Locally compact quantum groups

4. The dual of a locally compact quantum group

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Outline of lecture series

Outline of the series:

- The Haar weights on a locally compact quantum group
- The antipode of a locally compact quantum group
- The main theory
- Duality
- Miscellaneous topics

Outline of the present lecture

Outline of this lecture:

Introduction

We have a locally compact quantum group (M, Δ) with a unique left and right Haar weight φ and ψ. The modular automorphism groups are denoted by (σ_t) and (σ'_t) resp. There is also the scaling group (τ_t) and the polar decomposition of the antipode $S = R\tau_{-\frac{i}{2}}$ where R is an involutive, *-anti-isomorphism of M that flips the coproduct.

All these automorphism groups commute with each other. The anti-isomorphism R commutes with the scaling automorphisms, but not with the modular automorphisms. We have $R \circ \sigma_t = \sigma'_{-t} \circ R$.

The relation with the coproduct is as follows

$$\Delta(\sigma_t(x)) = (\tau_t \otimes \sigma_t)\Delta(x) \qquad \Delta(\sigma'_t(x)) = (\sigma'_t \otimes \tau_{-t})\Delta(x) \quad (1)$$

$$\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x) \qquad \Delta(\tau_t(x)) = (\sigma_t \otimes \sigma'_{-t})\Delta(x). \quad (2)$$

Relative invariance of the Haar weights

We have a strictly positive number ν, called the scaling constant, satisfying

$$\varphi \circ \tau_t = \nu^{-t}\varphi \qquad \psi \circ \tau_t = \nu^{-t}\psi \tag{3}$$

$$\psi \circ \sigma_t = \nu^{-t}\psi \qquad \varphi \circ \sigma'_t = \nu^{t}\varphi \tag{4}$$

Finally, there is the modular element δ. It is a non-singular, positive self-adjoint operator, affiliated with M and relating the left with the right Haar weight as $\psi = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. This operator satisfies $\sigma_t(\delta) = \nu^t\delta$ and $\sigma'_t(\delta) = \nu^{-t}\delta$. Is is invariant under the automorphisms (τ_t) and $R(\delta) = \delta^{-1}$. We also have the relation $\sigma'_t(x) = \delta^{it}\sigma_t(x)\delta^{-it}$.

We will work further with the left regular representation W. We write $\mathcal{H}$ for $\mathcal{H}_\varphi$ and Λ for Λ_φ. Formally we have

$$W^*(\xi \otimes \Lambda(x)) = \sum x_{(1)}\xi \otimes \Lambda(x_{(2)}).$$

The dual $(\widehat{M}, \widehat{\Delta})$

First we define the dual von Neumann algebra $\widehat{M}$.

Definition

Let $\widehat{M}$ be the σ-weak closure of the subspace $\{(\omega \otimes \iota)W \mid \omega \in M_*\}$ of $\mathcal{B}(\mathcal{H})$.

From the pentagon equation, it follows that the space $\{(\omega \otimes \iota)W \mid \omega \in M_*\}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$. To show that its closure in *-invariant, we use the formula

$$((\omega \otimes \iota)W)^* = (\omega_1 \otimes \iota)W$$

that holds for nice elements ω in M_* and where ω_1 is defined by $\omega_1(x) = \omega(S(x)^*)^- = \overline{\omega}(S(x))$ when $x \in \mathcal{D}(S)$.

Proposition

$\widehat{M}$ is a von Neumann algebra and $W \in M \otimes \widehat{M}$.

Topologies on a von Neumann algebra

Let M act on $\mathcal{H}$.

- The weak operator topology on M is the weakest topology making the linear maps $x \mapsto \langle x\xi, \eta\rangle$ continuous for all $\xi, \eta \in \mathcal{H}$.
- The σ-weak topology is the weakest topology making the linear maps
$$x \mapsto \sum \langle x\xi_i, \eta_i\rangle$$
continuous whenever $\sum \|\xi_i\|\|\eta_i\| < \infty$.

These two topologies coincide on bounded sets.

The space of σ-weakly continuous functionals on M is denoted by M_* and called the predual of M. The dual of M_* is M.

