Z}\Gamma$-modules with finite torsion, we have

$$\rho_p(M) = \rho_p(M') + \rho_p(M'')$$

for any prime p, and

$$\rho_\infty(M) = \rho_\infty(M') + \rho_\infty(M'').$$

Thank you for your attention.