Long time existence for water wave models

Jean-Claude Saut
Université Paris-Sud

Ocean Waves Dynamics
Fields Institute
May 6-11, 2013

- The (surface or internal) water wave equations are too complicated to hope to describe the long time dynamics of the solutions except in trivial situations (perturbations of the flat surface).
- A natural idea is to "zoom" at some specific regimes of wavelengths, amplitudes, steepness,..., in order to derive asymptotic models that will describe interesting dynamics.
- One has first to define one or several "small" parameters and then to expand ad hoc quantities with respect to them.
- The aim of the talk is to briefly explain how this process can be made (mathematically) rigorous and to focus on one step in this process, namely the long time existence for the asymptotic systems.

Actually this idea goes back to Lagrange (1781) who :

- Derived the water waves system for potential flows.
- Obtained at the first order approximation the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - gh\frac{\partial^2 u}{\partial x^2} = 0,$$

- The "KdV" equation was obtained by Boussinesq in 1871...

See O. Darrigol, *The world of flows*, CUP 2005.

- How to justify the asymptotic models ?

A not too bad analogy is the convergence theory of finite difference schemes (Lax-Richtmeyer theorem).

A finite difference scheme approximating a PDE problem $\mathcal{P}$ is convergent if and only if

- $\mathcal{P}$ is well-posed.
- The scheme is consistent with $\mathcal{P}$. This is not a dynamical notion (just algebraic).
- The scheme is stable. This is a dynamical notion (well-posedness).

For the water waves asymptotic models, the steps are :

- 1. Long time existence for the water waves system (with respect to the inverse of the small parameter) with suitable uniform bounds.
  This difficult step has been achieved for the KdV regime by W. Craig (1985) and for most regimes of surface waves (but not for the modulation one) by B. Alvarez-Samaniego and D. Lannes (Inventiones 2008), and for the modulation regime by S.Wu and N.Totz (2012).

- 2. Consistency. Roughly speaking this means that a given solution of the water waves system solves the asymptotic model up to a residual term (the precision). As for the notion of consistency of a finite difference scheme, this is not a dynamical notion and of course does not implies the convergence of the approximation.
- Can be obtained formally or rigorously by ad hoc expansions of the Dirichlet-Neumann operator.

- 3. Long time existence for the asymptotic models with uniform bounds.
  This is the aim of the present talk, see below.

- 4. Error estimates on the "correct" time scales.
  See lot of examples in the forthcoming book by David Lannes *The water problem : mathematical analysis and asymptotics*.
  For instance, in the Boussinesq regime (see below),

$$||U_{Euler} - U_{Bouss}|| = O(\epsilon^2 t), \quad \text{the optimal error.}$$

(Bona-Colin-Lannes 2005).

General comment on nonlinear dispersive equations and systems

- Most of nonlinear dispersive equations or systems are not derived from first principles but through some asymptotic expansion and are not supposed to be "good" models for all time. So the classical dichotomy *local well-posedness* versus *finite time blow-up* should be often replaced by questions on *long time existence* (with respect to some parameters). Then using only methods of dispersive equations (even the more sophisticated ones) seems to be useless... The situation for classical one-way propagation waves (KdV, KP,..) where local well-posedness in sufficiently large classes combined with the conservation of "charge" and energy implies *global* well-posedness **does not generalize** to the more (physically) relevant two-ways models which are *systems* and do not possess (in general) useful conserved quantities.
  Also, those two-ways models are not skew-adjoint perturbations of symmetric quasilinear hyperbolic systems...

- For instance, the KdV equation appears as

$$u_t + u_x + \epsilon(uu_x + u_{xxx}) = O(\epsilon^2),$$

  where ϵ is a small parameter which measures the effects (which occur at the same order) of weak nonlinearity and dispersion.
  So the KdV solutions are not relevant (as approximations of the original system) after time scales of order $1/\epsilon^2$.
- Global well-posedness thanks to local theory +conservation laws (L^2, $H^1, ...$).
  Also, KdV is a skew-adjoint perturbation of a conservation law $\Rightarrow$ trivial existence of $H^s, s > \frac{3}{2}$, solutions on time scales of order $1/\epsilon$...

