Wave turbulence in vibrating plates: can one hear a Kolmogorov spectrum?

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Wave turbulence.

- developed originally for water waves (ocean waves)
- stationary observed states: statistical description of wave systems?
- for waves, the linear order is crucial since one can make an expansion analysis with a small parameter (wave amplitude)
- assuming weak nonlinearities, a kinetic equation for the wave amplitudes can be obtained using asymptotic closures at large times
- although experimental evidences, many questions remain (ocean waves, range of validity, mathematical proof ...)

Wave turbulence in elastic plate

- elastic plates have dispersive waves and geometrical nonlinearities which suggest that wave turbulence can apply
- predicted theoretically in 2006 using classical wave turbulence arguments and shown in numerical simulations
- spectra of direct cascade of energy experimentally obtained with important differences with the theory.
- let know hear a Kolmogorov spectrum

Experiment movie by
N. Mordant (LPS-ENS)

Elastic plates

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)}\Delta^2 \zeta + \{\zeta, \chi\}$$

$$\frac{1}{E}\Delta^2 \chi = -\frac{1}{2}\{\zeta, \zeta\}$$

where E is the Young Modulus, h the plate thickness and:

$$\{f, g\} = f_{xx} g_{yy} + f_{yy} g_{xx} - 2 f_{xy} g_{xy}$$

and $\frac{1}{2}\{\zeta, \zeta\} = \zeta_{yy}\zeta_{xx} - \zeta_{xy}^2$ is the Gaussian curvature

Wave turbulence framework

- classical wave/weak turbulence machinery applies (DJR 2006)
- the dissipationless equation has a Hamiltonian structure

$$H = \int \left[\frac{h^2 E}{24(1-\sigma^2)} (\Delta\zeta)^2 - \frac{1}{2E} (\Delta\chi)^2 - \frac{1}{2} \chi\{\zeta,\zeta\} \right]$$

$$\omega_k = \sqrt{\frac{Eh^2}{12(1-\sigma^2)\rho}}\, |k|^2.$$

Using Fourier transform

$$H = h\int\left[\frac{1}{2\rho h^2}p_{k_1}p_{-k_1} + \frac{h^2E}{24(1-\sigma^2)}{k_1}^4\zeta_{k_1}\zeta_{-k_1}\right]d^2k_1 + \frac{h}{(2\pi)^2}\int T_{k_1k_2;k_3k_4}\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\delta(k_1+k_2+k_3+k_4)d^2k_{1234}$$

With

$$p_k = \rho h\dot{\zeta}_k$$

$$T_{k_1k_2;k_3k_4} = \frac{E}{8}\left(\frac{1}{2|k_1+k_2|^4} + \frac{1}{2|k_3+k_4|^4}\right)(k_1\times k_2)^2(k_3\times k_4)^2$$

Canonical variables: $\zeta_k = \frac{X_k}{\sqrt{2}}(A_k + A^*_{-k})$

$$X_k = \frac{1}{\sqrt{\omega_k \rho h}}.$$

$$p_k = -i\frac{X_k^{-1}}{\sqrt{2}}(A_k - A^*_{-k})$$

$$H = \int \omega_k A_k A_k^* dk + \frac{1}{4(2\pi)^2}\int X_{k_1}X_{k_2}X_{k_3}X_{k_4}T_{k_1k_2;k_3k_4}\sum_{s_1s_2s_3s_4} A_{k_1}^{s_1}A_{k_2}^{s_2}A_{k_3}^{s_3}A_{k_4}^{s_4}\delta(k_1+k_2+k_3+k_4)dk_{1234}$$

Equation for the cumulant: hierarchy. Need a closure (asymptotic arguments/random phase approximations).

Kinetic equation for elastic plates

$$\frac{d}{dt}n(l_2p_2) =$$

$$\varepsilon^4 12\pi l_2 \sum_{s_1s_2s_3} \int |J^{-l_2s_1s_2s_3}_{-p_2k_1k_2k_3}|^2 n(s_1k_1)n(s_2k_2)n(s_3k_3)n(l_2p_2)\left(\frac{l_2}{n(l_2p_2)} - \frac{s_1}{n(s_1k_1)} - \frac{s_2}{n(s_2k_2)} - \frac{s_3}{n(s_3k_3)}\right)$$

$$\times\delta(k_1+k_2+k_3-p_2)\delta(l_2\omega(p_2)-s_1\omega(k_1)-s_2\omega(k_2)-s_3\omega(k_3))dk_{123}$$

with $n(k) = < a_k a_k^* >$ related to the canonical variable

$$\frac{d}{dt}n(l_2\boldsymbol{p}_2) = \epsilon^4 12\pi l_2 \sum_{s_1s_2s_3}\int \left|J^{-l_2s_1s_2s_3}_{-\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\right|^2 n(s_1\boldsymbol{k}_1)n(s_2\boldsymbol{k}_2)n(s_3\boldsymbol{k}_3)n(l_2\boldsymbol{p}_2)\left(\frac{l_2}{n(l_2\boldsymbol{p}_2)} - \frac{s_1}{n(s_1\boldsymbol{k}_1)} - \frac{s_2}{n(s_2\boldsymbol{k}_2)} - \frac{s_3}{n(s_3\boldsymbol{k}_3)}\right) \times$$
$$\times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{p}_2)\delta(l_2\omega(\boldsymbol{p}_2) - s_1\omega(\boldsymbol{k}_1) - s_2\omega(\boldsymbol{k}_2) - s_3\omega(\boldsymbol{k}_3))d\boldsymbol{k}_{123}$$

Energy conservation

$$\mathcal{E} = \sum_{l_1}\int \omega(l_1\boldsymbol{p}_1)n(l_1\boldsymbol{p}_1,t)d\boldsymbol{p}_1$$

But no wave-action conservation!

H-Theorem

$$\mathcal{S}(t) = \sum_{l_1}\int \log[n(l_1\boldsymbol{p}_1,t)]d\boldsymbol{p}_1$$

