Locally compact quantum groups

5. Miscellaneous Topics

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Outline of the lecture series

Outline of the series:

- The Haar weights on a locally compact quantum group
- The antipode of a locally compact quantum group
- The main theory
- Duality
- Miscellaneous topics

All the slides are now on the webpage:
www.alfons-vandaele.be/fields2013.

I also included the pdf files of two articles containing material related with these lectures.

Outline of this lecture

Outline of this last lecture:

Summary of the previous lectures

- In the first lecture, we have concentrated on the Haar weights. We have seen how the modular theory of weights is used to pass from a C*-algebraic locally compact quantum group to a von Neumann algebraic locally compact quantum group.
- However, we still need to consider the way back. We will discuss this step in the present lecture.
- In the middle three lectures, we have developed the main theory. We started with the construction of the antipode and we needed the left and the right Haar weights, not for defining the antipode, but for proving that it was well-defined and densely defined.
- The polar decomposition of the antipode S (or rather of the operator K that implements it) is the main result from which many other results follow.

Summary of the previous lectures

- Again, for the construction of the dual, modular theory plays an important role.
- We learn from all this that the modular theory of weights (i.e. the theory of left Hilbert algebras, as developed by Tomita and Takesaki) provides the basic technical tool for the study of locally compact quantum groups.

We have already seen the variety of objects associated with a locally compact quantum group. And we have encountered several formulas relating these various objects.

In this last lecture, we will begin with some more formulas and see what the possible consequences are.

Many objects - many formulas

With any von Neumann algebraic quantum group (M, Δ) are associated many objects:

- There are the left and right Haar weights φ and ψ with their associated modular data: The Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$, the modular conjugations J_φ and J_ψ, the modular operators ∇_φ and ∇_ψ and the modular automorphisms (σ_t) and (σ'_t).
- Then there are the left and the right regular representations W and V.
- There are the components in the polar decomposition of the antipode S: The *-anti-isomorphism R and the scaling group (τ_t), together with their implementations.
- Finally, there is the modular element δ and its associated one-parameter group δ^{it}, relating the left with the right Haar weight.
- Next, we also have the dual $(\widehat{M}, \widehat{\Delta})$ and all the objects associated with this dual.

The use of a single Hilbert space

A first way to keep track of these various objects is to realize them all within the same Hilbert space, namely $\mathcal{H}_\varphi$. Keep in mind that this is like identifying $L^2(G)$ with $L^2(\widehat{G})$ by means of the Fourier transform in the case of an abelian locally compact group.

The next stept is the observation we made before:

Proposition

If J and ∇ and $\widehat{J}$ and $\widehat{\nabla}$ are the modular conjugation and the modular operator for the left Haar weight φ on M and the left Haar weight $\widehat{\varphi}$ on $\widehat{M}$ respectively, then

$$R(x) = \widehat{J}x^*\widehat{J} \qquad \tau_t(x) = \widehat{\nabla}^{it}x\widehat{\nabla}^{-it} \quad \text{for all } x \in M \tag{1}$$

$$\widehat{R}(y) = Jy^*J \qquad \widehat{\tau}_t(y) = \nabla^{it}y\nabla^{-it} \quad \text{for all } y \in \widehat{M} \tag{2}$$

The implementation of the scaling group

Proposition

There is a one-parameter group of unitaries P^{it} satisfying

$$P^{it}\Lambda_{\varphi}(x) = \nu^{\frac{1}{2}t}\Lambda_{\varphi}(\tau_t(x)) \quad \textit{and} \quad P^{it}\Lambda_{\widehat{\varphi}}(y) = \nu^{-\frac{1}{2}t}\Lambda_{\widehat{\varphi}}(\widehat{\tau}_t(y))$$

for all $x \in M$ and $y \in \widehat{M}$.

Observe that ν is the scaling constant for M and that the scaling constant for the dual $\widehat{M}$ is ν^{-1}.