- **Boussinesq regime for surface water waves (with flat bottom)** :
  $h =$ mean depth of the fluid layer, $a =$ typical amplitude of the wave and $\lambda =$ a typical (horizontal) wave length, one assumes that

$$\mu = a/h \sim (h/\lambda)^2 = \epsilon \ll 1.$$

- One starts from the Zakharov-Craig-Sulem formulation of the water waves problem for the elevation ζ of the wave and the trace ψ of the velocity potential at the free surface :

$$\begin{cases} \partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}(\frac{1}{\mu} + \epsilon^2|\nabla\zeta|^2)(Z_\mu(\epsilon\zeta)\psi)^2 = 0 \\ \partial_t \zeta + \epsilon\nabla\psi\cdot\nabla\zeta - (\frac{1}{\mu} + \epsilon^2|\nabla\zeta|^2)Z_\mu(\epsilon\zeta)\psi = 0, \end{cases} \tag{1}$$

- One gets (by expanding the Dirichlet to Neumann operator $Z_\mu(\epsilon\cdot)$ with respect to $\epsilon \sim \mu$), the (a, b, c, d) family of systems (Bona-Chen-S. 2002, Bona-Colin-Lannes 2005) for the amplitude ζ of the wave and an approximation $\mathbf{v}$ of the horizontal velocity taken at some height in the infinite strip domain :

$$\begin{cases} \partial_t \zeta + \nabla \cdot \mathbf{v} + \epsilon \nabla \cdot (\zeta \mathbf{v}) + \epsilon \big(a \nabla \cdot \Delta \mathbf{v} - b \Delta \partial_t \zeta \big) = 0, & (O(\epsilon^2)) \\ \partial_t \mathbf{v} + \nabla \zeta + \frac{\epsilon}{2} \nabla (|\mathbf{v}|^2) + \epsilon \big(c \nabla \Delta \zeta - d \Delta \partial_t \mathbf{v} \big) = 0, & (O(\epsilon^2)) \end{cases} \tag{2}$$

with the initial data

$$(\zeta, \mathbf{v})^T|_{t=0} = (\zeta_0, \mathbf{v}_0)^T \tag{3}$$

a, b, c, d are modelling parameters which satisfy the constraint $a + b + c + d = \frac{1}{3}$. Those three degrees of freedom arise from the height at which the horizontal velocity is taken and from a double use of the *BBM trick*.

Advantages of the (a, b, c, d) systems :

- Obtain well-posed systems (see the original Boussinesq system which is ill-posed...and is the one obtained by expanding the Hamiltonian).
- Choose (a, b, c, d) for better mathematical/numerical properties.
- Choose (a, b, c, d) to fit better the water waves dispersion relation.

As in the KdV case, it is clear that the Boussinesq systems cease to be accurate models for time scales of order $O(1/\epsilon^2)$. Actually the error should be at best $O(\epsilon^2 t)$ (Bona-Colin-Lannes 2005). The problem is then to prove *long time* well-posedness of the Cauchy problem, that is at least on time scales of order $1/\epsilon$, with uniform bounds.

Discarding the dispersion part, the Boussinesq system becomes the *hyperbolic* Saint-Venant (shallow water) system :

$$\begin{cases} \partial_t \zeta + \nabla \cdot \mathbf{v} + \epsilon \nabla \cdot (\zeta \mathbf{v}) = 0 \\ \partial_t \mathbf{v} + \nabla \zeta + \frac{\epsilon}{2} \nabla (|\mathbf{v}|^2) = 0, \end{cases} \tag{4}$$

with the initial data

$$(\zeta, \mathbf{v})^T|_{t=0} = (\zeta_0, \mathbf{v}_0)^T \tag{5}$$

Hyperbolic in the region

$$1 + \epsilon\zeta > 0.$$

- This system is symmetrizable : the symmetrizer is in 1D

$$S(U) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\epsilon\zeta \end{pmatrix},$$

  and by classical methods in symmetrizable quasilinear hyperbolic systems one gets well-posedness on time scales $O(1/\epsilon)$ for data in $H^s(\mathbb{R}^d)$, $s > \frac{d}{2}+1$.
- But of course the symmetrizer does not fit well with the dispersive part !

