Shoaling of nonlinear water waves

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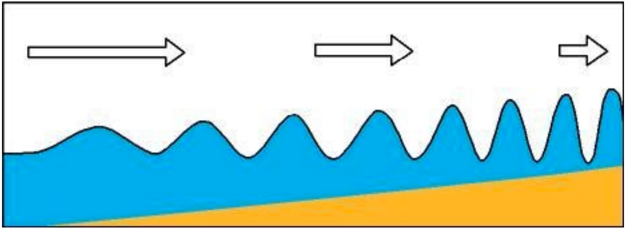
Shoaling waves



Classical Green's law for linear shallow water waves:
Speed $c = (gh)^{1/2}$,
Wavenumber $k = \omega/c$, where ω (wave frequency) is fixed ,
Amplitude $a \propto h^{-1/4}$, due to conservation of wave energy flux, $\propto ca^2$.

Shoaling waves: Undular bore

Undular bore of the 2004 Indian Ocean tsunami reaching the island of Koh Jum, Thailand.

Korteweg-de Vries equation: 1

Weakly nonlinear unidirectional long waves over variable topography are governed by the **variable-coefficient Korteweg-de Vries equation**,

$$A_t + cA_x + \frac{c_x}{2}A + \frac{3c}{2h}AA_x + \frac{ch^2}{6}A_{xxx} = 0\,. \qquad (1)$$

This is in non-dimensional form, based on a length scale h_0 (depth) and a time scale $(h_0/g)^{1/2}$. $A(x,t)$ is the free surface elevation above the undisturbed non-dimensional depth $h(x)$, and $c(x) = \sqrt{h(x)}$ is the non-dimensional linear long wave phase speed. The first two terms in (1) are the dominant terms, and by themselves describe the propagation of a linear long wave with speed c. The remaining terms represent, respectively, the effect of varying depth, weakly nonlinear effects and weak dispersion. It can be derived as an asymptotic long-wave small-amplitude reduction of the full Euler system using the usual balance in which $\partial/\partial t \sim \partial/\partial x \sim \epsilon \ll 1, A \sim \epsilon^2$ with weak inhomogeneity so that c_x/c **scales as** ϵ^3 .

Korteweg-de Vries equation: 2

We can cast (1) into the asymptotically equivalent form

$$A_\tau + \frac{h_\tau}{4h}A + \frac{3}{2h}AA_X + \frac{h}{6}A_{XXX} = 0\,, \tag{2}$$

$$\text{where} \quad \tau = \int_0^x \frac{dx'}{c(x')}, \qquad X = \tau - t\,. \tag{3}$$

Here $h = h(\tau)$ explicitly depends on the variable τ which describes evolution along the path of the wave. Formally we write $A(x,t) = \tilde{A}(X,\tau)$ and $h(x) = \tilde{h}(\tau)$ but then omit the "tilde" in (2). The balance of terms in (2) is ensured by $\partial/\partial\tau \sim \epsilon^3$, $\partial/\partial X \sim \epsilon$, $A \sim \epsilon^2$. Thus, unlike in the original variable-coefficient KdV equation (1), where both independent variables x and t vary on the same scale $\sim 1/\epsilon$, in (2) the "time" τ is a slow variable relative to the "spatial" coordinate X. We stress that equations (1) and (2) are **asymptotically equivalent**. They differ with respect to terms of $O(\epsilon^7)$, which is the same as the error term in both equations.

Korteweg-de Vries equation: initial condition

We shall suppose that

$$h(x) = 1 \text{ for } x < 0\,, \quad h(x) = h_1 < 1 \text{ for } x > x_1\,, \tag{4}$$

and varies monotonically in $0 \leq x \leq x_1$, where $x_1 \gg 1$. For times $t < 0$ an initial condition is imposed in $x < 0$, that is

$$A(x, t = 0) = A_0(x) \text{ for } x < x_0 < 0\,, \quad A(x, t = 0) = 0 \text{ for } x > x_0\,. \tag{5}$$

Thus initially we generate a solution of the constant coefficient KdV equation, and the aim is to see how this develops in $x > 0$. **The special case when A_0 is a constant for $x < x_0$ generates an undular bore, which then moves into shallower water**.

In terms of the "signalling" variables (3) the initial condition (5) becomes

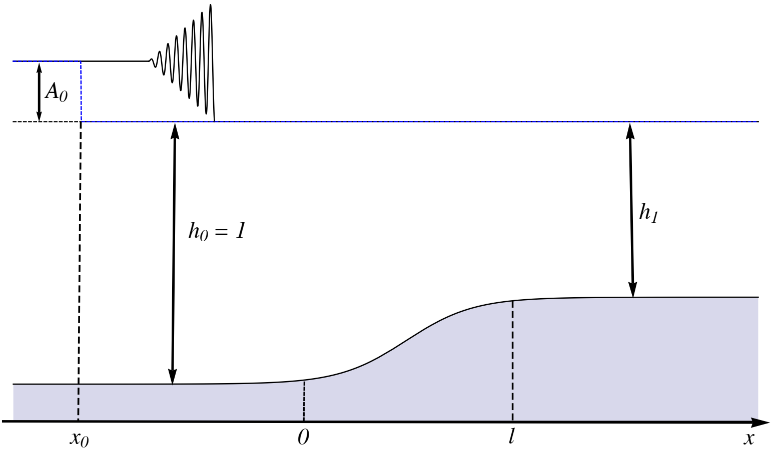
$$\tilde{A}(X(t = 0) = \tau, \tau) = A_0(x)\,, \tag{6}$$

where $x(\tau)$ is given by (3). However, because τ is a slow variable relative to X, $\tilde{A}(X, \tau) = \tilde{A}(X, 0) + O(\epsilon^3)$ so asymptotically,

$$A(X, \tau = 0) = A_0(X)\,, \tag{7}$$

where we have again omitted the "tilde" for A and used the fact that $A_0(x)$ is only non-zero in $x < 0$ where $\tau = x$, and so $X = x$ at $t = 0$.