Proposition

$$\nabla^{it} = (\widehat{J}\widehat{\delta}^{it}\widehat{J})\,P^{it} \quad \textit{and} \quad \widehat{\nabla}^{it} = (J\delta^{it}J)\,P^{it}$$

These are the modular operators for the left Haar weights. One gets the modular operators for the right Haar weights by implementation with $\widehat{J}$ and J respectively.

Some more formulas and some consequences

Proposition

We have $\widehat{\delta}^{is}\delta^{it} = \nu^{-ist}\delta^{it}\widehat{\delta}^{is}$ *for all* $s, t \in \mathbb{R}$. *Also* $\widehat{J}J = \nu^{\frac{i}{4}}J\widehat{J}$.

Proposition

$$P^{-2it} = \delta^{it}\,(J\delta^{it}J)\,\widehat{\delta}^{it}\,(\hat{J}\widehat{\delta}^{it}\hat{J})$$

This formula corresponds to what is called in Hopf algebra theory Radford's formula for the fourth power of the antipode! It has some nice consequences.

- First it means that all data are determined essentially by the modular conjugations J and $\widehat{J}$ together with the modular elements δ and $\widehat{\delta}$.
- If $\delta = 1$ and $\widehat{\delta} = 1$, then all modular operators are trivial and all Haar weights have to be traces. Also the scaling group and hence the scaling constant are all trivial.

Recall: From C*-algebras to von Neumann algebras

Remember the following result from the first lecture:

Proposition

Assume that (A, Δ) is a locally compact quantum group in the C^*-framework. Let φ and ψ be a left and a right Haar weight on A. Define $M = \pi_\varphi(A)''$. Then Δ extends to a coproduct on M. Also the weights φ and ψ extend to a left and a right Haar weight on M. So (M, Δ) is a locally compact quantum group in the von Neumann algebra framework.

Remember that we did not need the density conditions saying that

$$(\omega \otimes \iota)\Delta(a) \qquad \text{and} \qquad (\iota \otimes \omega)\Delta(a)$$

each span a dense subspace of A.

Back: From von Neumann algebras to C*-algebras

Here is the reverse procedure.

Proposition

Let (M, Δ) be a locally compact quantum group in the von Neumann algebraic framework. Let φ and ψ be the left and the right Haar weight. Let A be the norm closure of the space $(\iota \otimes \omega)W$ where W is the left regular representation and $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$. Then A is a dense C*-subalgebra of M. The restriction of Δ to A is a coproduct on A satisfying the stronger density conditions. That is

$$\Delta(A)(1 \otimes A) \qquad \text{and} \qquad \Delta(A)(A \otimes 1)$$

are dense in $A \otimes A$. The restrictions of φ and ψ to A are a left and a right Haar weight on (A, Δ). Hence, (A, Δ) is a locally compact quantum group in the C*-algebraic sense.

Recovering the original C*-algebraic quantum group

There are two obvious results to verify.

Suppose we start with (A, Δ) and pass to the von Neumann algebra extension $\pi_\varphi(A)''$. If we go back to the C*-framework, we want to recover the original pair. This is essentially a consequence of the result saying that the norm closure of the space

$$\{(\iota \otimes \omega)W \mid \omega \in \mathcal{B}(\mathcal{H}_\varphi)_*\}$$

is the same as the norm closure of the space

$$\mathrm{sp}\{(\iota \otimes \omega)\Delta(a) \mid a \in A,\ \omega \in A^*\}.$$

This results has been proven for the σ-weak closures but one can check that the argument works for the norm closures as well.

Here we need to assume the density conditions in order to have that this last space is equal to the original C*-algebra.

The next step is restricting the various data (like the modular automorphism groups, the scaling group, ...) from the von Neumann algebra M to the C*-subalgebra A. Fortunately, this is all very straightforward.

The main reason why all this can be done, is a consequence of the fact that many formulas can be written with the operators implementing them. We give an example.

