Locally compact quantum groups

2. The antipode

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Outline of lecture series

Outline of the series:

- The Haar weights on a locally compact quantum group
- The antipode of a locally compact quantum group
- The main theory
- Duality
- Miscellaneous topics

Outline of the second lecture

Outline of this second lecture:

- Introduction
- The left and the right regular representations
- The antipode and its implementation
- Conclusions

Introduction

We will start with a locally compact quantum group (M, Δ) in the von Neumann algebraic framework. Recall that M is a von Neumann algebra and Δ a unital and normal *-homomorphism $M \to M \otimes M$ satisfying coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Also the existence of a left and of a right Haar weight is assumed. A left Haar weight is a faithful, normal semi-finite weight satisfying left invariance

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1.$$

Similarly for a right Haar weight.

The first step in the development of the theory is the construction of the left and of the right regular representations of the locally compact quantum group (M, Δ).

The next step is the construction of the antipode. This is a problem.

Introduction

In Hopf algebra theory, the antipode S is characterized with the formulas

$$m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \tag{1}$$
$$m(\iota \otimes S)\Delta(a) = \varepsilon(a)1 \tag{2}$$

where m stands for multiplication and where ε is the counit. The counit is characterized by

$$(\varepsilon \otimes \iota)\Delta(a) = a \tag{3}$$
$$(\iota \otimes \varepsilon)\Delta(a) = a. \tag{4}$$

In the operator algebra approach, this causes two difficulties:

- The definition of $S \otimes \iota$ and $S \otimes \iota$ on completed tensor products.
- The definition of the multiplication map on completed tensor products.

Introduction

In the traditional operator algebraic approaches, the antipode is characterized in connection with the left Haar weight:

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)).$$

The main difficulty with this formula is that it makes the definition of the antipode dependent on the choice of the left Haar weight.

The main feature of the approach I present here is the introduction of an antipode, without reference to the Haar weights. The Haar weights are used to show that the antipode is well-defined and that its domain is dense.

It is based on the Hopf algebra result:

$$a \otimes 1 = \sum \Delta(a_{(1)})(1 \otimes S(a_{(2)})) \tag{5}$$

$$S(a) \otimes 1 = \sum (1 \otimes a_{(1)})\Delta(S(a_{(2)})) \tag{6}$$

The right regular representation

Let (M, Δ) be a locally compact quantum group (in the von Neumann algebra setting). Let φ be a left Haar weight and let ψ be a right Haar weight.

Let $\mathcal{H}$ be the underlying Hilbert space of the von Neumann algebra M. Consider the GNS representation w.r.t. ψ. We will let act M directly on $\mathcal{H}_\psi$.

Proposition

There is an isometric operator V on $\mathcal{H}_\psi \otimes \mathcal{H}$ given (formally) by

$$V(\Lambda_\psi(x) \otimes \xi) = \sum \Lambda(x_{(1)}) \otimes x_{(2)}\xi.$$

This operator satisfies

- $V(x \otimes 1) = \Delta(x)V$
- $(\iota \otimes \Delta)V = V_{12}V_{13}$

The left regular representation

Proposition

There is a co-isometric operator W *on* $\mathcal{H} \otimes \mathcal{H}_\varphi$ *given (formally) by*

$$W^*(\xi \otimes \Lambda_\varphi(x)) = \sum x_{(1)}\xi \otimes \Lambda_\varphi(x_{(2)}).$$

This operator satisfies

- $(1 \otimes x)W = W\Delta(x)$
- $(\Delta \otimes \iota)W = W_{13}W_{23}$

Later we will show that V and W are actually unitary operators.

Also the left and right Haar weights will be shown to be unique.

Then W and V are called the left and the right regular representations of the locally compact quantum group (M, Δ).

The antipode - the von Neumann algebra level

We now introduce the antipode on the von Neumann algebra.

Definition

For an element $x \in M$ we say that $x \in \mathcal{D}_0$ if there is an element $x_1 \in M$ satisfying the following condition:

For all $\varepsilon > 0$ and vectors $\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n$ in $\mathcal{H}$, there exist elements $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m$ in M such that

$$\|x\xi_k \otimes \eta_k - \sum \Delta(p_j)(\xi_k \otimes q_j^* \eta_k)\| < \varepsilon \qquad (7)$$

$$\|x_1\xi_k \otimes \eta_k - \sum \Delta(q_j)(\xi_k \otimes p_j^* \eta_k)\| < \varepsilon. \qquad (8)$$

We will define $S_0 : \mathcal{D}_0 \to M$ by $S_0(x)^* = x_1$.

