Locally compact quantum groups

3. The main theory

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Outline of lecture series

Outline of the series:

- The Haar weights on a locally compact quantum group
- The antipode of a locally compact quantum group
- The main theory
- Duality
- Miscellaneous topics

Outline of the third lecture

Outline of this third lecture:

- Introduction - with a review of some previous results
- The polar decomposition of the antipode
- Uniqueness of the Haar weights
- The main results
- Conclusions

Summary of the previous lectures

Recall the definition of a locally compact quantum group in the von Neumann algebraic setting.

Definition

A locally compact quantum group is a pair (M, Δ) of a von Neumann algebra M with a coproduct $\Delta : M \to M \otimes M$ so that there exist left and right Haar weights.

The Haar weights are faithful, normal and semi-finite. A left Haar weight φ is left invariant:

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1$$

for all positive elements $x \in M$ so that $\varphi(x) < \infty$. Similarly for a right Haar weight ψ.

Some comments:

The definition in the case of von Neumann algebras is simpler than for C*-algebras:

- No need to work with multipliers.
- No need to impose extra density conditions.
- Theory of weights for von Neumann algebras is better known.

Left invariance:

Consider a locally compact group G and $M = L^\infty(G)$ with $\Delta(f)(p,q) = f(pq)$. Then

$$((\iota \otimes \varphi)\Delta(f))(p) = \int f(pq)dq = \int f(q)dq$$

and so $(\iota \otimes \varphi)\Delta(f) = \varphi(f)1$.

The left and right regular representations

Formally, the left regular representation W and the right regular representation V are defined by

$$V(\Lambda_\psi(x) \otimes \xi) = \sum \Lambda_\psi(x_{(1)}) \otimes x_{(2)}\xi \qquad (1)$$

$$W^*(\xi \otimes \Lambda_\varphi(x)) = \sum x_{(1)}\xi \otimes \Lambda_\varphi(x_{(2)}) \qquad (2)$$

These are unitary operators satisfying:

Proposition

- $\Delta(x) = V(x \otimes 1)V^*$
- $\Delta(x) = W^*(1 \otimes x)W$
- $(\iota \otimes \Delta)V = V_{12}V_{13}$
- $(\Delta \otimes \iota)W = W_{13}W_{23}$

About V and W:

$$((\iota \otimes \langle \cdot \xi, \eta \rangle)V)\Lambda_\psi(x) = \Lambda_\psi((\iota \otimes \langle \cdot \xi, \eta \rangle)\Delta(x))$$
$$((\langle \cdot \xi, \eta \rangle \otimes \iota)W^*)\Lambda_\varphi(x) = \Lambda_\varphi((\langle \cdot \xi, \eta \rangle \otimes \iota)\Delta(x))$$

Remark that by invariance:

- $(\iota \otimes \omega)\Delta(x) \in \mathfrak{N}_\psi$ if $x \in \mathfrak{N}_\psi$
- $(\omega \otimes \iota)\Delta(x) \in \mathfrak{N}_\varphi$ if $x \in \mathfrak{N}_\varphi$

The antipode

The antipode S_0 is a closed linear map, with dense domain $\mathcal{D}_0$ characterized by the following result.

Proposition

Let $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$ and $x = (\iota \otimes \omega)W$ and $x_1 = (\iota \otimes \overline{\omega})W$, then $x \in \mathcal{D}_0$ and $x_1 = S_0(x)^*$.

We can write this as

$$(S_0 \otimes \iota)W = W^*$$

or as

$$S_0((\iota \otimes \varphi)(\Delta(x)(1 \otimes y)) = (\iota \otimes \varphi)((1 \otimes x)\Delta(y))$$

with the right choice for the elements x and y. The operator $x \mapsto S_0(x)^*$ is implemented by the operator K, formally satisfying

$$K(\Lambda_\psi(x)) = \Lambda_\psi(S_0(x)^*)$$

again for suitable elements x in M.

Remark that

$$(\iota \otimes \langle \cdot \Lambda_\varphi(x), \Lambda_\varphi(y) \rangle) W^* = (\iota \otimes \varphi)((1 \otimes y^*)\Delta(x))$$

and hence, we can rewrite the formula $(S_0 \otimes \iota)W = W^*$ as

$$S_0((\iota \otimes \varphi)(\Delta(y^*)(1 \otimes x)) = (\iota \otimes \varphi)((1 \otimes y^*)\Delta(x))$$

Formula with two left regular representations

Let φ_1 and φ_2 be two left Haar weights on (M, Δ). Denote the associated left regular representations by W_1 and W_2.