$$\frac{d\mathcal{S}}{dt} \geq 0$$

2 types of stationnary solutions

- Rayleigh-Jeans equilibrium

$$n_k^{eq} = \frac{T}{\omega_k} = \frac{T}{hck^2}$$

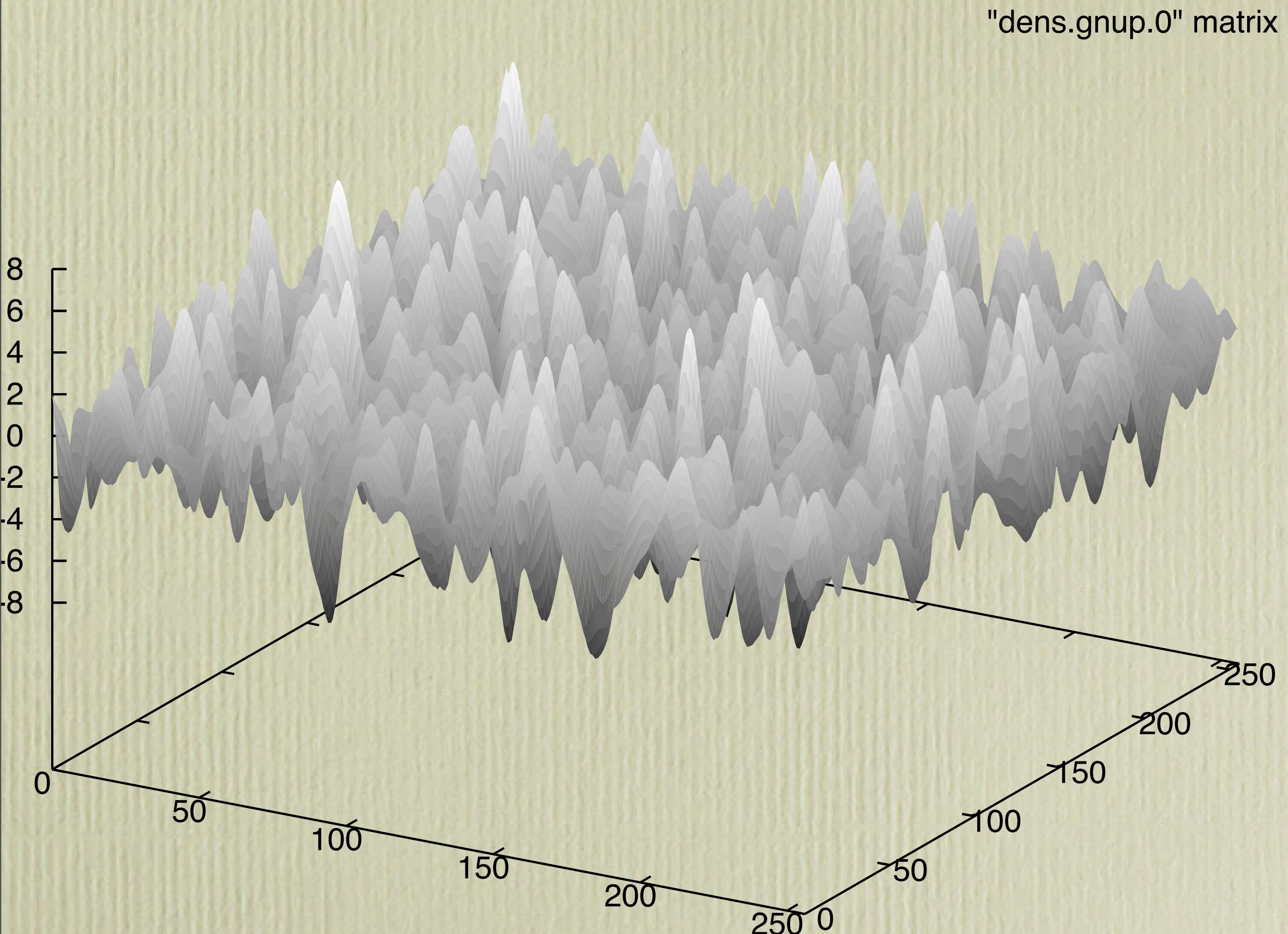
- K-Z cascade of energy

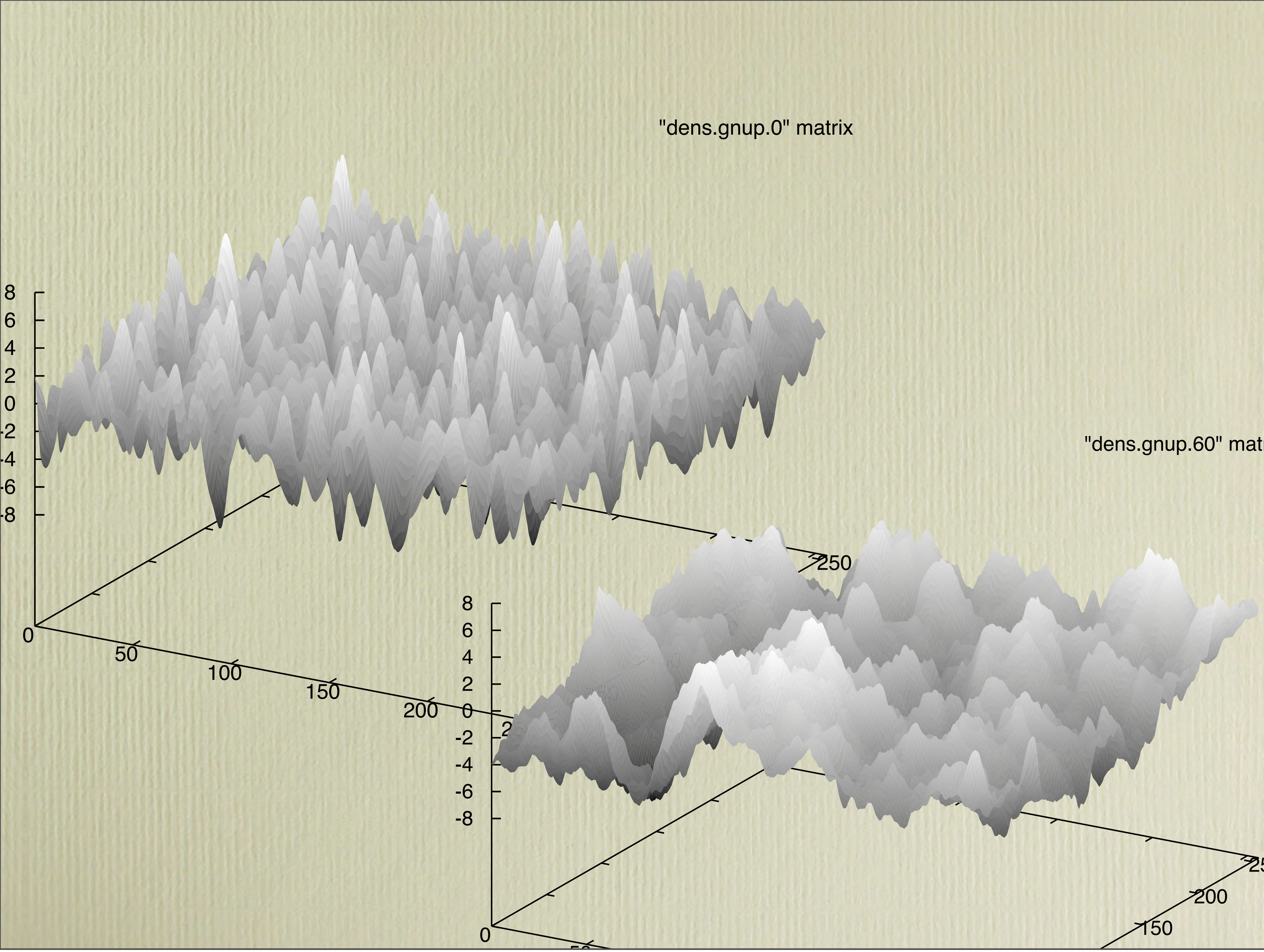
$$n_k^{eq} = C\frac{P^{1/3}}{k^2} log^{1/3}\left(\frac{k}{k_c}\right)$$

Numerical simulations

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)} \Delta^2 \zeta + \{\zeta, \chi\} + f_{in} + f_{diss}$$

- pseudo-spectral method
- periodic boundary conditions
- Adams-Bashford scheme


"dens.gnup.0" matrix
8
6
4
2
0
-2
-4
-6
-8
0
50
100
150
200
250
0
50
100
150
200
250