The antipode - the von Neumann algebra level

We will have the following properties.

Proposition

- *If* $x \in \mathcal{D}_0$, *then* $S_0(x)^* \in \mathcal{D}_0$ *and* $S_0(S_0(x)^*)^* = x$.
- *If* $x, y \in \mathcal{D}_0$, *then* $xy \in \mathcal{D}_0$ *and* $S_0(xy) = S_0(y)S_0(x)$.
- *The map* $x \to S_0(x)^*$ *is closed for the strong operator topology on* M.

What are the problems and what are the solutions?

- We need $x_1 = 0$ if $x = 0$ to have S_0 well defined.
- We need the density of $\mathcal{D}_0$.

The right Haar weight is used to solve the first problem. The left Haar weight is used to prove the density.

The antipode - the Hilbert space level

We now define the map $x \mapsto S(x)^*$ on the Hilbert space level.

Definition

Let $\xi \in \mathcal{H}_\psi$. We say that $\xi \in \mathcal{D}(K)$ if there is a vector $\xi_1 \in \mathcal{H}_\psi$ satisfying the following condition:

For all $\varepsilon > 0$ and vectors $\eta_1, \eta_2, \ldots, \eta_n$ in $\mathcal{H}_\psi$, there exist elements $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m$ in $\mathcal{N}_\psi$ such that

$$\|\xi \otimes \eta_k - V(\sum \Lambda_\psi(p_j) \otimes q_j^* \eta_k)\| < \varepsilon \tag{9}$$

$$\|\xi_1 \otimes \eta_k - V(\sum \Lambda_\psi(q_j) \otimes p_j^* \eta_k)\| < \varepsilon. \tag{10}$$

We will show that $\xi_1 = 0$ if $\xi = 0$. Then we can define $K(\xi) = \xi_1$.

The antipode - implementation

Here is the relation between the two operators.

Proposition

- *If* $\xi \in \mathcal{D}(K)$*, then* $K\xi \in \mathcal{D}(K)$ *and* $K(K\xi)) = \xi$.
- K *is a closed operator.*

Proposition

Let $x \in \mathcal{D}_0$ *and assume that* x_1 *is as before. If* $\xi \in \mathcal{D}(K)$ *then* $x\xi \in \mathcal{D}(K)$ *and* $Kx\xi = x_1 K\xi$.

If we can show that the domain of K is dense, it follows that S_0 will be well-defined. Also $Kx\xi = S_0(x)^* K\xi$ when $x \in \mathcal{D}_0$.

The proof of the density of $\mathcal{D}_0$ and of $\mathcal{D}(K)$ is similar and uses the left Haar weight.

The operator K is well-defined

Proposition

The operator K is well-defined.

Proof.

Assume that

$$\sum \Lambda_\psi(p_j) \otimes q_j^* \eta \to V^*(\xi \otimes \eta) \quad \text{and} \quad \sum \Lambda_\psi(q_j) \otimes p_j^* \eta \to 0.$$

Take the scalar product of the first expression with a vector $\pi'(\zeta)\zeta' \otimes \eta'$ where ζ and η' are right bounded. Then

$$\sum \langle \Lambda_\psi(p_j) \otimes q_j^* \eta, \pi'(\zeta)^* \zeta' \otimes \eta' \rangle = \sum \langle \zeta \otimes \pi'(\eta')^* \eta, p_j^* \zeta' \otimes \Lambda_\psi(q_j) \rangle$$

This proves that $\langle V^*(\xi \otimes \eta), \pi'(\zeta)^* \zeta' \otimes \eta' \rangle = 0$ and hence $V^*(\xi \otimes \eta) = 0$. □

The operator K is densely defined

If $c \in \mathcal{N}_\psi$ and $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$ one can show that

$$(\iota \otimes \omega(c\,\cdot\,))W \in \mathcal{N}_\psi$$

where W is the left regular representation.