Korteweg-de Vries equation: initial condition



Korteweg-de Vries equation: alternatives

The governing equation (2) can be cast into several exactly equivalent forms. The most common is the **variable-coefficient KdV equation**,

$$B = h^{1/4}A\,, \quad \text{so that} \quad B_\tau + \frac{3}{2h^{5/4}}BB_X + \frac{h}{6}B_{XXX} = 0\,. \qquad (8)$$

This form shows that equation (2) has two integrals of motion with the densities proportional to $B = h^{1/4}A$ **and** $B^2 = h^{1/2}A^2$. That is, conservation of "mass" and "momentum" (**wave action flux**). Or, recast (2) into a **perturbed KdV equation**,

$$u = \frac{3}{2h^2}A = \frac{3}{2h^{9/4}}B\,, \quad S = \frac{1}{6}\int_0^\tau h(\tau')d\tau' = \frac{1}{6}\int_0^x h(x')^{1/2}dx'\,. \qquad (9)$$

$$\text{so that} \quad u_S + 6uu_X + u_{XXX} = -\frac{9h_S}{4h}u\,. \qquad (10)$$

Yet another convenient form for (2) is obtained by putting

$$T = \frac{1}{6}\int_0^\tau \frac{d\tau'}{h^{5/4}(\tau')} = \frac{1}{6}\int_0^x \frac{dx'}{h^{7/4}(x')}\,, \qquad (11)$$

$$U = \frac{3B}{2} \quad U_T + 6UU_X + \beta(T)U_{XXX} = 0\,, \quad \beta(T) = h^{9/4}\,. \qquad (12)$$

Slowly varying solitary wave

A solitary wave propagating over slowly varying topography will deform **adiabatically** so that its amplitude varies as h^{-1} . To show this, use the conservation law for wave action flux. Thus, from (12),

$$\frac{d}{dT}\int_{-\infty}^{\infty} U^2 dX = 0\,. \tag{13}$$

The slowly varying solitary wave for (12) is

$$U \sim a\,\mathrm{sech}^2\{\gamma(X-\Phi(T))\}\,, \quad V = \Phi_T = 2a = 4\beta\gamma^2\,. \tag{14}$$

where the amplitude a is a slowly varying function of T. Substitution into (13) gives

$$\frac{a^2}{\gamma} = 4\beta^2\gamma^3 = \text{constant}\,, \tag{15}$$

and so $\gamma \sim \beta^{-2/3}$ and $a \sim \beta^{-1/3}$. Thus, since $\beta = h^{9/4}$, we have $a \sim h^{-3/4}$ and noting that $U \sim h^{1/4}A$, the result follows. Note that mass is not conserved by the deforming solitary wave, and this is compensated by the generation of a trailing shelf.

Undular bore: 1

When the bottom is flat, $\beta = 1$ in (12), assume that the initial condition is $U(X, T = 0) = H(-X)U_0$, $U_0 = 3A_0/2 > 0$. Then the decay of the initial discontinuity at $X = 0$ leads to the development of an undular bore, an **expanding slowly modulated periodic wavetrain**, asymptotically described by **Whitham modulation theory**. The local wave form of the undular bore is given by the cnoidal wave

$$U = a\{b(m) + \mathrm{cn}^2(\frac{q}{\beta^{1/2}}(X - X_0 - VT); m)\} + d \,, \tag{16}$$

$$\text{where} \quad b = \frac{1-m}{m} - \frac{E(m)}{mK(m)} \,, \quad a = 2mq^2 \,,$$

$$\text{and} \quad V = 6d + 2a\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\} \,. \tag{17}$$

Here $\mathrm{cn}(x; m)$ is the Jacobi elliptic function of modulus m $(0 < m < 1)$ and $K(m), E(m)$ are the elliptic integrals of the first and second kind respectively, a is the wave amplitude, d is the mean level, V is the wave speed, and X_0 is a constant defining the initial phase.

Undular bore: 2

Note that we have retained β in (16) and (18), in order to include the case when $\beta \neq 1$ on the shelf. The spatial period (wavelength) is

$$L = \frac{2K(m)\beta^{1/2}}{q}\,. \tag{18}$$

When $1 - m \ll 1$, $L \gg \beta^{1/2}/q$. This family of solutions contains three free parameters, which are chosen from the set $\{a, q, V, d, m\}$. **As $m \to 1$, $\mathbf{cn}(x; m) \to \mathbf{sech}(x)$ and then the cnoidal wave (16) becomes a solitary wave, riding on a background level d**. On the other hand, as $m \to 0$, $\mathrm{cn}(x; m) \to \cos x$ and so the cnoidal wave (16) collapses to a linear sinusoidal wave (in this limit $a \to 0$).

Whitham modulation theory assumes that the expression (16) describes a modulated wave in which the amplitude a, the mean level d, the speed V and the modulus m are all slowly varying functions of X and T. The outcome is a set of three nonlinear hyperbolic equations for three of the available free parameters, chosen from the set (a, q, V, d, m), or rather better, from an appropriate combination of them.