"dens.gnup.0" matrix
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"dens.gnup.60" matr
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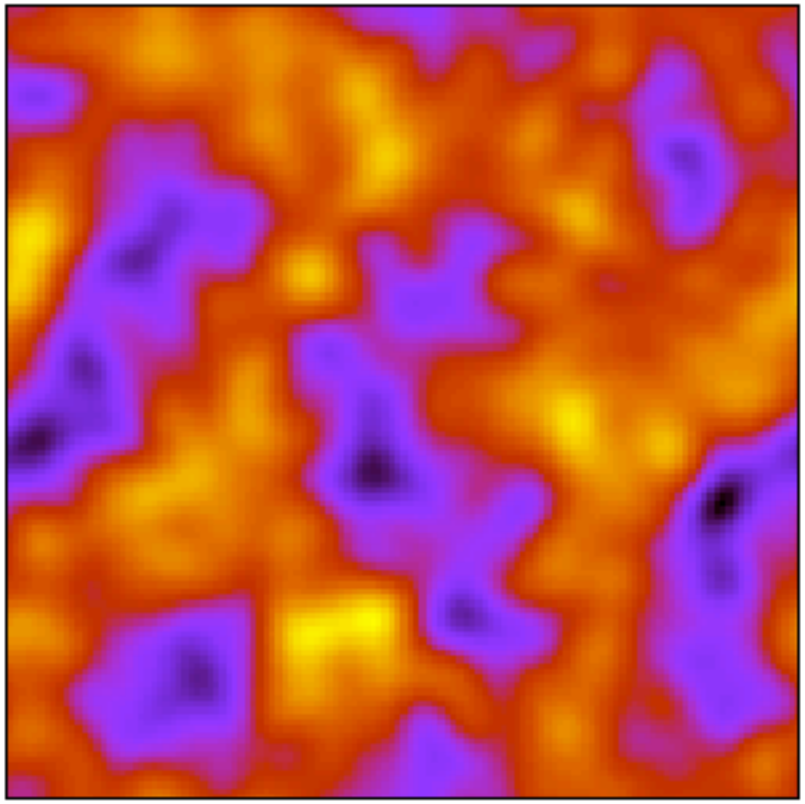
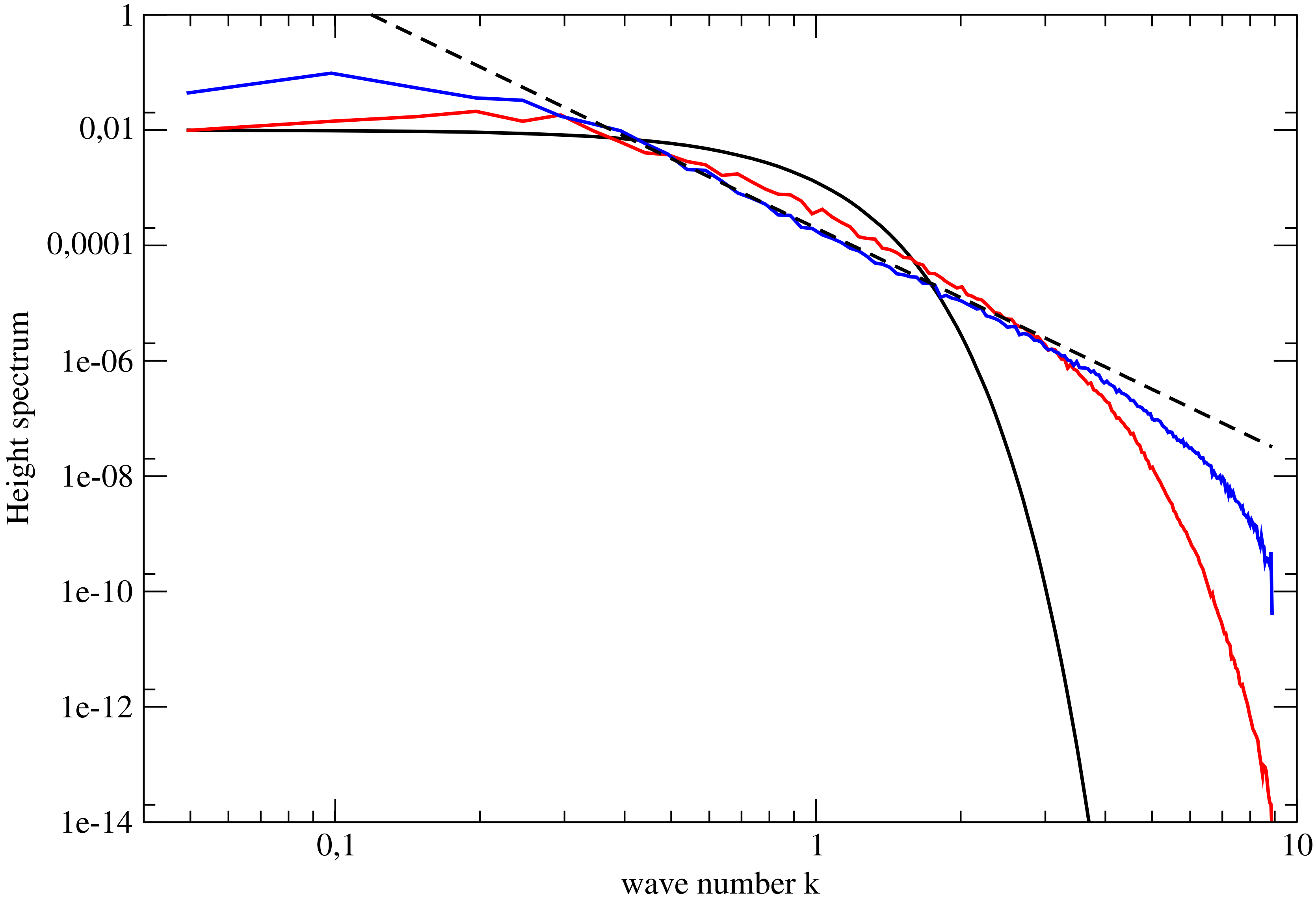
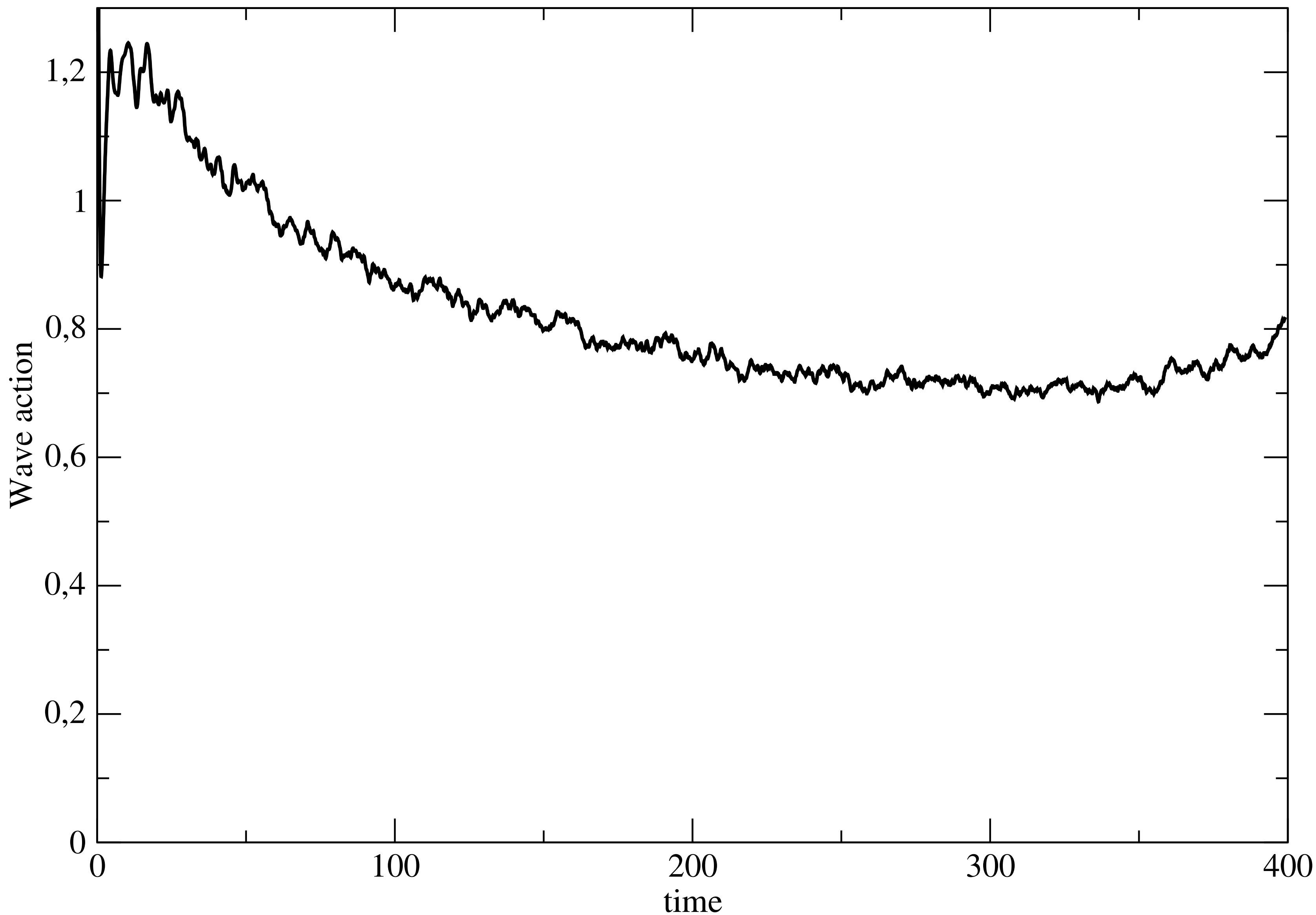
-2
-4
-6
-8
0
150
200