Example

Start from the formula $\Delta(\sigma_t(x)) = (\tau_t \otimes \sigma_t)\Delta(x)$. The automorphisms (σ_t) are implemented by the unitary operators ∇^{it}. From this we obtain that

$$(\tau_t \otimes \iota)W = (1 \otimes \nabla^{-it})W(1 \otimes \nabla^{it})$$

for all t. It follows from this formula that the scaling group τ_t leaves the C*-algebra invariant and that $t \mapsto \tau_t(a)$ is norm continuous for all $a \in A$.

The other loop

We also have to verify the other loop. This is the easier case.

If we start with a von Neumann algebraic quantum group, pass to the dense C*-algebra containing it and if we then go back, we should arrive at the original case.

Again, this part is very straightforward. Indeed, if we start with the von Neumann algebra M, we know that the C*-algebra A is defined as the norm closure of $\{(\iota \otimes \omega)W \mid \omega \in \mathcal{B}(\mathcal{H})_*\}$. But as M is the σ-weak closure of this space, we see that A is generating M. And of course the original coproduct on M will be recovered if we first restrict it to A and then extend it again to M.

This shows that the two approaches are indeed equivalent to one another.

Examples

- There are the basic examples $C_0(G)$ and $C_r^*(G)$ for any locally compact group G. For these examples, many of the data are too trivial to illustrate the rich theory.
- Lately, there has been a lot of research about special cases of compact quantum groups. These are the ones where the underlying C*-algebra has an identity and where the Haar weights are finite. For these examples, we necessarily have $\varphi = \psi$ and $\delta = 1$. The scaling group may be non-trivial, but the scaling constant ν will be 1.
- Most interesting, from a theoretical point of view, are those examples where non of the objects are trivial, or trivially related with each other.
- To construct such examples with greater complexity, is in general quite involved.

Examples

Many of these examples start with a relatively simple Hopf *-algebra that is lifted to an operator algebra setting. This often causes serious problems (by the use of unbounded operators as generators).

Consider e.g. the relation $ab = \lambda ba$ of self-adjoint elements in a *-algebra. Necessarily we have that λ is a complex number of modulus 1. It is non-trivial to find self-adjoint operators satisfying such commutation relations. Moreover, a coproduct give by $\Delta(a) = a \otimes a$ and $\Delta(b) = a \otimes b + b \otimes 1$ presents major difficulties when one tries to lift it to the operators.

The bicrossproduct construction

Here is an other example:

Consider a locally compact group G with two closed subgroups H and K. Assume that the map $(h,k) \mapsto hk$ is a homeomorphism from the Cartesian product $H \times K$ to G.

Notation

For any two elements $h \in H$ and $k \in K$, there is a unique way to write hk as a product $k'h'$ with $h' \in H$ and $k' \in K$. We will use $h \triangleright k$ for k' and $h \triangleleft k$ for h'. We get a left action of the group H on the set K and a right action of the group K on the set H. These actions are related in a very specific way.

We can associate two C*-algebras in duality as follows.

The bicrossproduct construction

Definition

Consider the C*-algebra $C_0(K)$ and let H act on it by $(f \triangleleft h)(k) = f(h \triangleright k)$. Similarly consider the C*-algebra $C_0(H)$ and let K act on it by $(k \triangleright f)(h) = f(h \triangleleft k)$. Let A be the reduced crossed product of $C_0(K)$ with this right action of H and let B be the reduced crossed product of $C_0(H)$ with this left action of K. Each of these algebras has the space $C_c(H, K)$ as a dense *-subalgebra A_0 and B_0 (with different products).

Proposition

The product in A_0 is given by

$$(f_1 f_2)(h, k) = \int_H f_1(u, (u^{-1}h) \triangleright k) f_2(u^{-1}h, k)\, du$$

and the involution by $f^*(h, k) = \delta_H(h^{-1})\bar{f}(h^{-1}, h \triangleright k)$.