Undular bore: 3

The relevant asymptotic solution is then, using a similarity variable,

$$\frac{X}{T} = 2U_0 \left\{1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)}\right\} \quad \text{for} \quad -6U_0 < \frac{X}{T} < 4U_0 \,, \tag{19}$$

$$a = 2U_0 m \,, \quad d = U_0 \left\{m - 1 + \frac{2E(m)}{K(m)}\right\} \,, \quad q = {U_0}^{1/2} \,. \tag{20}$$

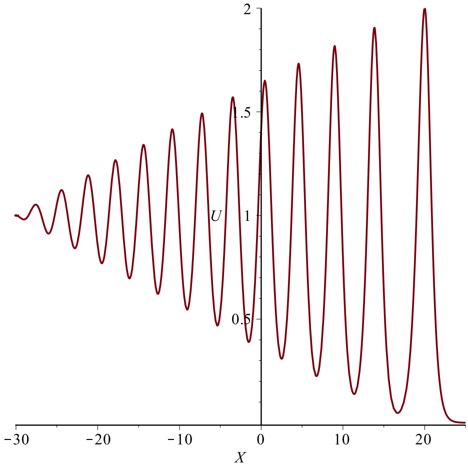
Note that this solution does not depend on the value of β, which affects only the wavelength of the underlying periodic wave (16). The wavenumber distribution in the undular bore is then given by

$$k = \frac{2\pi}{L} = \frac{\pi U_0^{1/2}}{\beta^{1/2} K(m)} \,. \tag{21}$$

Ahead, where $X/T > 4U_0$, $U = 0$, $m \to 1$, $a \to 2U_0$ and $d \to 0$; **the leading wave is a solitary wave of amplitude $2U_0$ relative to a mean level of 0**. Behind, where $X/T < -6U_0$, $U = U_0$ and at this end $m \to 0$, $a \to 0$, and $d \to U_0$; the wavetrain is now sinusoidal with the wavelength $L = \pi(\beta/U_0)^{1/2}$.

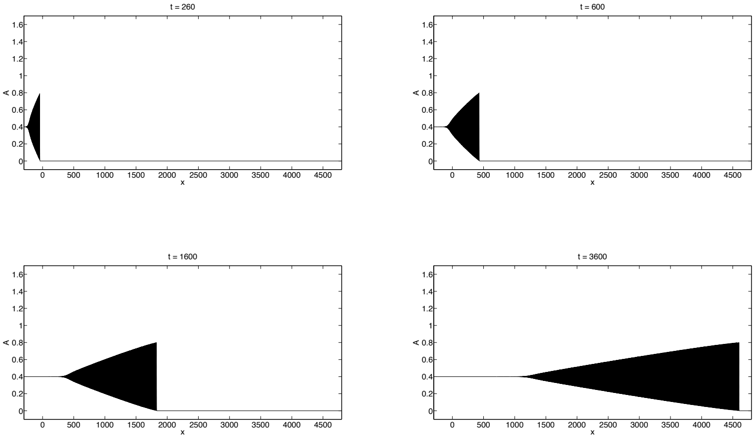
Undular bore: 4

Plot of an undular bore (19) for $U_0 = 1$ at $T = 5$.



Undular bore: evolution on a flat bottom

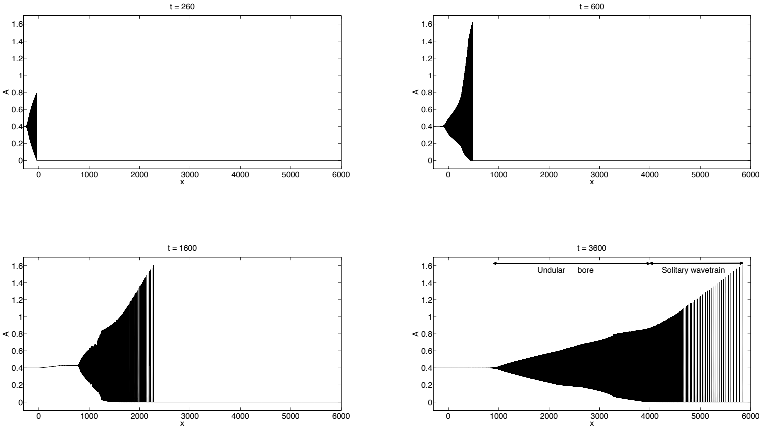
$h(x) = 1$, $t = 260, 600, 1600, 3600$ (left to right, top to bottom)



Undular bore: evolution on a slope, numerical

$$h(x) = 1\,(x < 0)\,, \quad 1 - \alpha x\,(0 < x < 400)\,, \quad 0.64\,(x > 400)\,, \quad \alpha = 0.0009\,.$$

$t = 260, 600, 1600, 3600$ (left to right, top to bottom)



Undular bore: evolution on a slope, theory

$$U_T + 6UU_X + \beta(T)U_{XXX} = 0 \quad \text{where} \quad \beta(T) = h^{9/4}\,.$$

where $\beta(T) = 1$ for $T < T_0$ and varies monotonically to β_1 for $T > T_1$. If the slope is sufficiently gentle, one might expect that the undular bore undergoes an adiabatic change **retaining its structure as a slowly modulated nonlinear periodic wavetrain with a soliton at the leading edge and the linear vanishing amplitude wavepacket at the trailing edge**. Let the structure be confined $X_a(T) < X < X_b(T)$ so that $[U] = U(X_b) - U(X_a)$. Then $[U]_T = \frac{\partial}{\partial T}\int_{X_a}^{X_b} U_X\,dX = 0$ provided $U_X = U_{XXX} = 0$ at $X = X_{a,b}(T)$. Thus, since the wavetrain advances into the undisturbed depth region, $U(X_b) = 0$, one has