There are various other topologies on a von Neumann algebra, all weaker than the norm topology, and stronger than the weak operator topology.

The Pentagon equation

We have

- $\Delta(x) = W^*(1 \otimes x)W$
- $(\Delta \otimes \iota)W = W_{13}W_{23}$

If we combine this with the fact that M is the left leg of W, we can summarize these two formulas and find

$$W_{13}W_{23} = W_{12}^*W_{23}W_{12}.$$

Rewriting it, we find the so-called Pentagon equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$. Now apply $\omega \otimes \omega' \otimes \iota$ and we find that the space

$$\{(\omega \otimes \iota)W \mid \omega \in M_*\}$$

is a subalgebra of $\mathcal{B}(\mathcal{H})$.

The norm closure of $\{(\omega \otimes \iota)W \mid \omega \in M_*\}$

In general, $\{(\omega \otimes \iota)W \mid \omega \in M_*\}$ is not *-invariant. We have that

$$((\omega \otimes \iota)W)^* = (\omega_1 \otimes \iota)W$$

provided $\omega_1(x) = \overline{\omega}(S(x))$.

We can produce enough such functionals by taking integrals

$$x \mapsto \int f(t)\omega(\tau_t(x))\,dt$$

with the appropriate choice of functions f. Recall that $S = R\tau_{-\frac{i}{2}}$.

The dual coproduct $\widehat{\Delta}$ on $\widehat{M}$

Proposition

For all $y \in \widehat{M}$ *we have* $W(y \otimes 1)W^* \in \widehat{M} \otimes \widehat{M}$. *If we define*

$$\widehat{\Delta}(y) = \chi W(y \otimes 1)W^*$$

where χ *is the flip,* $\widehat{\Delta}$ *is a coproduct on* $\widehat{M}$

All this is an easy consequence of the Pentagon equation for W, now written as

$$W_{23}W_{12}W_{23}^* = W_{12}W_{13}$$

and the fact that $W \in M \otimes \widehat{M}$. The use of the flip map is just a convention.

This is the easy part of the construction. The more difficult step is the construction of the dual Haar weights.

Construction of a left Hilbert algebra

To construct the dual left Haar weight $\hat{\varphi}$ on $(\hat{M}, \hat{\Delta})$, we need a left Hilbert algebra.

Definition

Let $\hat{\mathfrak{N}}$ be the set of elements $y \in \hat{M}$ so that there is a $\omega \in M_*$ and a vector $\xi \in \mathcal{H}$ satisfying

$$\omega(x^*) = \langle \xi, \Lambda(x) \rangle \quad \text{for all } x \in \mathfrak{N}_\varphi \quad \text{and} \quad y = (\omega \otimes \iota)W.$$

We set $\hat{\Lambda}(y) = \xi$.

Observe that, given y, the element ω and the vector ξ are unique, if they exist. Also $\hat{\Lambda}$ is linear and injective.

What is the possible motivation for such a definition?

Motivation of this definition

The Fourier transform $\widehat{z}$ of an element z is defined as the linear functional $\omega = \varphi(\,\cdot z)$ (provided this makes sense). Now, it is known from the algebraic theory that the multiplicative unitary W is essentially the duality. So, formally, we have $\widehat{z} = (\omega \otimes \iota)W$ when $\omega = \varphi(\,\cdot z)$. The 'spaces' $L^2(G)$ and $L^2(\widehat{G})$ are identified which means (again formally) that we want $\Lambda_{\widehat{\varphi}}(\widehat{z}) = \Lambda_\varphi(z)$. This formula is rewritten as

$$\langle \Lambda_{\widehat{\varphi}}(\widehat{z}), \Lambda_\varphi(x) \rangle = \langle \Lambda_\varphi(z), \Lambda_\varphi(x) \rangle = \varphi(x^* z) = \omega(x^*)$$

whenever $x \in \mathcal{N}_\varphi$.

We get the formulas from the previous definition with $y = \widehat{z}$ and $\xi = \Lambda_{\widehat{\varphi}}(\widehat{z})$.

The *- algebra $\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*$

First we need to show that $\hat{\Lambda}(\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*)$ is dense in $\mathcal{H}$.