Notation

Denote by T_r the closure of the map $\Lambda_{\varphi_1}(x) \mapsto \Lambda_{\varphi_2}(x^*)$, defined for $x \in \mathfrak{N}_{\varphi_1} \cap \mathfrak{N}^*_{\varphi_2}$.

Then one can show, (by a careful argument):

Proposition

$$(K \otimes T_r)W_1 = W_2^*(K \otimes T_r)$$

Remark that $K \otimes T_r$ is a closed, unbounded operator from (a dense domain in) $\mathcal{H}_\psi \otimes \mathcal{H}_{\varphi_1}$ to $\mathcal{H}_\psi \otimes \mathcal{H}_{\varphi_2}$ and that W_1 and W_2 are unitaries on $\mathcal{H}_\psi \otimes \mathcal{H}_{\varphi_1}$ and $\mathcal{H}_\psi \otimes \mathcal{H}_{\varphi_2}$ respectively.

About the preclosedness of the map $\Lambda_{\varphi_1}(x) \mapsto \Lambda_{\varphi_2}(x^*)$.

Consider the case $\varphi_1 = \varphi_2 = \varphi$.

Take right bounded elements ξ, η.

$$\langle \pi'(\xi)^*\eta, \Lambda_\varphi(x^*)\rangle = \langle \eta, \pi'(\xi)\Lambda_\varphi(x^*)\rangle = \langle \eta, x^*\xi\rangle$$

$$\langle \pi'(\eta)^*\xi, \Lambda_\varphi(x)\rangle = \langle \xi, \pi'(\eta)\Lambda_\varphi(x)\rangle = \langle \xi, x\eta\rangle$$

and so

$$\langle \pi'(\xi)^*\eta, \Lambda_\varphi(x^*)\rangle = \langle \Lambda_\varphi(x), \pi'(\eta)^*\xi\rangle$$

The case with two different left Haar weights is treated with a 2×2 matrix trick.

Polar decompositions

We now consider the case where φ_1 and φ_2 are the same left Haar weight φ. We use T for the operator T_r in this case. And W for the left regular representation.

Notation

We use $K = IL^{\frac{1}{2}}$ and $T = J\nabla^{\frac{1}{2}}$ for the polar decompositions of the operators K on $\mathcal{H}_\psi$ and T on $\mathcal{H}_\varphi$.

Remark that the last one is generally written as $S = J\Delta^{\frac{1}{2}}$ but we have to use an other notation for obvious reasons. The following is then an immediate consequence of the formula $(K \otimes T)W = W^*(K \otimes T)$:

Proposition

- $(I \otimes J)W(I \otimes J) = W^*$
- $(L^{it} \otimes \nabla^{it})W(L^{-it} \otimes \nabla^{-it}) = W$ *for all* $t \in \mathbb{R}$.

Some density results

The following results should have been considered earlier. The two results are proven together.

Proposition

Let φ be any left Haar weight and W the associated regular representation. Then

$$\{(\iota \otimes \omega)W \mid \omega \in \mathcal{B}(\mathcal{H}_\varphi)_*\}$$

is σ-weakly dense in M

Proposition

The spaces

$$sp\{(\iota \otimes \omega)\Delta(x) \mid x \in M,\ \omega \in M_*\} \tag{3}$$

$$sp\{(\omega \otimes \iota)\Delta(x) \mid x \in M,\ \omega \in M_*\} \tag{4}$$

are σ-weakly dense in M.

The scaling and modular automorphisms

We have the modular automorphisms on M given by $\sigma_t : x \mapsto \nabla^{it} x \nabla^{-it}$. But we also have the scaling group.

Definition

We define $R : M \to M$ by $R(x) = Ix^*I$ and $\tau_t : M \to M$ by $\tau_t(x) = L^{it} x L^{-it}$ for all $t \in \mathbb{R}$.

Definition

The polar decomposition of the antipode is $S = R\tau_{-\frac{i}{2}}$ where $\tau_{-\frac{i}{2}}$ is the analytic extension of (τ_t) to the point $-\frac{i}{2}$.