- The question is then : what is the existence time for the Boussinesq systems ? Will it be the same, or larger than the "hyperbolic" one ?
- Related questions : what is the influence of a (possibly weak) dispersive perturbation on solutions of quasilinear hyperbolic systems (existence time, blow-up issues, spaces of resolution,...) ? Addressed in Linares-Pilod-S (SIMA 2012+ arXiv 2013+ work in in progress).

Concerning the Burgers -Hilbert equation

$$\partial_t u + \mathcal{H} u + \epsilon u \partial_x u = 0, \qquad (6)$$

where $\mathcal{H}$ is the Hilbert transform.
J.K. Hunter and M.Ifrim have obtained the rather unexpected result (SIMA 2012).
See also another proof in Hunter-Ifrim-Tataru-Wang 2013 :

Theorem

Suppose that $u_0 \in H^2(\mathbb{R})$. There are constants $k > 0$ and $\epsilon_0 > 0$, depending only on $|u_0|_{H^2}$, such that for every ϵ with $|\epsilon| \leq \epsilon_0$, there exists a solution $u \in C(I_\epsilon; H^2(\mathbb{R}) \cap C^1(I_\epsilon; H^1(\mathbb{R}))$ of BH defined on the time-interval $I_\epsilon = [-k/\epsilon^2, k/\epsilon^2]$.

So the existence time is enhanced thanks to the order zero operator $\mathcal{H}$.

Dispersive perturbations of hyperbolic systems.

$$\partial_t U + \epsilon\, \mathcal{A}(U, \nabla U) + \epsilon \mathcal{L} U = 0, \tag{7}$$

where $\mathcal{L}$ is a linear (non necessarily skew-adjoint) dispersive linear operator and $\epsilon > 0$ is a small parameter which measures the (comparable) nonlinear and dispersive effects.
Boussinesq systems for surface water waves are closely related to such systems.
When $\mathcal{L} = 0$ one has a quaslinear hyperbolic system and if it is symmetrizable one obtains a lifespan of order $1/\epsilon$ for the solutions of the associated Cauchy problem.

A basic question (in particular to justify the validity of (7) as an asymptotic model) is to analyze how this life span is modified by the presence of the dispersive term $\epsilon\mathcal{L}$.

We will first study this question in the context of the Boussinesq systems, and the in a more complicated "Full dispersion" system.
A natural approach is to try to apply the *dispersive techniques* (based on various dispersive properties of the dispersive linear part) to see if one can enhance the existence time (F. Linares , D. Pilod, JCS, SIMA 2012).
We focus on the *more dispersive* Boussinesq systems.

- Dispersion matrix for the Boussinesq systems :

$$D = \epsilon \begin{pmatrix} -b\Delta\partial_t & a\Delta\nabla \\ c\Delta\nabla\cdot & -d\Delta\partial_t \end{pmatrix}.$$

- The corresponding non zero eigenvalues are

$$\lambda_\pm(\xi) = \pm i|\xi| \left(\frac{(1-\epsilon a|\xi|^2)(1-\epsilon c|\xi|^2)}{(1+\epsilon d|\xi|^2)(1+\epsilon b|\xi|^2)} \right)^{\frac{1}{2}}.$$

- The linearized system around the null solution is well-posed provided that

$$a \le 0, \quad c \le 0, \quad b \ge 0, \quad d \ge 0, \tag{8}$$

$$\text{or} \quad a = c > 0, \quad b \ge 0, \quad d \ge 0. \tag{9}$$

Note that the eigenvalues $\lambda(\xi)$ can have order $3, 2, 1, 0, -1$.

- Order 3 : "KdV" type.
- Order 2 : "Schrödinger" type.