$$[U] = -U_0 \qquad \text{for all} \quad T > 0\,. \tag{22}$$

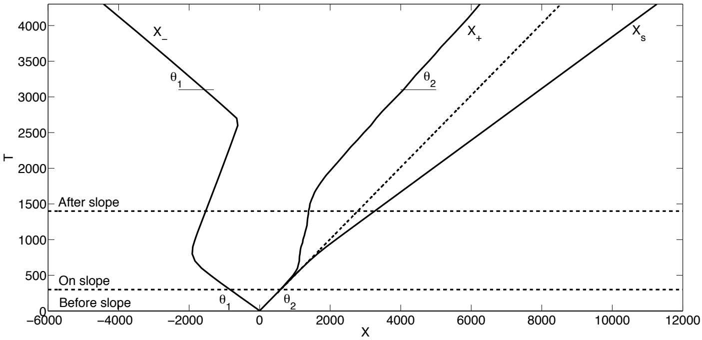
This result is not affected by $\beta(T)$ and the actual form of the structure. Thus if a single undular bore emerges onto the shelf with $\beta = \beta_1$ then (22) implies that relevant modulation solution for $T > T_1$ will have the same form (19) but with X generally replaced by $X - X_0(m)$, and so the modulation solution cannot remain a centred fan but must become a more general, simple-wave solution of the Whitham equations.

Undular bore: evolution on a slope, theory

But if the leading solitary wave in the undular bore evolves over the slope as an isolated solitary wave (the "weak interaction scenario", an assumption to be confirmed), then its amplitude must vary adiabatically, $a = 2U_0\beta^{-1/3}$ to conserve the action flux, so that for $T > T_1$ the **leading solitary wave amplitude is $2U_0\beta_1^{-1/3} > 2U_0$, which is clearly inconsistent with the scenario based on the assumption of a single undular bore emerging onto the shelf**. To resolve the above inconsistency an additional solitary wavetrain at the front of the undular bore is needed to provide the gradual increase of the amplitude from $2U_0$ at the undular bore leading edge to the value $2U_0\beta_1^{-1/3}$.

Thus, the propagation of an undular bore over a broad region of slowly decreasing depth leads to a **non-adiabatic effect, the generation of a solitary wavetrain in front of the bore**. The adiabatic deformation of the bore itself is twofold: (i) the change of the characteristic scale of the oscillations in the bore due to the change of the dispersion coefficient β in (12); (ii) the occurrence of the additional slow "modulation phase shift" $X_0(m)$ throughout the bore so that the relevant modulation solution generally represents a non-centred simple wave of the Whitham equations.

Undular bore: evolution on a slope, theory



X-T plane of the evolution of an undular bore according to the KdV equation (12). The undular bore and the solitary wavetrain are confined to $[X_-(T), X_+(T)]$ and $[X_+(T), X_s(T)]$ respectively. The dashed line shows an extrapolation of the leading edge $X_+(T)$ of the initial undular bore so one can see the spatial shift of $X_+(T)$ due to the interaction with the slope.

Undular bore: evolution on a slope, theory

This plot reveals another feature of the undular bore propagation on a slope, related to the dynamics of the trailing edge $X_-(T)$, that is **non-monotonic behaviour of the edge $X_-(T)$** in the interval $300 \lesssim T \lesssim 3000$, which apparently is related to the occurrence of the earlier mentioned spatial shift $X_0(m)$ in the modulation solution (20) leading to the "de-centring" of the expansion fan. This results, for large T, in the stationary shift for the curve $X_-(T)$ relative to its initial behaviour. The multiphase behaviour continues for some time after the bore emerges onto the shelf, and at sufficiently large times, $T \gtrsim 3000$ the single-phase slowly modulated wave behaviour throughout the whole wavetrain is restored. This transient multiphase dynamics near the trailing edge does not affect the front, "solitary-wave", part of the bore which retains a regular single-phase structure at all times.

Three main conclusions:

(i) the generation of the solitary wavetrain ahead of the bore;
(ii) the undular bore edge speeds, $X'_\pm(T)$, asymptotically restore their original values $X'_-(T) = \cot(\theta_1)$ and $X'_+ = \cot(\theta_2)$ on the shelf;
(iii) the spatial shifts in the positions of the transformed undular bore edges $X_\pm(T)$ relative to those that would have taken place in the original bore in the absence of the variable topography.

Solitary wavetrain: 1

The slowly varying solitary wave for (12)

$$U \sim a\,\mathrm{sech}^2(\gamma\Theta)\,, \quad \Theta_T = -\kappa V\,, \quad \Theta_X = \kappa\,, \tag{23}$$

$$\text{where} \quad V = 2a = 4\beta\gamma^2\kappa^2\,, \tag{24}$$

which is an extension of (14) to allow for both X, T variation. The modulation equations for the amplitude a and the soliton wavetrain wavenumber κ are

$$\left\{\frac{a^2}{\kappa\gamma}\right\}_T + V\left\{\frac{a^2}{\kappa\gamma}\right\}_X = 0\,, \tag{25}$$

$$\kappa_T + (V\kappa)_X = 0\,. \tag{26}$$

This system (25), (26) can be obtained as a reduction of the Whitham equations for a modulated cnoidal wave, in the limit when the modulus $m \to 1$, or can be obtained directly, using averaging of the KdV "mass" and "momentum" conservation laws.