Free spectrum



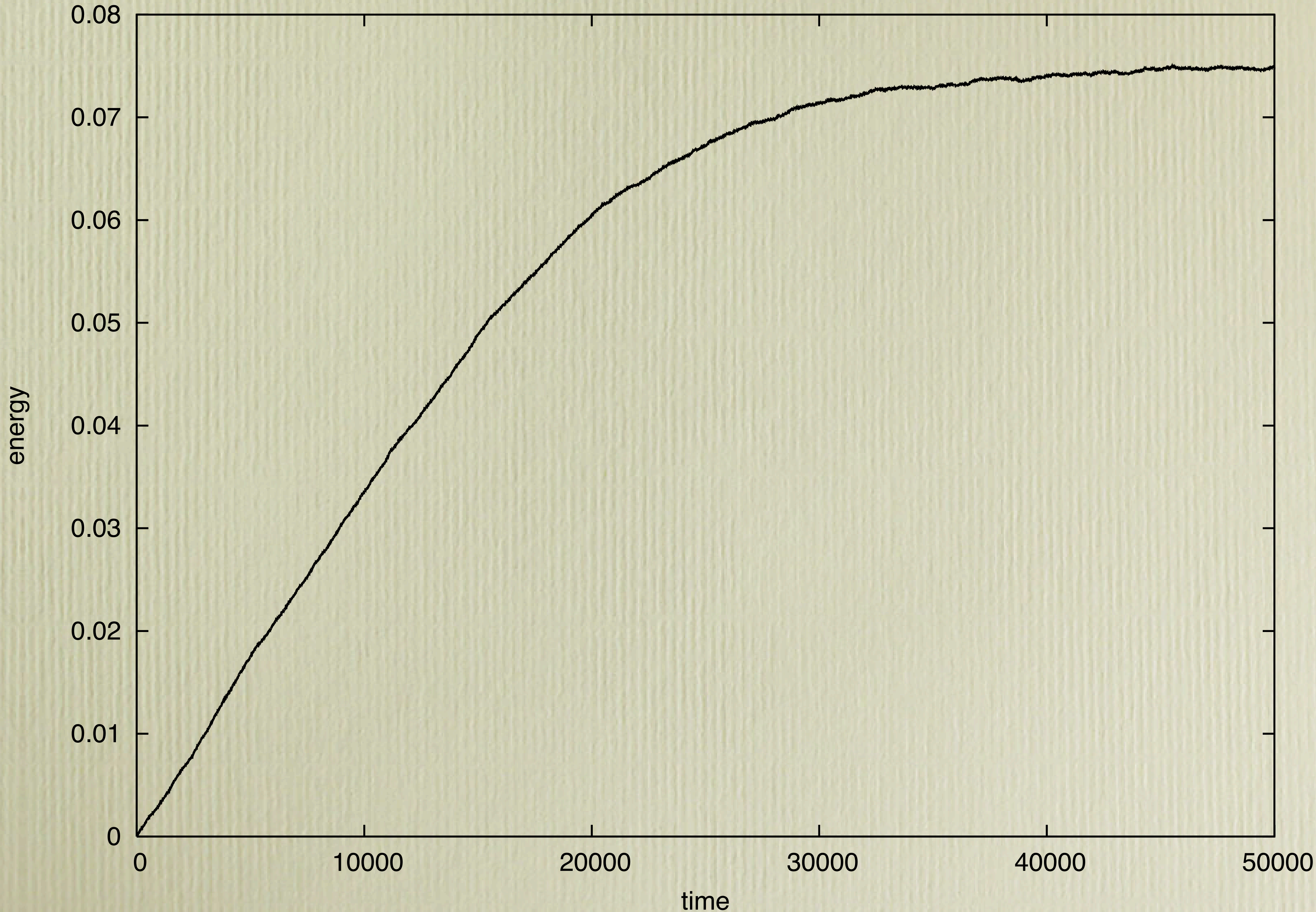
Wave action

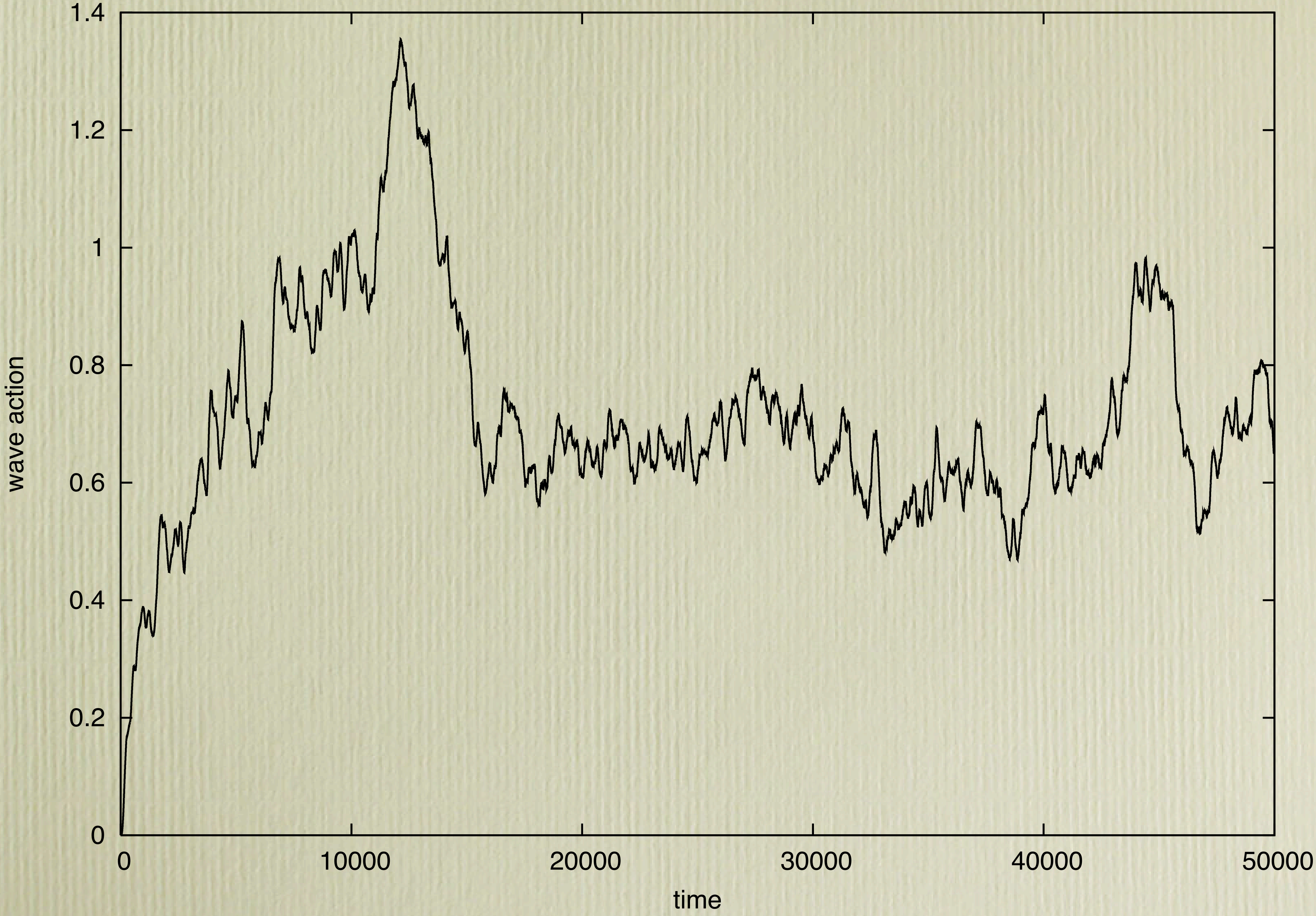


Forced turbulence

- injection at large scale (white noise in a narrow wavenumber window)
- dissipation above a critical wavenumber kc
- wave action pumping (or not) at large scale to avoid numerical instability

Stationnary state?