- When $b = d$ the Boussinesq systems are Hamiltonian with Hamiltonian (recall that generically $a, c \leq 0$) :

$$H(\zeta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^d}[-c|\nabla\zeta|^2 - a|\nabla\mathbf{v}|^2 + \zeta^2 + |\mathbf{v}|^2(1+\zeta)].$$

- Useless in 2D (and in 1D except with a smallness condition).

When surface tension is taken into account, one obtains a similar class of Boussinesq systems (Daripa-Das 2003) with a changed into $a - \tau$ where $\tau \geq 0$ is the Bond number which measures surface tension effects. The constraint on the parameters a, b, c, d is the same and condition (13) reads now

$$\begin{cases} a - \tau \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \\ \text{or} \\ a - \tau = c > 0, \quad b \geq 0, \quad d \geq 0. \end{cases} \tag{10}$$

The *more dispersive* Boussinesq system, corresponds to $b = d = 0$, $a = c = 1/6$ (similar result for $b = d = 0$, $c < 0, d < 0$). Though not the more relevant for modeling purposes due in particular to its computational difficulties and its "bad" behavior to short waves, it is mathematically challenging (this is one of the few physically relevant 2D versions of the KDV equation...) and show the limitations of the purely *dispersive* method to get long time well-posedness results.

- Joint with F. Linares and D. Pilod (SIMA 2012).

$$\begin{cases} \partial_t \zeta + \operatorname{div} \mathbf{v} + \epsilon \operatorname{div} (\zeta \mathbf{v}) + \epsilon \operatorname{div} \Delta \mathbf{v} = 0 \\ \partial_t \mathbf{v} + \nabla \zeta + \epsilon \frac{1}{2} \nabla (|\mathbf{v}|^2) + \epsilon \nabla \Delta \zeta = 0 \end{cases}, \quad (x_1, x_2) \in \mathbb{R}^2, \ t \in \mathbb{R}, \tag{11}$$

The results.

- Well posedness in the Sobolev space $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ on some interval $[0, C/\sqrt{\epsilon})$.
- Note that the existence time $1/\sqrt{\epsilon}$ is worse than the hyperbolic one !
- The exponent $3/2$ is strictly smaller than the "hyperbolic" one 2.
- No non cavitation assumption needed.

The proof consists in implementing a fixed point argument on the Duhamel formulation by using various dispersive estimates on the linear part : Strichartz (with smoothing), Kato type smoothing, maximal function estimate.

- It seems that this "KdV-KdV" system cannot (apparently) be solved by "elementary" energy methods.
- For all other Bousinesq systems one obtains also a life span of order $O(1/\sqrt{\epsilon})$

- **Existence on long time**.

In Bona-Colin-Lannes (BCL 2005) a new class of *fully symmetric* Boussinesq systems was introduced. The idea was, when the dispersive part is skew-adjoint, that is when $a = c$, to perform the change of variables

$$\tilde{\mathbf{v}} = \mathbf{v}(1 + \frac{\epsilon}{2}\zeta)$$

to get (up to terms of order ϵ^2) systems having a *symmetric* nonlinear part. Those transformed systems can be viewed as skew-adjoint perturbations of *symmetric* first order hyperbolic systems (when $\mathbf{v}$ is curlfree up to $0(\epsilon)$) and existence on time scales of order $1/\epsilon$ follows classically. But of course the transformed systems are not members of the a, b, c, d class of Boussinesq systems and this does not solve the long time existence (though this justifies in some sense this class as a *good* one to approximate the full Euler system with free boundary, see BCL).

- We now aim to proving long time $(1/\epsilon)$ existence for the original Boussinesq systems.
- (Li Xu, JCS, J.Math.Pures Appl. 2012). This approach was also used by Li Xu (Nonlinearity 2012) for *intermediate long waves* systems for internal waves).

Definition

For any $s \in \mathbb{R}$, $k \in \mathbb{N}$, $\epsilon \in (0,1)$, the Banach space $X^s_{\epsilon^k}(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm :

$$|u|^2_{X^s_{\epsilon^k}} = |u|^2_{H^s} + \epsilon^k |u|^2_{H^{s+k}}. \quad (12)$$

k, and later k' are positive numbers which depend on (a, b, c, d). For instance $(k, k') = (3, 3)$ when $a, c < 0$, $b, d > 0, b \neq d$.