Solitary wavetrain: 2

Using the relations (23, 24), equations (25), (26) can be written in the form

$$\mathcal{A}_\sigma + 2\mathcal{A}\mathcal{A}_X = 0\,, \qquad \mathcal{A} = \left\{\frac{a^2}{\sqrt{2}\kappa\gamma}\right\}^{2/3} = a\beta^{1/3}\,, \tag{27}$$

$$\kappa_\sigma + (2\mathcal{A}\kappa)_X = 0\,, \tag{28}$$

$$\text{where} \quad \sigma = \int_0^T \beta(T')^{-1/3} dT'\,. \tag{29}$$

Remarkably, the system (27, 28) has the same form as system (25, 26) in the case when $\beta = \beta_0 = constant$, that is for the constant-coefficient KdV equation. When there is no X-variation, $\mathcal{A}, \kappa$ are constants, and the result (15) is recovered. The general solution of (27, 28) is given by,

$$\mathcal{A} = \text{constant}\,, \quad \text{on} \quad \frac{dX}{d\sigma} = 2\mathcal{A}\,, \tag{30}$$

$$\text{and} \quad \frac{d\kappa}{d\sigma} = -2\mathcal{A}_X\kappa = \frac{\mathcal{A}_\sigma}{\mathcal{A}}\kappa\,. \tag{31}$$

Note that the system (30), (31) has only one multiple characteristic family and all the characteristics are straight lines in the X-σ plane.

Solitary wavetrain: 3

The trailing edge of the solitary wavetrain is $X = X_+(T)$ where $a = 2U_0$, and initially $X_+'(0) = 4U_0$. However, $X = X_+(T)$ is not associated with the trajectory of a particular solitary wave, as the solitary waves must be allowed to cross this boundary to enable the formation of the advancing modulated solitary wavetrain. Hence

$$0 < X_+'(T) < 4U_0 \quad \text{for} \quad T_0 < T < T_1 \,. \tag{32}$$

The boundary condition for (27) is,

$$\mathcal{A} = 2U_0\beta^{1/3} \quad \text{on} \quad X = \bar{X}(\sigma)\,, \tag{33}$$

where $\bar{X}(\sigma) = X_+(T(\sigma))$, where $T(\sigma)$ is the inverse of $\sigma = \sigma(T)$ (29). The solution for $\mathcal{A}$ is

$$\mathcal{A} = \mathcal{A}_0(\sigma_0) = 2U_0\beta^{1/3}(T(\sigma_0))\,, \quad X - \bar{X}(\sigma_0) = 2\mathcal{A}_0 \cdot (\sigma - \sigma_0)\,, \tag{34}$$

$\sigma_0 \in [0, \sigma_1]$ being a parameter on the curve $X = \bar{X}(\sigma)$. Elimination of the parameter σ_0 from (34) yields $\mathcal{A}$ as a function of X, σ. The solution (34) is defined for $\bar{X}(\sigma) < X < X_s$, where $X_s = 4U_0\sigma$ is the trajectory of the leading wave in the solitary wavetrain, having the amplitude $a = 2U_0\beta^{-1/3}$, that is $\mathcal{A} = 2U_0$. For $X > 4U_0\sigma$ we have $\mathcal{A} = 0$.

Solitary wavetrain: 4

Calculating the derivative $\mathcal{A}_X$ we obtain:

$$\mathcal{A}_X = \frac{\mathcal{A}_0'(\sigma_0)}{2\mathcal{A}_0'(\sigma_0)(\sigma - \sigma_0) + [\bar{X}'(\sigma_0) - 2\mathcal{A}_0(\sigma_0)]} \,. \tag{35}$$

Owing to (32) $[\bar{X}'(\sigma_0) - 2\mathcal{A}_0(\sigma_0)] < 0$, therefore to guarantee the existence of the obtained solution for all X, σ one must have $\mathcal{A}_0' < 0$. This, by (34), (29) implies $\beta'(T) < 0$. This this solution represents a rarefaction fan emanating from the curve $X = \bar{X}(\sigma_0)$. The condition $\beta'(T) < 0$ (decreasing depth) can be viewed as the condition of the formation of an expanding solitary wavetrain in front of the bore. Decreasing depth "promotes" the detachment of solitary waves at the leading edge of the undular bore. This also confirms our initial assumption that the leading solitary wave of the undular bore behaves as an isolated KdV solitary wave.

Solitary wavetrain: 5

Using $A(X, \sigma)$ defined by (34) the corresponding general solution for κ is then found from (31), that is

$$\kappa = \kappa_0 \left\{ 1 + \frac{2\mathcal{A}_0'(\sigma_0)(\sigma - \sigma_0)}{\bar{X}'(\sigma_0) - 2\mathcal{A}_0(\sigma_0)} \right\}^{-1} . \quad (36)$$

where κ_0 is the value of κ on the curve $X = X_+(T(\sigma_0)) = \bar{X}(\sigma_0)$ and $\sigma_0(X, \sigma)$ is defined by (34). Generally, to find the curve $X = X_+(T)$ for $T_0 < T < T_1$ one needs to solve the full perturbed modulation system. However, it is instructive to assume that