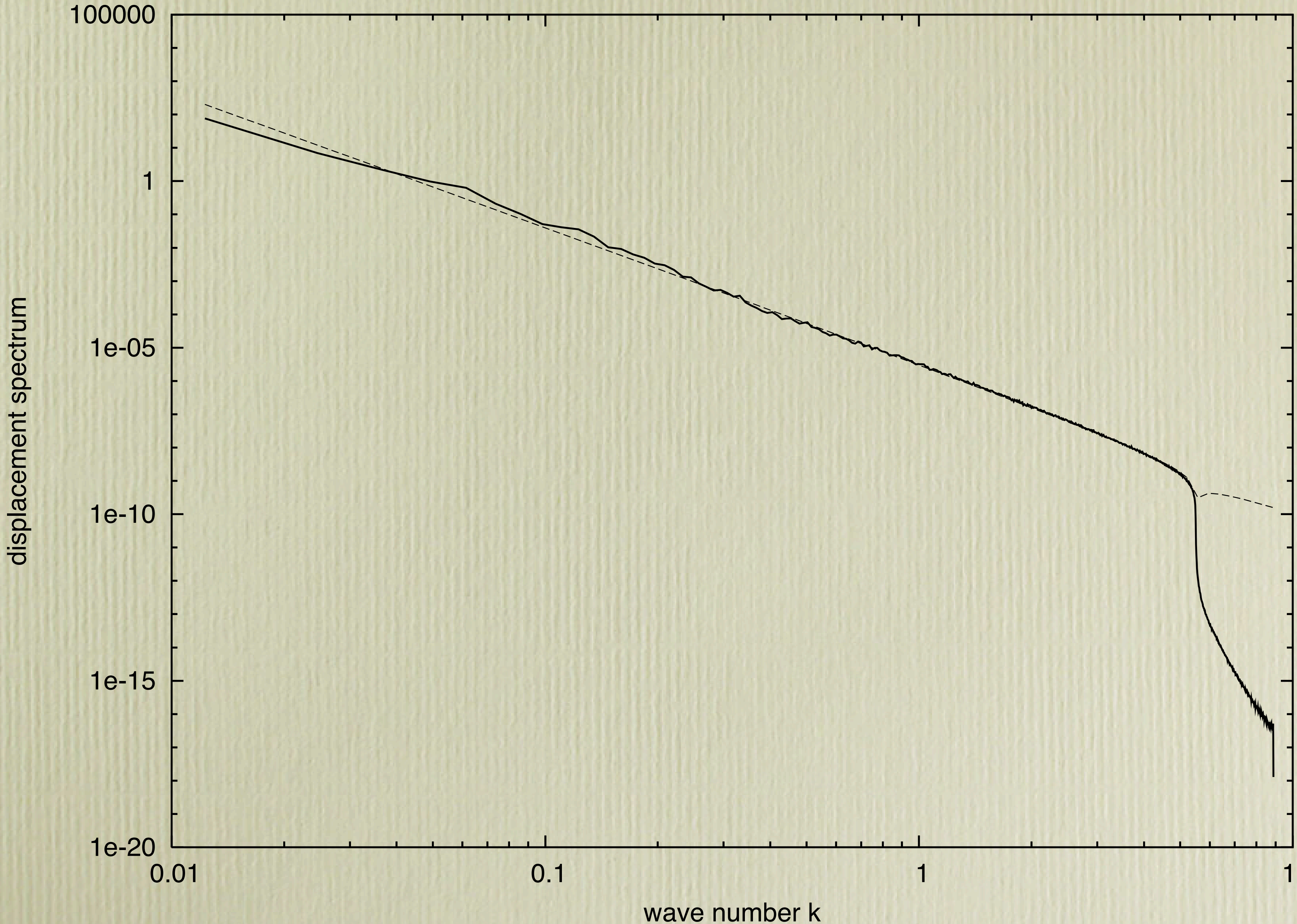


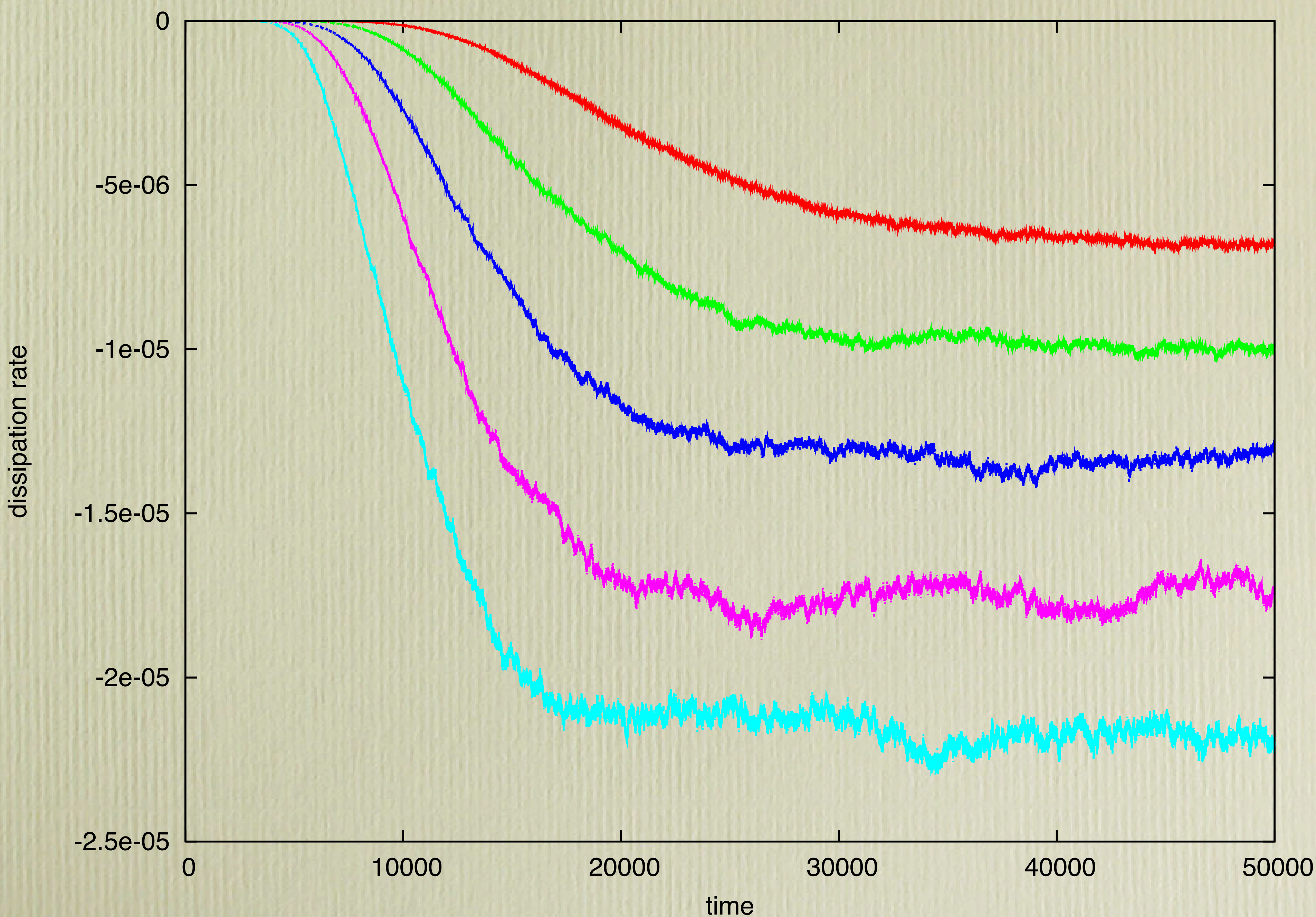


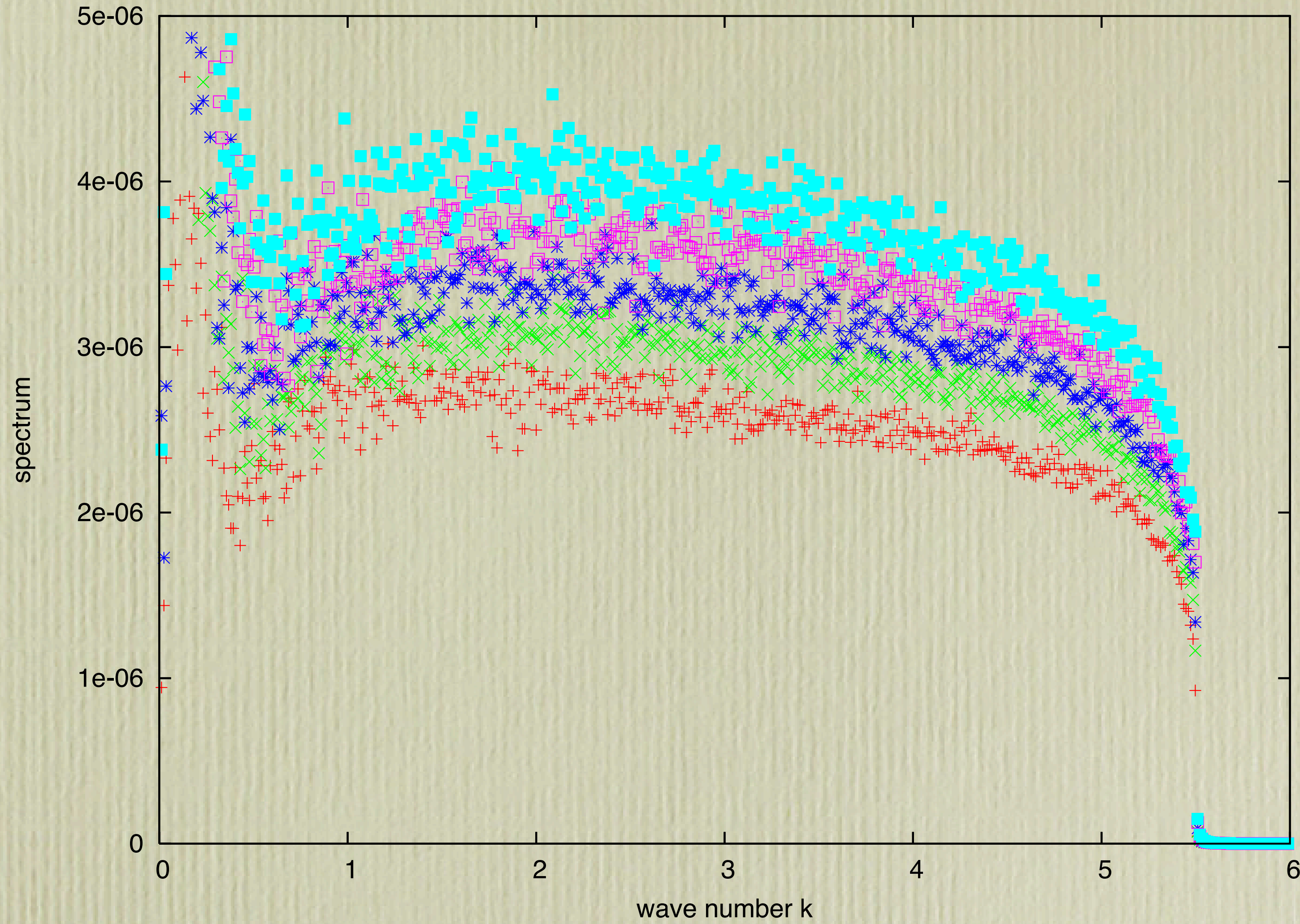
1.4
1.2
1
0.8
0.6
0.4
0.2
0
wave action
0
10000
20000
30000
40000
50000
time