Recall that the (a, b, c, d) systems are linearly well-posed when

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \quad (13)$$

$$\text{or} \quad a = c > 0, \quad b \geq 0, \quad d \geq 0. \quad (14)$$

Theorem

Let $t_0 > \frac{n}{2}$, $s \geq t_0 + 2$ *if* $b + d > 0$, $s \geq t_0 + 4$ *if* $b = d = 0$. *Let* a, b, c, d *satisfy the condition* (13). *Assume that* $\zeta_0 \in X^s_{\epsilon^k}(\mathbb{R}^n), \mathbf{v}_0 \in X^s_{\epsilon^{k'}}(\mathbb{R}^n)$ *satisfy the (non-cavitation) condition*

$$1 - \epsilon\zeta_0 \geq H > 0, \quad H \in (0,1), \tag{15}$$

Then there exists a constant $\tilde{c}_0$ *such that for any* $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\zeta_0|_{X^s_{\epsilon^k}} + |\mathbf{v}_0|_{X^s_{\epsilon^{k'}}})}$, *there exists* $T > 0$ *independent of* ϵ *and a unique solution* $(\zeta, \mathbf{v})^T$ *with* $\zeta \in C([0, T/\epsilon]; X^s_{\epsilon^k}(\mathbb{R}^n))$ *and* $\mathbf{v} \in C([0, T/\epsilon]; X^s_{\epsilon^{k'}}(\mathbb{R}^n))$. *Moreover,*

$$\max_{t\in[0,T/\epsilon]} (|\zeta|_{X^s_{\epsilon^k}} + |\mathbf{v}|_{X^s_{\epsilon^{k'}}}) \leq \tilde{c}(|\zeta_0|_{X^s_{\epsilon^k}} + |\mathbf{v}_0|_{X^s_{\epsilon^{k'}}}). \tag{16}$$

Here $\tilde{c} = C(H^{-1})$ *and* $\tilde{c}_0 = C(H^{-1})$ *are nondecreasing functions of their argument*

- The idea of the proof is to perform a suitable symmetrization (up to lower order terms) of a linearized system and then to implement a energy method on an approximate system. The method is of "hyperbolic" spirit.
- This is why we need the non cavitation condition (the hyperbolicity condition for the Saint-Venant system). Note that no such condition was needed in the proofs using dispersion.

Setting $\mathbf{V} = (\zeta, \mathbf{v})^T$, $\mathbf{U} = (\eta, \mathbf{u})^T = \epsilon \mathbf{V}$, we rewrite (4) as

$$\begin{cases} (1 - b\epsilon\Delta)\partial_t \eta + \nabla \cdot \mathbf{u} + \nabla \cdot (\eta \mathbf{u}) + a\epsilon \nabla \cdot \Delta \mathbf{u} = 0, \\ (1 - d\epsilon\Delta)\partial_t \mathbf{u} + \nabla \eta + \frac{1}{2}\nabla(|\mathbf{u}|^2) + c\epsilon\nabla\Delta\eta = 0. \end{cases} \tag{17}$$

with the initial data

$$(\eta, \mathbf{u})^T|_{t=0} = (\epsilon\zeta_0, \epsilon\mathbf{v}_0)^T \tag{18}$$

Let $g(D) = (1 - b\epsilon\Delta)(1 - d\epsilon\Delta)^{-1}$. Then (31) is equivalent after applying $g(D)$ to the second equation to the condensed system :

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U} + M(\mathbf{U}, D)\mathbf{U} = 0, \qquad (19)$$

where

$$M(\mathbf{U}, D) = \begin{pmatrix} \mathbf{u} \cdot \nabla & (1 + \eta + a\epsilon\Delta)\partial_{x_1} & (1 + \eta + a\epsilon\Delta)\partial_{x_2} \\ g(D)(1 + c\epsilon\Delta)\partial_{x_1} & g(D)(u_1\partial_{x_1}) & g(D)(u_2\partial_{x_1}) \\ g(D)(1 + c\epsilon\Delta)\partial_{x_2} & g(D)(u_1\partial_{x_2}) & g(D)(u_2\partial_{x_2}) \end{pmatrix}. \qquad (20)$$