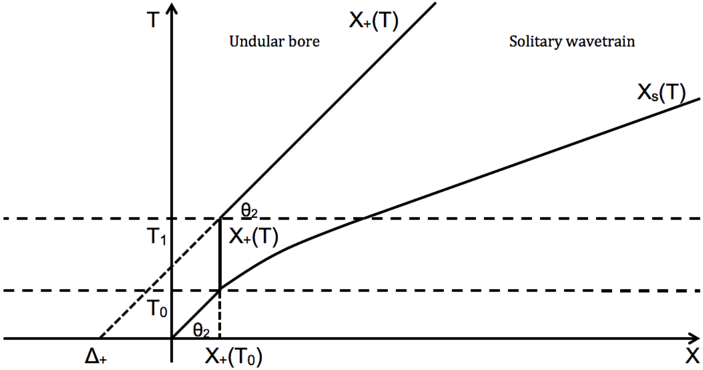
$$X_+'(T) \ll 4U_0 \quad \text{for} \quad T_0 < T < T_1 \, , \quad (37)$$

and thus $\bar{X}'(\sigma_0) \ll 2\mathcal{A}_0(\sigma_0)$. This is valid when $\beta(T)$ varyies sufficiently fast on a typical time scale of the solitary wavetrain modulations, but still is slow on the time scale of a single solitary wave. Thus we get that

$$X_+(T) \simeq X_+(T_0) = 4U_0 T_0 \quad \text{for} \quad T_0 < T < T_1 \, . \quad (38)$$

Solitary wavetrain: 5

Our numerical simulations confirm that (37, 38) can be safely used in the solution (36) for κ for a broad range of the slope values specified in terms of $\beta(T)$. The schematic behaviour of the boundaries $X_+(T)$ and $X_s(T)$ illustrating the asymptotic generation of the solitary wavetrain is shown in this plot.



Solitary wavetrain: 6

Thus, using (37) we have to leading order

$$\kappa \simeq \kappa_0 \left\{ 1 - \frac{\mathcal{A}_0'(\sigma_0)(\sigma - \sigma_0)}{\mathcal{A}_0(\sigma_0)} \right\}^{-1} = \kappa_0 \left\{ 1 - \frac{2}{3} \frac{\beta'(\sigma_0)}{\beta(\sigma_0)} (\sigma - \sigma_0) \right\}^{-1}, \tag{39}$$

where $\beta(\sigma_0) \equiv \beta(T(\sigma_0))$, $\beta'(\sigma_0) = \beta_T \beta^{1/3}(\sigma_0) < 0$ and so (39) exists for all X, σ. Then the **leading edge of the undular bore, that is also the trailing edge of the solitary wavetrain, emerging on the shelf is** $X_+(T) \simeq 4U_0T_0 + 4U_0(T - T_1)$ **for** $T > T_1$ and the phase shift $\Delta_+ = X_0(m = 1))$ can be estimated as $\Delta_+ \simeq -4U_0(T_1 - T_0)$.
Let $\sigma_1 = \sigma(T_1)$. On the shelf where $T > T_1, \sigma > \sigma_1$ we have $\beta = \beta_1$, $\sigma = \sigma_1 + (T - T_1)\beta_1^{-1/3}$. **The leading edge of the solitary wavetrain on the shelf is** $X_s = 4U_0\sigma = 4U_0(\sigma_1 + (T - T_1)\beta_1^{-1/3})$. For $T > T_1$ both boundaries $X_+(T)$ and $X_s(T)$ confining the expansion fan are characteristics and the total number of solitary waves in the train for $T > T_1$ is constant. Examining the undular bore structure, we get that

$$\kappa_0(\sigma_0) \simeq \frac{U_0^{1/2}}{4\beta^{1/2}(\sigma_0)} I, \quad \text{where} \quad I \approx 0.6569, \tag{40}$$

Solitary wavetrain: 7

Finally, from (24), (27), the wavenumber $\gamma(X,T) = \kappa^{-1}\beta^{-1/3}(\mathcal{A}/2)^{1/2}$ and so the slowly varying solitary wavetrain (23) is fully defined. As $T \to \infty$, $\sigma \sim T\beta_1^{-1/3}$, $X_+ \sim 4U_0T$, $X_s \sim 4U_0\beta_1^{-1/3}T$ and the asymptotic solution is,

$$4U_0T < X < 4U_0\beta_1^{-1/3}T : \qquad \mathcal{A} \sim \frac{X}{2\sigma}\,, \quad \text{or} \quad a \sim \frac{X}{2T}\,, \tag{41}$$

$$\kappa \sim \frac{g(X/(2T))}{\sigma} \sim \frac{g(\mathcal{A})}{\sigma}\,, \quad \text{or} \quad \kappa \sim \frac{G(a)}{T}\,. \tag{42}$$

Here $g(\mathcal{A}) = 3\kappa_0\beta(\sigma_0)/(2\beta'(\sigma_0))$, where $\sigma_0(\mathcal{A})$ is found from the solution $\mathcal{A} = 2U_0\beta^{1/3}(\sigma_0)$ (see (34)). The function $G(a) = g(a\beta_1^{1/3})$ is the distribution function over amplitude in the solitary wavetrain so that $G(a)da$ is the number of solitons with amplitudes in the interval $[a, a+da]$. Since the total number of solitons N in the train remains constant for $T > T_1$ it can be estimated by the formula

$$N \simeq \int_{2U_0}^{2U_0\beta_1^{-1/3}} G(a)\, da\,. \tag{43}$$

Solitary wavetrain: 8

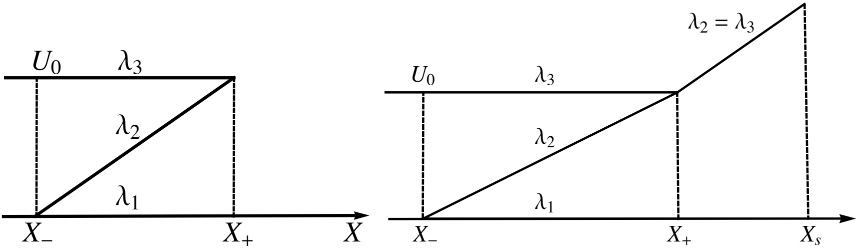
The outcome of the analysis can be described in a using a diagram showing the behaviour of the Riemann invariants $\lambda_j(X,T),\ j=1,2,3$ of the Whitham modulation equations in the combined modulation solution. These Riemann invariants $\lambda_1 \le \lambda_2 \le \lambda_3$ can be expressed in terms of any three independent modulation parameters f rom the set $\{a, q, V, d, m\}$.