Proposition

Let $\xi, \eta \in \mathcal{H}$ and assume that η is right bounded. Let $\omega = \langle \cdot \xi, \eta \rangle$ and $y = (\omega \otimes \iota)W$. Then $y \in \hat{\mathfrak{N}}$ and $\hat{\Lambda}(y) = \pi'(\eta)^* \xi$. In particular, the space $\hat{\Lambda}(\hat{\mathfrak{N}})$ is dense in $\mathcal{H}$ and also $\hat{\mathfrak{N}}$ is σ-weakly dense in $\hat{M}$.

Doing this construction a little more careful, we find the density of $\hat{\Lambda}(\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*)$ in $\mathcal{H}$ and of $\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*$ in $\hat{M}$.

The following will provide the multiplication.

Proposition

Let $\omega, \omega_1 \in M_*$ and $y = (\omega \otimes \iota)W$ and $y_1 = (\omega_1 \otimes \iota)W$. If $y \in \hat{\mathfrak{N}}$, then also $y_1 y \in \hat{\mathfrak{N}}$ and $\hat{\Lambda}(y_1 y) = y_1 \hat{\Lambda}(y)$.

Assume that ξ is any vector in the Hilbert space $\mathcal{H}$ and that η is right bounded. Then

$$\langle \pi'(\eta)^* \xi, \Lambda(x) \rangle = \langle \xi, \pi'(\eta)\Lambda(x) \rangle = \langle \xi, x\eta \rangle$$

and we see that y, defined as $y = (\omega \otimes \iota)W$ where $\omega = \langle \cdot \xi, \eta \rangle$ will satisfy $\widehat{\Lambda}(y) = \pi'(\eta)^* \xi$.

The left Hilbert algebra $\hat{\Lambda}(\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*)$

Proposition

Let $\mathfrak{A} = \hat{\Lambda}(\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*)$. We can equip $\mathfrak{A}$ with the *-algebra structure inherited from $\hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*$. If we denote y by $\pi(\xi)$ when $y \in \hat{\mathfrak{N}} \cap \hat{\mathfrak{N}}^*$ and $\xi = \hat{\Lambda}(y)$, then we have:

- *$\mathfrak{A}$ and $\mathfrak{A}^2$ are dense in $\mathcal{H}$,*
- *$\pi(\xi)$ is continuous for all $\xi \in \mathfrak{A}$,*
- *π is a *-representation of $\mathfrak{A}$,*
- *The *-operation on $\mathfrak{A}$, denoted as $\xi \mapsto \xi^\sharp$, is preclosed.*

Theorem

There exists a normal faithful semi-finite weight $\hat{\varphi}$ on $\hat{M}$ such that the G.N.S.-representation can be realized in $\mathcal{H}$, satisfying $\hat{\mathfrak{N}} \subseteq \mathfrak{N}_{\hat{\varphi}}$ and such that the canonical map $\Lambda_{\hat{\varphi}}$ is the closure of $\hat{\Lambda}$ on $\hat{\mathfrak{N}}$.

Left invariance of the dual left Haar weight

Proposition

Define the unitary $\widehat{W} = \Sigma W^* \Sigma$ *on* $\mathcal{H} \otimes \mathcal{H}$. *We use* Σ *for the flip operator on* $\mathcal{H} \otimes \mathcal{H}$. *Then* $(\omega \otimes \iota)\widehat{\Delta}(y) \in \mathfrak{N}_{\widehat{\varphi}}$ *and*

$$((\omega \otimes \iota)\widehat{W}^*)\Lambda_{\widehat{\varphi}}(y) = \Lambda_{\widehat{\varphi}}((\omega \otimes \iota)\widehat{\Delta}(y))$$

whenever $y \in \mathfrak{N}_{\widehat{\varphi}}$ *and* $\omega \in \mathcal{B}(\mathcal{H})_*$.

The proof is rather straightforward. At the end one uses that $\Lambda_{\widehat{\varphi}}$ on $\mathfrak{N}_{\widehat{\varphi}}$ is the closure of $\widehat{\Lambda}$ on $\widehat{\mathfrak{N}}$.