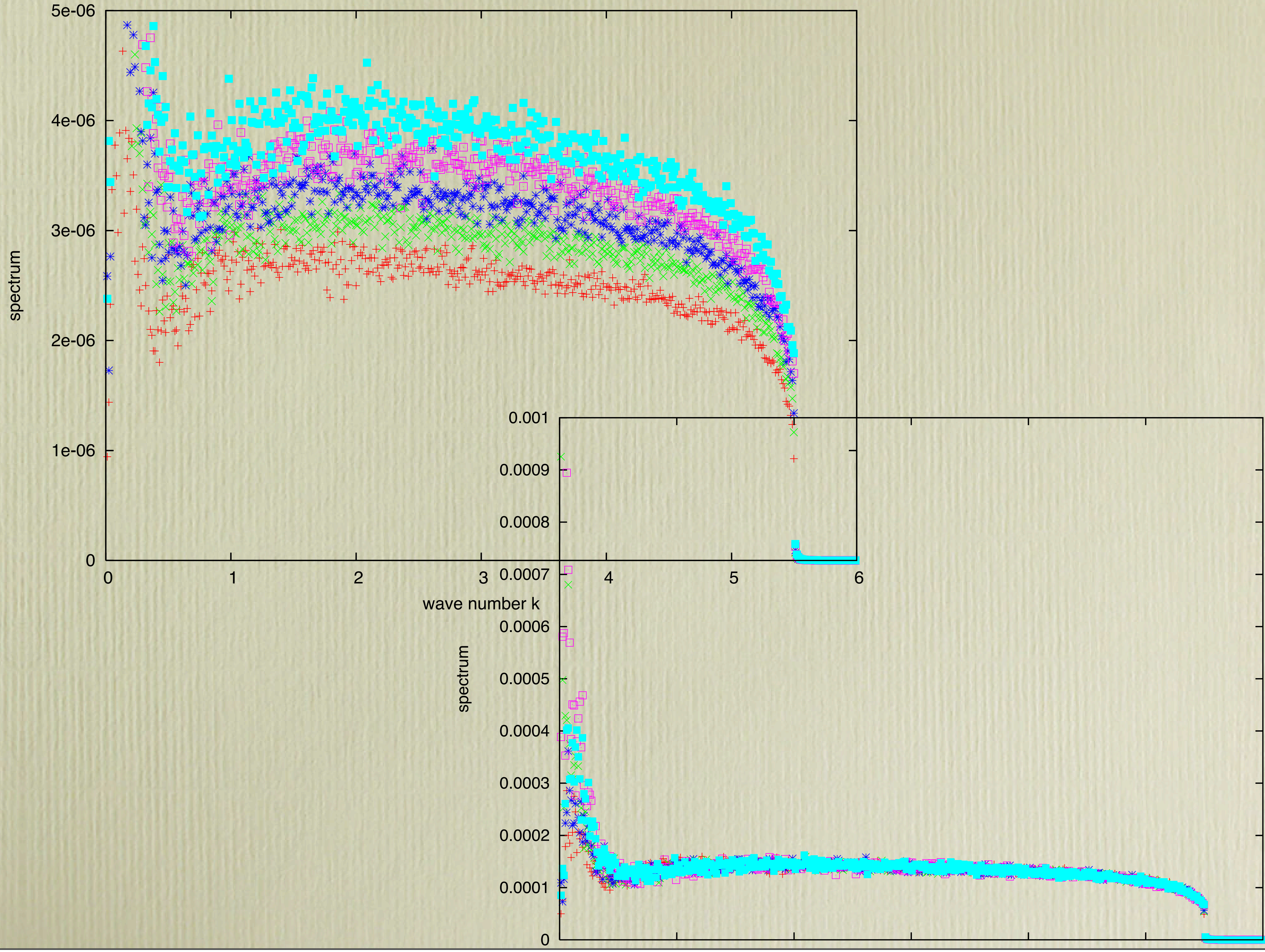
Spectrum

$$<|\zeta_k|^2> = \frac{n_k^{KZ}}{\rho\omega_k} \propto \frac{P^{1/3} log(kd/k)^{1/3}}{k^4}$$




0
-5e-06
-1e-05
-1.5e-05
-2e-05
-2.5e-05
dissipation rate
0
10000
20000
30000
40000
50000
time


5e-06
4e-06
3e-06
2e-06
1e-06
0
spectrum
0
1
2
3
4
5
6
wave number k

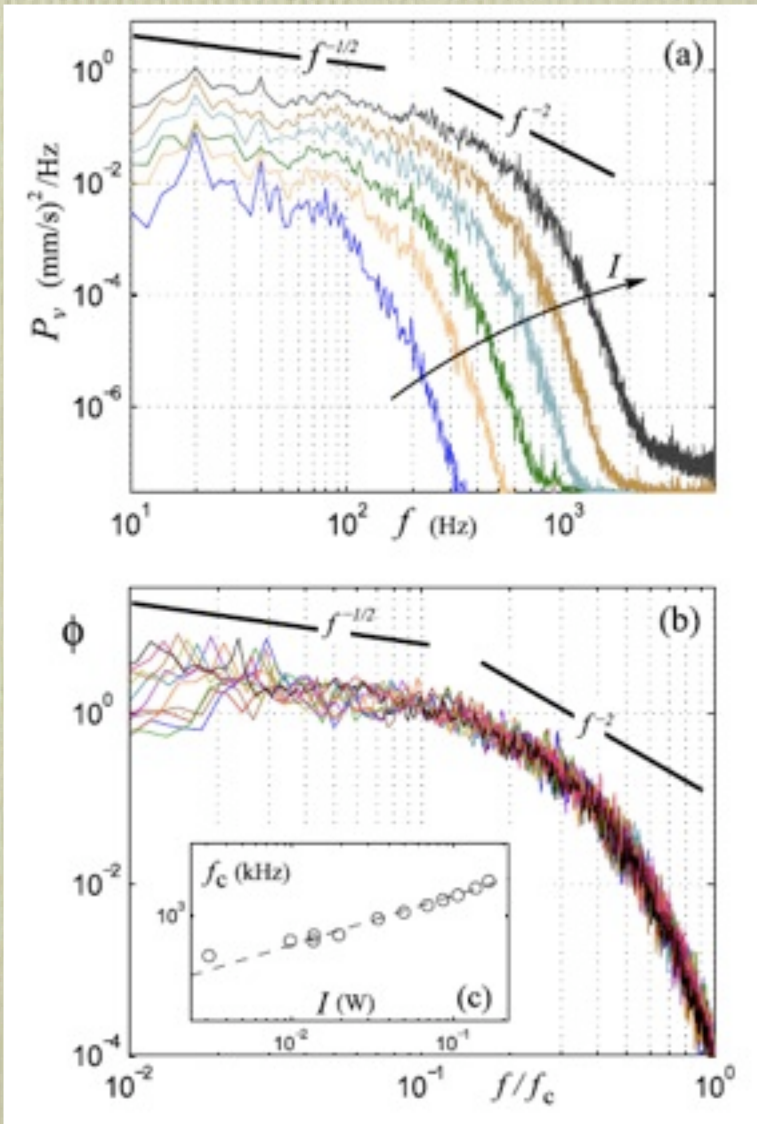


mardi 7 mai 2013

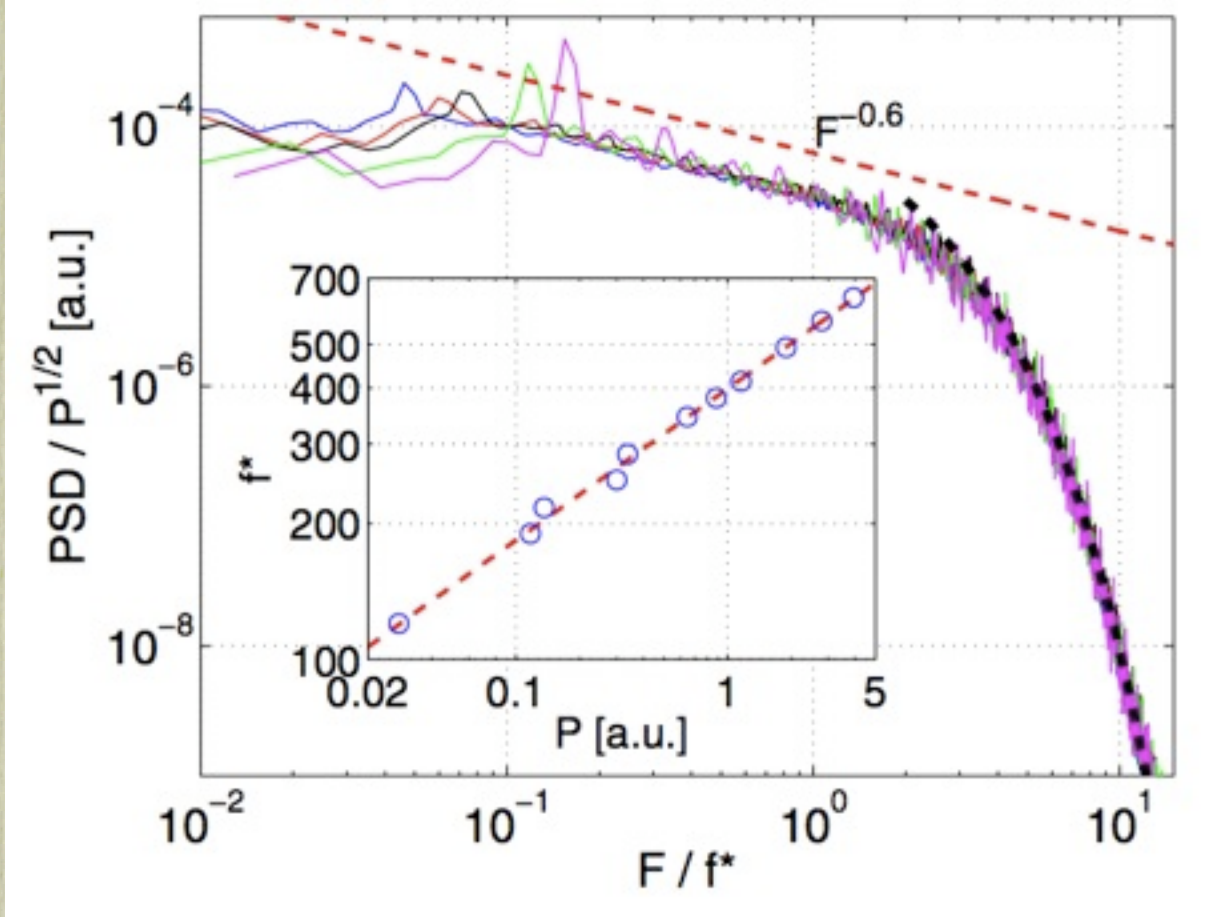
Experiment movie by
N. Mordant (LPS-ENS)