$$
\begin{gathered}
a = 2(\lambda_2 - \lambda_1)\,, \quad q = \sqrt{\lambda_3 - \lambda_1}\,, \quad V = 2(\lambda_1 + \lambda_2 + \lambda_3)\,, \\
d = \lambda_1 + \lambda_2 - \lambda_3 + 4(\lambda_3 - \lambda_1)\frac{E(m)}{K(m)}\,.
\end{gathered} \tag{44}
$$

Then the Gurevich-Pitaevskii modulation solution for the undular bore is given by $\lambda_1 = 0$, $\lambda_3 = U_0$, while $\lambda_2 = U_0 m$ is defined by the same expression (19). In the solitary wavetrain we have $\lambda_2 = \lambda_3$ to leading order so the asymptotic solution (41) as $T \to \infty$ assumes the form $\lambda_3 \sim X/(4T)$ in the interval $4U_0T \le X \le 4U_0T\beta_1^{-1/3}$.

Solitary wavetrain: 9

Schematic behaviour of the Riemann invariants in the modulation solution. Left: regular undular bore (before the slope, $T < T_0$); Right: undular bore with an advancing soliton train confined to $[X_+, X_s]$ (after the slope, $T \gg T_1$)



Numerical results: 1

The developed theory has three main assumptions:
(i) the undular bore on a slope can be described by a slowly modulated periodic solution of the KdV equation;
(ii) the "weak interaction scenario" ensuring the behaviour of the leading solitary wave in the undular bore on a slope as an isolated KdV soliton;
(iii) the wave structure forming in front of the undular bore is indeed a solitary wavetrain.
We use the variable-coefficient KdV equation in the form (8),

$$B_\tau + \frac{3}{2h^{5/4}} B B_X + \frac{h}{6} B_{XXX} = 0\,,$$

with the dependence $h(\tau)$ corresponding to the depth profile,

$$1\,(\tau < 400)\,, 1 - \frac{\alpha(\tau - 400)^2}{2}\,(400 < \tau < 844.44)\,, 0.64\,(\tau > 844.44)\,,$$

where $\alpha = 0.0009$. Equation (8) is exactly equivalent to the equation (12) used for our asymptotic analysis. The initial condition is taken in the form of a smooth step,

$$B(X, 0) = \frac{1}{4}(1 - \tanh(X/10))\,. \tag{45}$$

Numerical results: 1

Assuming that the locally, undular bore can be described by the periodic solution (16) we use the numerical solutions to extract the parameters corresponding to the modulation Riemann invariants λ_j introduced in (44),or in terms of the basic wave parameters b_1, b_2 and b_3

$$\lambda_3 = \frac{b_2 + b_3}{2} \qquad \lambda_2 = \frac{b_1 + b_3}{2} \qquad \lambda_1 = \frac{b_1 + b_2}{2} . \tag{46}$$

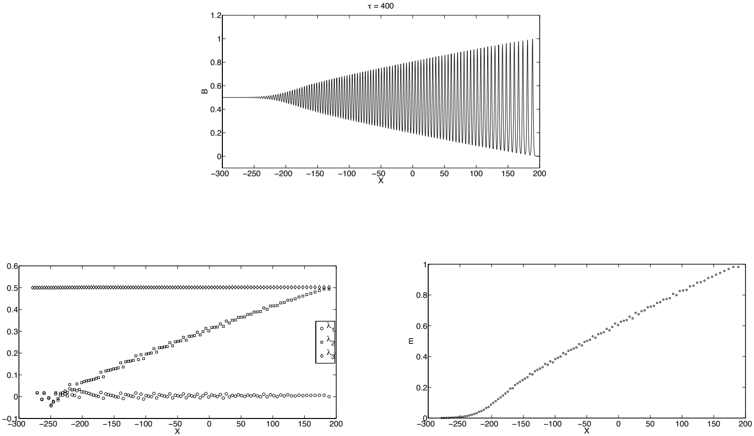
Here $b_2 \equiv B_{\min}$ and $b_3 \equiv B_{\max}$ are easily found from the numerical data. The third parameter b_1 can be obtained from the numerical values of the local spatial period (wavelength) L, which for the variable-coefficient KdV equation (8) is given by the formula

$$L = \frac{4h^{9/8}K(m)}{\sqrt{3(b_3 - b_1)}} \quad \text{where} \quad m = \frac{b_3 - b_2}{b_3 - b_1} . \tag{47}$$

We expect that the variables λ_j will demonstrate the qualitative behaviour shown above, which will be a confirmation of the validity of the modulation analysis.

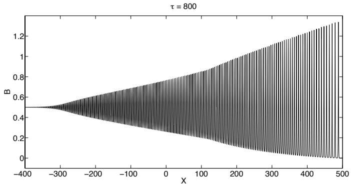
Numerical results: 3

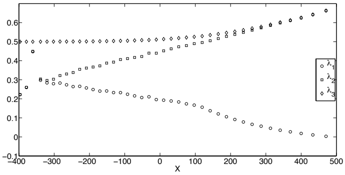
Initial undular bore (before slope), $\tau = 400$. In sequence, $B(X)$; the modulation Riemann variables $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the modulus $m = (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$ as a function of X.