Proposition

Let $c, d \in \mathcal{N}_\psi$ and $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$ and define

$$\xi = \Lambda_\psi((\iota \otimes \omega(c \,\cdot\, d^*))W).$$

Then $\xi \in \mathcal{D}(K)$ and $K\xi = \Lambda_\psi((\iota \otimes \overline{\omega}(d \,\cdot\, c^*))W)$.

Proof.

We take $\omega = \langle \cdot\, \xi', \eta' \rangle$, an orthonormal basis (ξ_j) and

$$p_j = (\iota \otimes \langle \cdot\, \xi_j, c^*\eta' \rangle)W \qquad \text{and} \qquad q_j = (\iota \otimes \langle \cdot\, \xi_j, d^*\xi' \rangle)W.$$

Then $p_j, q_j \in \mathcal{N}_\psi$ and they will give the required elements. □

The operator K is densely defined

Define

$$\mathcal{K} = \overline{\mathrm{sp}}\{\Lambda_\psi((\iota \otimes \omega(c\,\cdot\,))W) \mid c \in \mathcal{N}_\psi,\ \omega \in \mathcal{B}(\mathcal{H}_\varphi)_*\}.$$

One can show that also

$$\mathcal{K} = \overline{\mathrm{sp}}\{\Lambda_\psi((\iota \otimes \omega)\Delta(x)) \mid x \in \mathcal{N}_\psi,\ \omega \in M_*\}.$$

Furthermore, it is possible to show that

$$\mathcal{K} \otimes \mathcal{H}_\varphi \subseteq V(\mathcal{K} \otimes \mathcal{H}_\varphi) \quad \text{and} \quad V(\mathcal{H}_\psi \otimes \mathcal{H}_\varphi) \subseteq \mathcal{K} \otimes \mathcal{H}_\varphi.$$

All these properties together give the following results.

Proposition

- *V is unitary.*
- *$\mathcal{D}(K)$ is dense in $\mathcal{H}_\psi$.*

By symmetry, also W will be unitary.

$\mathcal{D}_0$ is dense en S_0 is well-defined

Because K is densely defined, S_0 is well-defined.

Proposition

Let $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$ *and* $x = (\iota \otimes \omega)W$ *and* $x_1 = (\iota \otimes \overline{\omega})W$*, then* $x \in \mathcal{D}_0$ *and* $x_1 = S_0(x)^*$*.*

Proof.

Assume that $\omega = \langle \cdot\, \xi, \eta \rangle$. Take an orthonormal basis (ξ_j) in $\mathcal{H}_\varphi$. Define

$$p_j = (\iota \otimes \langle \cdot\, \xi_j, \eta \rangle)W \qquad \text{and} \qquad q_j = (\iota \otimes \langle \cdot\, \xi_j, \xi \rangle)W.$$

Using the formula $(\Delta \otimes \iota)W = W_{13}W_{23}$, one can show that these are the elements we need. □

Conclusions

- In the first lecture we discussed the passage from a C*-algebraic locally compact quantum group to a von Neumann algebraic one.
- In this lecture, we introduced the left and the right regular representations W and V associated with a left and a right Haar weight φ and ψ.
- We defined the antipode without reference to the Haar weights.
- We used the left and the right Haar weights to show that (1) the antipode is well-defined and (2) it is densely defined.
- This approach differs from other approaches where the Haar weights are used to define the antipode. This causes a problem because at the beginning of the development, it is not shown yet that the Haar weights are unique.

References

- G. Pedersen: *C^*-algebras and their automorphism groups* (1979).
- M. Takesaki: *Theory of Operator Algebras II* (2001).
- J. Kustermans & S. Vaes: *Locally compact quantum groups*. Ann. Sci. Éc. Norm. Sup. (2000).
- J. Kustermans & S. Vaes: *Locally compact quantum groups in the von Neumann algebra setting*. Math. Scand. (2003).
- A. Van Daele: *Locally compact quantum groups: The von Neumann algebra versus the C^*-algebra approach*. Preprint KU Leuven (2005). Bulletin of Kerala Mathematics Association (2006).
- A. Van Daele: *Locally compact quantum groups. A von Neumann algebra approach*. Preprint University of Leuven (2006). Arxiv: math/0602212v1 [math.OA].