Experiments

$< |\zeta_\omega|^2 > \propto E_0$

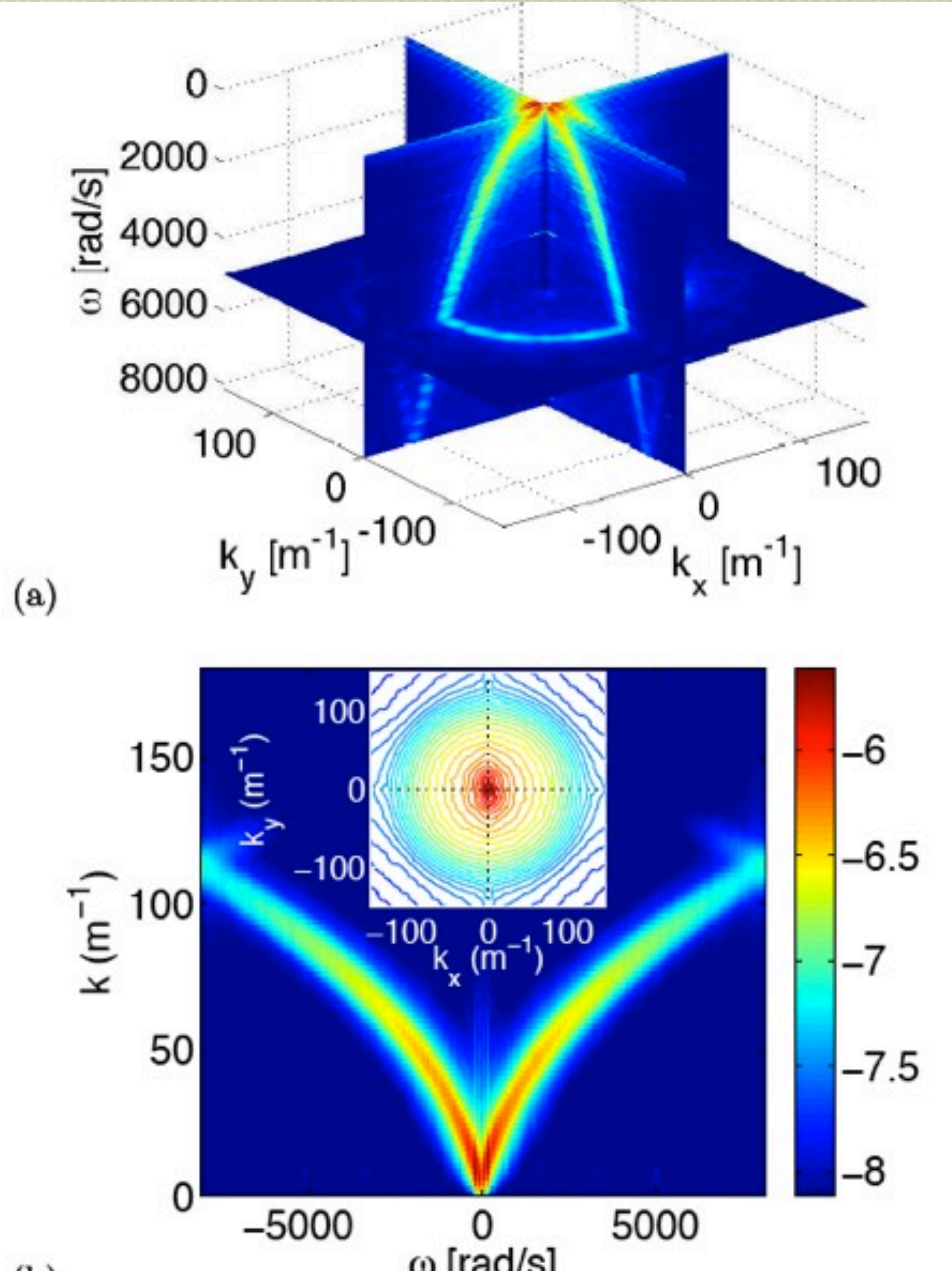


A. Boudaoud, O. Cadot, B. Odille & C. Touzé, PRL 234504 (2008).



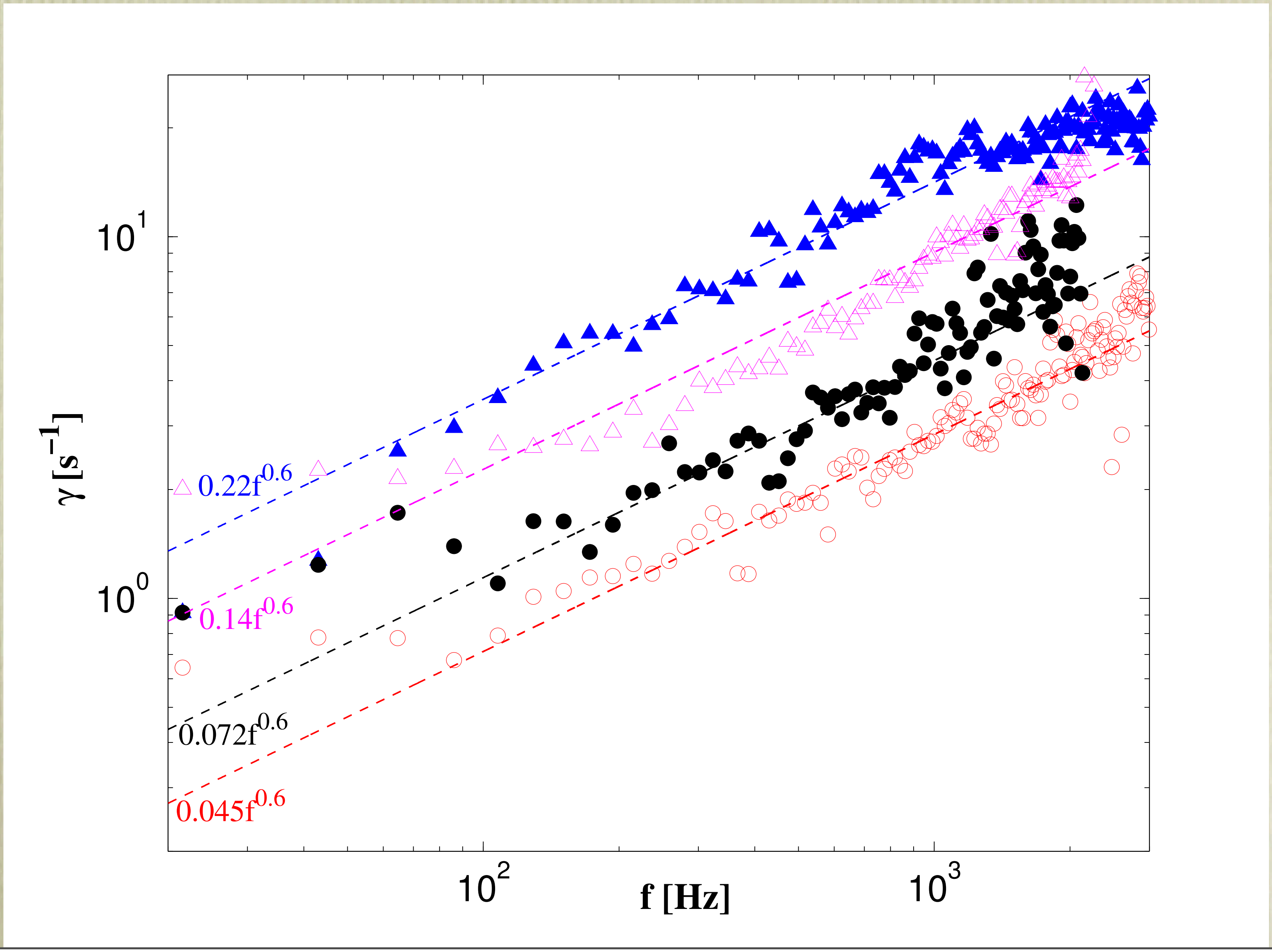
N. Mordant, PRL 234505 (2008).