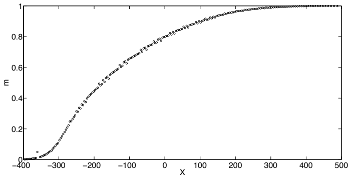


Numerical results: 4

Undular bore on the slope, $\tau = 800$. In sequence, $B(X)$; the modulation Riemann variables $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the modulus $m = (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$ as a function of X.

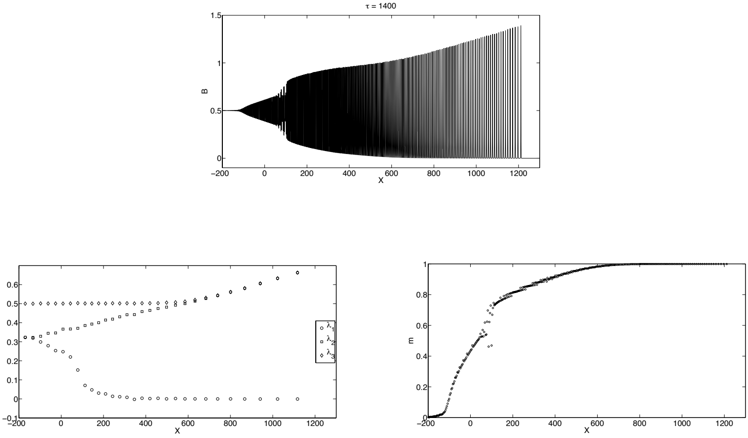






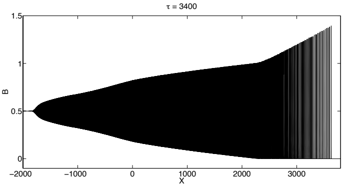
Numerical results: 5

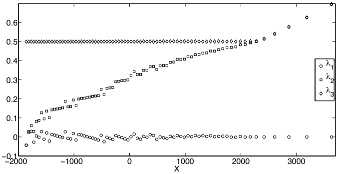
Undular bore after the slope, $\tau = 1400$. In sequence, $B(X)$; the modulation Riemann variables $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the modulus $m = (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$ as a function of X.

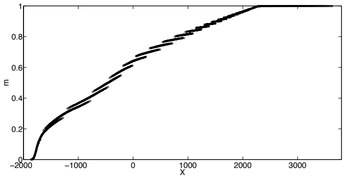


Numerical results: 6

Undular bore on shelf, $\tau = 3400$. In sequence, $B(X)$; the modulation Riemann variables $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the modulus $m = (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$ as a function of X.





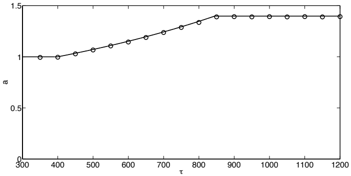


Numerical results: 7

Note the behaviour of the Riemann variables λ_2 and λ_3 near the leading edge of the undular bore in all plots, where $\frac{d}{dX}\lambda_2 < \infty$ and $\frac{d}{dX}\lambda_3 < \infty$, which corresponds to the"weak interaction" scenario. The figure shows a comparison for the theoretical adiabatic amplitude of on an isolated solitary wave on a slope with the numerical values of the leading solitary wave amplitude in the modulated wavetrain. For the variable-coefficient KdV equation (8) the adiabatic variations of the solitary wave amplitude are given by formula, see (15)

$$a = a_0 \left(\frac{h_0}{h(\tau)} \right)^{3/4}, \tag{48}$$

where h_0 and a_0 are the initial depth and the solitary wave amplitude respectively.

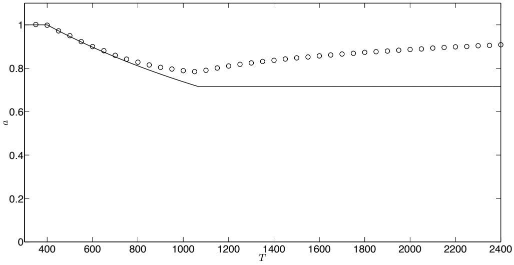


Increasing depth

Suppose that $\beta'(T) > 0$.The lead solitary wave amplitude implied by the "local" scenario,

$$a(T) = 2U_0\beta^{-1/3}(T)\,.$$

Comparison with numerical simulations. Solid line: adiabatic theory for an isolated solitary wave. Symbols: numerical values for the undular bore lead solitary wave amplitude.



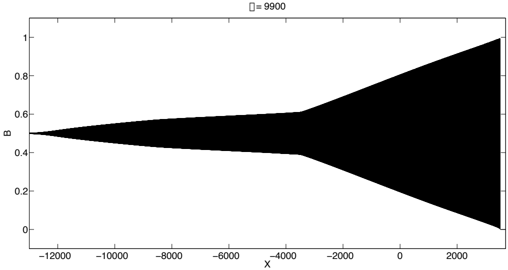
Hence: **non-local (strong interaction) scenario for $\beta'(T) > 0$**.

Increasing depth: non-adiabatic response

Let β vary monotonically from $\beta(T < T_0) = 1$ to $\beta(T > T_1) = \beta_1 > 1$.
At the **trailing edge** of the undular bore, we initially $(T < T_0)$ have a linear wave packet propagating with the group speed $s^- = -6U_0$, which must remain the same for $T > T_1$ since the jump $[U]$ across the bore is conserved.
On the other hand, the adiabatic evolution of the (initially) same linear wave packet implies for $T > T_1$

$$T > T_1 : \qquad c_g = 6U_0(1 - 2\beta_1) < s^- \, . \tag{49}$$

Therefore, one must insert a small-amplitude wavetrain behind the bore, **a non-adiabatic response**.



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