In order to solve the system (4)-(18), we consider the following linear system in $\mathbf{U}$

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U} + M(\underline{\mathbf{U}}, D)\mathbf{U} = \mathbf{F}, \tag{21}$$

together with the initial data

$$\mathbf{U}|_{t=0} = \mathbf{U}_0, \tag{22}$$

The key point to solve the linear system (34)-(22) is to search a "symmetrizer" $S_{\underline{\mathbf{U}}}(D)$ of $M(\underline{\mathbf{U}}, D)$ such that the principal part of $iS_{\underline{\mathbf{U}}}(\xi)M(\underline{\mathbf{U}}, \xi)$ is self-adjoint and $((S_{\underline{\mathbf{U}}}(\xi)\mathbf{U}, \mathbf{U}))^{1/2}$ defines a norm under a smallness assumption on $\underline{\mathbf{U}}$. It is not difficult to find that :

(i) if $b = d$, $g(D) = 1$, $S_{\underline{\mathbf{U}}}(D)$ is defined by

$$\begin{pmatrix} 1 + c\epsilon\Delta & \underline{u}_1 & \underline{u}_2 \\ \underline{u}_1 & 1 + \underline{\eta} + a\epsilon\Delta & 0 \\ \underline{u}_2 & 0 & 1 + \underline{\eta} + a\epsilon\Delta \end{pmatrix}; \qquad (23)$$

(ii) if $b \neq d$, $S_{\underline{\mathbf{U}}}(D)$ is defined by

$$\begin{pmatrix} (1+c\epsilon\Delta)^2 g(D) & g(D)(\underline{u}_1(1+c\epsilon\Delta)) & g(D)(\underline{u}_2(1+c\epsilon\Delta)) \\ g(D)(\underline{u}_1(1+c\epsilon\Delta)) & (1+\underline{\eta}+a\epsilon\Delta)(1+c\epsilon\Delta) & 0 \\ g(D)(\underline{u}_2(1+c\epsilon\Delta)) & 0 & (1+\underline{\eta}+a\epsilon\Delta)(1+c\epsilon\Delta) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{u}_1\underline{u}_1 & \underline{u}_1\underline{u}_2 \\ 0 & \underline{u}_1\underline{u}_2 & \underline{u}_2\underline{u}_2 \end{pmatrix}(g(D)-1). \tag{24}$$

Note that $S_{\underline{\mathbf{U}}}(D)$ is not self-adjoint since at least its diagonal part is not.

Then we define the energy functional associated with (34) as

$$E_s(\mathbf{U}) = \big((1 - b\epsilon\Delta)\Lambda^s \mathbf{U}, S_{\underline{\mathbf{U}}}(D)\Lambda^s \mathbf{U}\big)_2. \tag{25}$$

One can show that $E_s(\mathbf{U})$ defined in (35) is trully a energy functional equivalent to (the square of) some $X^s_{\epsilon^k}(\mathbb{R}^2)$ norm provided a smallness condition is imposed on $\underline{\mathbf{U}}$, which is satisfied (for ϵ small enough) if

$$1+\underline{\eta} \geq H > 0, \quad |\underline{\mathbf{U}}|_\infty \leq \kappa(H, a, b, c, d), \quad |\underline{\mathbf{U}}|_{H^s} \leq 1, \quad \text{for} \quad t \in [0, T']. \tag{26}$$

For the nonlinear system, if ϵ is small enough, this smallness condition holds for its solution $(\zeta, \mathbf{v})^T$, i.e., $(\eta, \mathbf{u})^T \equiv \epsilon(\zeta, \mathbf{v})^T$ satisfies (26).

- One has then (painful) task to derive a priori estimates on the linearized system (which use in particular the commutator estimates of D. Lannes 2006), in the various cases.