We can define a pairing on these subalgebras by the formula:

Theorem

For $f \in A_0$ and $g \in B_0$ we define

$$\langle f, g \rangle = \iint_{H \times K} f(h,k)g(h,k)\,dh\,dk$$

where we integrate over the left Haar measures on H and K. There exists coproducts on the C^*-algebras A and B, induced by the products via this pairing. They give a dual pair of locally compact quantum groups.

The right Haar weight on A is given by $\psi(f) = \int_K f(e,k)d^r k$ for $f \in A_0$ where e is the identity in H and where we integrate over the right Haar measure over K.

With these formulas, we have enough information to calculate all data.
Unfortunately, the scaling group is trivial for this example.

Almost matched pairs

Consider the $ax + b$-group. It is the group G of 2×2- matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and with $a > 0$. We can consider two subgroups H and K given by matrices of the form

$$\begin{pmatrix} a & a-1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

respectively. We do not get a genuine matched pair as the map $(h, k) \mapsto hk$ is a homeomorphism from $H \times K$ to an open and dense (but proper) subset of G. Still, the bicrossproduct construction is possible as a locally compact quantum group. We can get examples this way where the scaling group is non-trivial. However, the scaling constant will still be trivial.

Recent developments - Generalizations

There are several directions with recent developments.

Let me focus on just one generalization: Locally compact quantum groupoids.

The first step towards this theory is the following notion.

Definition

A weak Hopf algebra is a pair (A, Δ) of a unital algebra A with a coproduct $\Delta : A \to A \otimes A$ that is not necessarily unital. However, the idempotent E, defined as $\Delta(1)$ in $A \otimes A$ has to satisfy certain properties. It is assumed that there is a counit and an antipode. The axioms for the counit are as for Hopf algebras, but the axioms for the antipode are different.

Weak multiplier Hopf algebras

There is a generalization of this concept to the non-unital case.

- The algebra A is no longer assumed to be unital.
- The coproduct Δ maps A to $M(A \otimes A)$.
- There is an idempotent $E \in M(A \otimes A)$ satisfying certain conditions, making it unique. After extending the coproduct to $M(A)$ we have $\Delta(1) = E$.
- There is a counit satisfying the usual conditions.
- And there is an antipode satisfying the same conditions as for weak Hopf algebras.

This gives a weak multiplier Hopf algebra. Basic examples come from a (discrete) groupoid.

Also integrals (these are like the Haar weights in the case of a locally compact quantum group) are studied. A special case is that of a weak multiplier Hopf *-algebra with positive integrals.

Locally compact quantum groupoids

One can proceed in two directions.

- One can start with such a weak multiplier Hopf *-algebra with positive integrals and lift this structure to an operator algebraic framework. This is too restrictive.
- One can start with a weak multiplier Hopf *-algebra with positive integrals and first pass to a Hopf algebroid formulation. This is done by considering the canonical maps

$$a \otimes b \mapsto \Delta(a)(1 \otimes b) \qquad \text{and} \qquad a \otimes b \mapsto (a \otimes 1)\Delta(b)$$

  as bijective maps between appropriate balanced tensor products. And then further lifting to an operator algebra approach.

This should eventually lead to a theory of locally compact quantum groupoids, very much as the theory of locally compact quantum groups as presented in this series of lectures.

Conclusions

- We have treated the theory of locally compact quantum groups.
- It is not so important from which set of axioms you start. The final objects are the same.
- This is in particular true for the two basic approaches: The C^*-algebraic approach and the von Neumann algebraic approach. They yield the same objects and the approaches are equivalent with each other.
- In the general theory, one has to assume the existence of the Haar weights. This is not nice from a theoretical point of view. However, in examples, the Haar weights are mostly immediately available.
- There are now plenty of non-trivial examples.
- The focus of the present research in the field is on compact quantum groups in one direction and on locally compact quantum groupoids in another direction.

References

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