Theorem

The weight $\widehat{\varphi}$ *is left invariant on* $(\widehat{M}, \widehat{\Delta})$.

The dual right Haar weight

Recall the formula $(I \otimes J)W(I \otimes J) = W^*$ where J is the modular conjugation associated with the original left Haar weight φ on (M, Δ).

Proposition

Define $\widehat{R}$ on $\widehat{M}$ by $\widehat{R}(y) = Jy^*J$. Then $\widehat{R}$ is an involutive *-anti-automorphism of $\widehat{M}$ that flips the coproduct $\widehat{\Delta}$.

We can now define $\widehat{\psi}$ on $(\widehat{M}, \widehat{\Delta})$ by $\widehat{\psi} = \widehat{\varphi} \circ \widehat{R}$. This will be a right invariant weight. Hence we find that $(\widehat{M}, \widehat{\Delta})$ is a locally compact quantum group. It is called the dual of (M, Δ).

It is not hard to show that the dual of $(\widehat{M}, \widehat{\Delta})$ is canonically isomorphic with the original locally compact quantum group(M, Δ).

More formulas

We have lots of operators and other objects, related with a locally compact quantum group (M, Δ) and its dual $(\widehat{M}, \widehat{\Delta})$.

Due to the relative invariance of the Haar weights, the automorphism groups are implemented by unitaries.

Proposition

There exist continuous one-parameter groups of unitaries (u_t), (v_t) and (w_t) on $\mathcal{H}$ given by

- $u_t \Lambda_\varphi(x) = \Lambda_\varphi(\sigma_t(x))$
- $v_t \Lambda_\varphi(x) = \nu^{\frac{1}{2}t} \Lambda_\varphi(\tau_t(x))$
- $w_t \Lambda_\varphi(x) = \nu^{-\frac{1}{2}t} \Lambda_\varphi(\sigma'_t(x))$

when $x \in \mathfrak{N}_\varphi$. They all commute and implement the associated automorphism groups.

We also have the one parameter groups (δ^{it}) and $(\widehat{\delta}^{it})$.

Proposition

The modular conjugation $\widehat{J}$ and the modular operator $\widehat{\nabla}$ for the dual left Haar weight $\widehat{\varphi}$ are given by

$$\widehat{J}\Lambda_{\varphi}(x) = \Lambda_{\varphi}(R(x)^{*}\delta^{\frac{1}{2}}) \quad (5)$$

$$\widehat{\nabla}^{it}\Lambda_{\varphi}(x) = \Lambda_{\varphi}(\tau_{t}(x)\delta^{-it}) \quad (6)$$

where $x \in \mathfrak{N}_{\varphi}$.

Proposition

$$R(x) = \widehat{J}x^{*}\widehat{J} \qquad \tau_{t}(x) = \widehat{\nabla}^{it}x\widehat{\nabla}^{-it} \quad \text{for all } x \in M \quad (7)$$

$$\widehat{R}(y) = Jy^{*}J \qquad \widehat{\tau}_{t}(y) = \nabla^{it}y\nabla^{-it} \quad \text{for all } y \in \widehat{M} \quad (8)$$

Proposition

We have

- $\hat{\Delta}(\hat{\delta}^{it}) = \hat{\delta}^{it} \otimes \hat{\delta}^{it}$ *for all* t.
- $\Delta(\delta^{it}) = \delta^{it} \otimes \delta^{it}$ *for all* t.

Remark that the second formula is proven by duality, from the first one.

Proposition

- $\nabla^{it} = (\hat{J}\hat{\delta}^{it}\hat{J})\,P^{it}$
- $\hat{\nabla}^{it} = (J\delta^{it}J)\,P^{it}$

We have written P^{it} for ν^{it}, introduced earlier. We get similar formulas for the modular operators of the right Haar weights.

Conclusions

- We have associated a dual locally compact quantum group $(\widehat{\Delta}, \widehat{M})$ to any locally compact quantum group (M, Δ).
- We obtained many formulas connecting the multitude of objects that come with such a pair of quantum groups.
- However, we seem to have forgotten to go back to the C^*-algebras.
- This is one of the topics we plan to treat in the last lecture.

References

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