- experiments show important differences with the theoretical predictions
- dispersion relation (mean curvature, three wave interactions, finite amplitude effects)
- boundary conditions
- dissipation
- an-isotropy-homogenous assumption
- still a question of debate

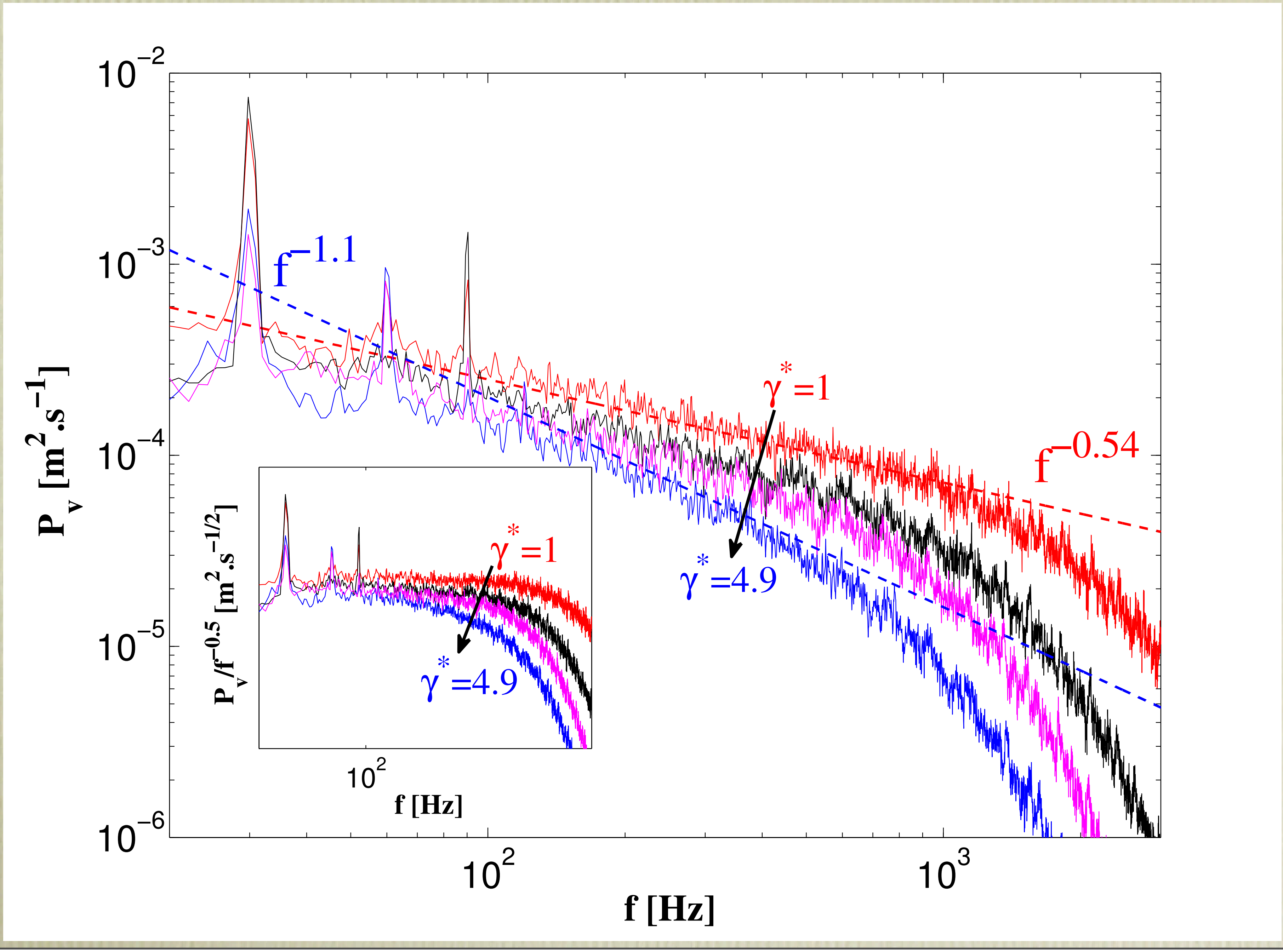


P. Cobelli, P. Petitjeans, A. Maurel, V. Pagneux & N. Mordant, PRL 103, 204301 (2009).

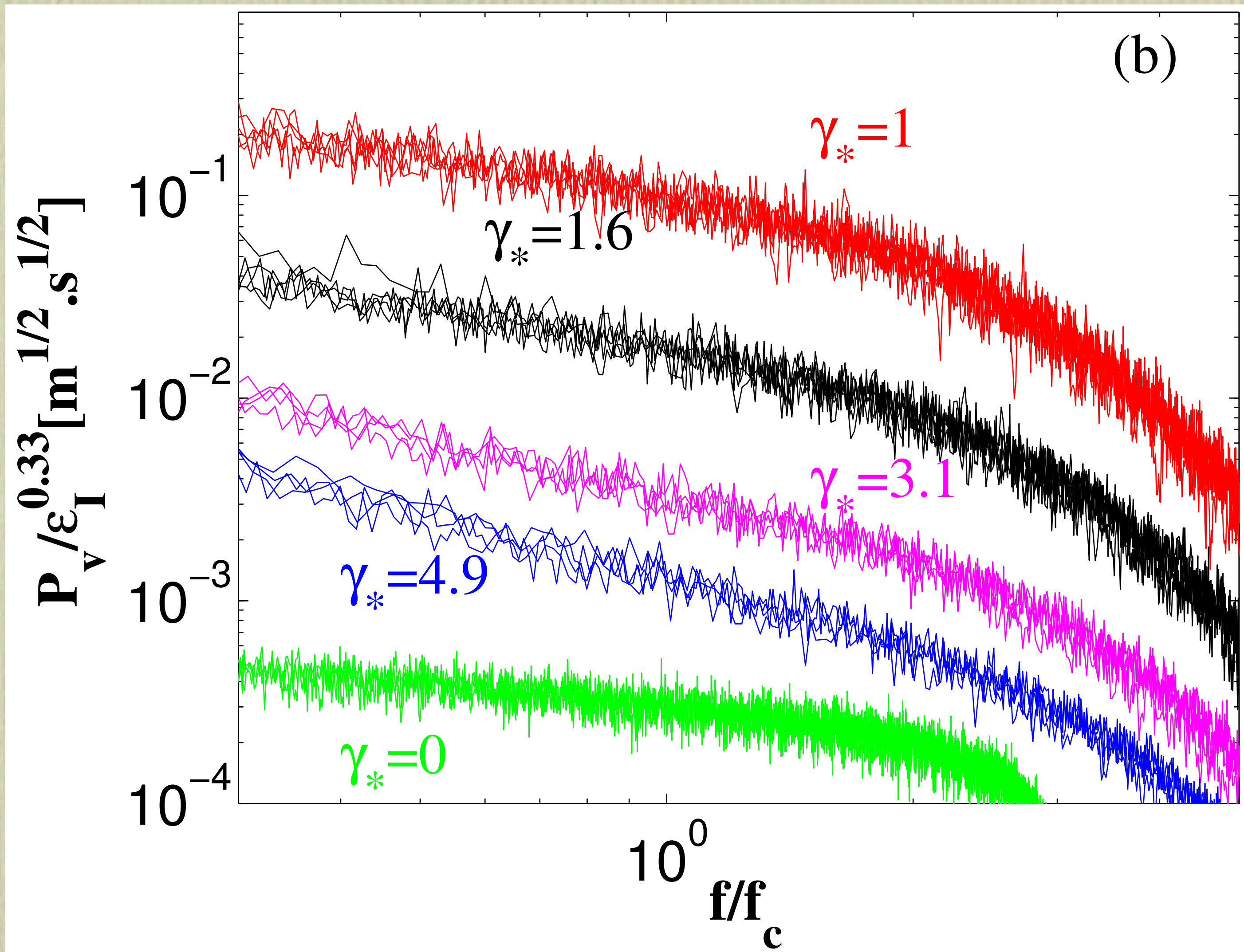
Changing dissipation



Changes slope!



Numerically also!



Conclusion

- vibrating plates is a great tool for investigating WT concepts
- WT applies and KZ spectrum is predicted
- However experiments show different spetra
- dissipation (at all scales) is a good candidate to explain the difference: need a modified WT therory to account for dissipation at all scales

Perspectives

- model accounting for dissipation at every scale (local modal in wavenumber)
- forcing inverse cascade? It works.
- high amplitude forcing: d-cone dynamics!