One may have to redefine S by this formula. Still, we have $(\iota \otimes \omega)W \in \mathcal{D}(S)$ and

$$S((\iota \otimes \omega)W) = (\iota \otimes \omega)W^*.$$

The first important consequences

If we combine these results with the earlier formulas

- $\Delta(x) = W^*(1 \otimes x)W$ and
- $(\Delta \otimes \iota)W = W_{13}W_{23}$,

we find the following important formulas:

Proposition

For all $x \in M$ and $t \in \mathbb{R}$ we have:

- $\Delta(\sigma_t^\varphi(x)) = (\tau_t \otimes \sigma_t^\varphi)\Delta(x)$,
- $\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x)$,
- $\Delta(R(x)) = (R \otimes R)\Delta'(x)$ *where* Δ' *is obtained from* Δ *by applying the flip.*

Uniqueness of the Haar weights

Consider two left invariant weights φ_1 and φ_2 with associated data. Recall the operator T_r, defined as the closure of the map $\Lambda_{\varphi_1}(x) \mapsto \Lambda_{\varphi_2}(x^*)$ where $x \in \mathfrak{N}_{\varphi_1} \cap \mathfrak{N}^*_{\varphi_2}$. Recall that

$$(K \otimes T_r)W_1 = W_2^*(K \otimes T_r).$$

Proposition

Let $T_r = J_r \nabla_r^{\frac{1}{2}}$ denote the polar decomposition of T_r. Let $u_t = \nabla_1^{it} \nabla_r^{-it}$. Then

$$(1 \otimes u_t)W_1(1 \otimes u_t^*) = W_1$$

The proof follows from the two formulas

- $(L^{it} \otimes \nabla_1^{it})W_1(L^{-it} \otimes \nabla_1^{-it}) = W_1$
- $(L^{it} \otimes \nabla_r^{it})W_1(L^{-it} \otimes \nabla_r^{-it}) = W_1$

Uniqueness of the Haar weights

Proposition

If $x \in M$ and $\Delta(x) = 1 \otimes x$ then x is a scalar multiple of 1.

If a right Haar weight ψ would be bounded, one could obtain $\psi(x)1 = \psi(1)x$ and the result would follow. The idea also works in general, but one has to be more careful.

Theorem

The Haar weights on a locally compact quantum group are unique (up to a scalar).

As we found $(1 \otimes u_t)W_1(1 \otimes u_t^*) = W_1$, we get $\Delta(u_t) = 1 \otimes u_t$ for all t. Hence, these unitaries are multiples of 1 and this implies that φ_2 is a scalar multiple of φ_1.

Formulas involving the automorphism groups

Denote by (σ_t) and (σ'_t) the modular automorphisms of the left and the right Haar weight. Denote by (τ_t) the scaling automorphisms.

Proposition

All these automorphisms mutually commute. Moreover $R(\tau_t(x)) = \tau_t(R(x))$ *and* $R(\sigma_t(x)) = \sigma'_{-t}(R(x))$ *for all* x.

Proposition

For all $x \in M$ *it holds:*

- $\Delta(\sigma_t(x)) = (\tau_t \otimes \sigma_t)\Delta(x)$,
- $\Delta(\sigma'_t(x)) = (\sigma'_t \otimes \tau_{-t})\Delta(x)$,
- $\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x)$,
- $\Delta(\tau_t(x)) = (\sigma_t \otimes \sigma'_{-t})\Delta(x)$.

Relative invariance of the Haar weights

If we combine these results with the uniqueness of the Haar weights, we find:

Proposition

There exists a strictly positive number ν *so that*

- $\varphi \circ \tau_t = \nu^{-t}\varphi$,
- $\psi \circ \tau_t = \nu^{-t}\psi$,
- $\psi \circ \sigma_t = \nu^{-t}\psi$,
- $\varphi \circ \sigma'_t = \nu^{t}\varphi$,

for all $t \in \mathbb{R}$.

The modular element

We finish with the modular element, relating the left with the right Haar weight.

Proposition

There exists a unique, non-singular, positive self-adjoint operator δ, affiliated with M such that $\psi = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. This operator satisfies $\sigma_t(\delta) = \nu^t \delta$ and $\sigma'_t(\delta) = \nu^{-t}\delta$. It is invariant under the automorphisms (τ_t) and $R(\delta) = \delta^{-1}$. We also have the relation $\sigma'_t(x) = \delta^{it}\sigma_t(x)\delta^{-it}$.

For the proof one uses that ψ is relatively invariant under the modular automorphisms of φ. One also has the formula $\Delta(\delta) = \delta \otimes \delta$, but that seems more difficult to obtain.

Conclusions

- In the first lecture, we passed form C^*-algebras to von Neumann algebras.
- In the second lecture, we studied the regular representations and the antipode.
- In this lecture we used the polar decomposition of the operator K, implementing the antipode.
- And relative modular theory to obtain uniqueness of the Haar weights.
- Then the rest of the theory with the main formulas follows quickly.
- The next lecture is devoted to the study of the dual.

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