- Construction of the nonlinear solution by an iterative scheme :

We construct the approximate solutions $\{\mathbf{V}^n\}_{n\geq 0} = \{(\eta^n, \mathbf{v}^n)^T\}_{n\geq 0}$ with $\mathbf{U}^{n+1} = \epsilon \mathbf{V}^n$ solution to the linear system

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U}^{n+1} + M(\mathbf{U}^n, D)\mathbf{U}^{n+1} = 0, \quad \mathbf{U}^{n+1}|_{t=0} = \epsilon \mathbf{V}_0 \equiv \mathbf{U}_0, \tag{27}$$

and with $\mathbf{U}^0 = \mathbf{U}_0$. Given $\mathbf{U}^n$ satisfying the above assumption, the linear system (27) is unique solvable.

- The proofs using dispersion (that is high frequencies) do not take into account the algebra (structure) of the nonlinear terms. They allows initial data in relatively big Sobolev spaces but seem to give only existence times of order $O(1/\sqrt{\epsilon})$.
- The existence proofs on existence times of order $1/\epsilon$ are of "hyperbolic" nature. They do not take into account the dispersive effects (treated as perturbations).
- Is it possible to go till $O(1/\epsilon^2)$, or to get global existence. Plausible in one D (the Boussinesq systems should evolves into an uncoupled system of KdV equations). Not so clear in 2D... One should there use dispersion. Use of a normal form technique (*à la Germain-Masmoudi-Shatah*) ? Possible difficulties due to the dispersion relation.

- We consider now a regime leading to a less dispersive system (joint work with Li Xu, JMP 2012), a version of a full dispersion system.

A full dispersion system.

As above we will denotes h the typical height of the fluid, a a typical amplitude of the wave and λ a typical horizontal wavelength. Two crucial parameters are the shallowness parameter $\mu = (\frac{h}{\lambda})^2$ and the nonlinearity parameter $\epsilon = \frac{a}{h}$. For instance the Boussinesq regime corresponds to $\mu \sim \epsilon$, leading to Boussinesq type systems studied previously. In deeper water, the shallowness parameter μ need not to be small and the Boussinesq regime is no more valid.

Asymptotic expansion can be then made assuming that the *steepness parameter* $\varepsilon = \epsilon\sqrt{\mu} = \frac{a}{\lambda}$ is small, leading to the *small steepness, or full-dispersion* system (Matsuno 1992, Choï 1995). The dispersion is the same as the linearized water waves system one.


栏 杆 边 请
注 意 安 全
CAUTION DEEP WATER

$$\begin{cases} \partial_t \zeta - \dfrac{1}{\sqrt{\mu}\nu}\mathcal{H}_\mu \mathbf{v} + \dfrac{\varepsilon}{\nu}\big(\mathcal{H}_\mu(\zeta \nabla \mathcal{H}_\mu \mathbf{v}) + \nabla\cdot(\zeta \mathbf{v})\big) = 0, \\ \partial_t \mathbf{v} + \nabla\zeta + \dfrac{\varepsilon}{\nu}\big(\dfrac{1}{2}\nabla(|\mathbf{v}|^2) - \sqrt{\mu}\nu \nabla\zeta \mathcal{H}_\mu \nabla\zeta\big) = \mathbf{0}, \end{cases} \tag{28}$$

where $\mathbf{v} = \mathbf{v}(t,x)$ ($x \in \mathbb{R}^d, d = 1,2$) is an $O(\epsilon^2)$ approximation of the horizontal velocity at the surface and $\zeta = \zeta(t,x)$ is the deviation of the free surface.
Following Lannes (2012), ν is a smooth function of μ such that $\nu \sim 1$ when $\mu \ll 1$ (shallow water) and $\nu \sim \frac{1}{\sqrt{\mu}}$ when $\mu = O(1)$ or $\mu \gg 1$ (deeper water), for instance $\nu = \frac{\tanh(\sqrt{\mu})}{\sqrt{\mu}}$. $\mathcal{H}_\mu$ is a Fourier multiplier defined as

$$\forall \mathbf{v} \in \mathcal{S}(\mathbb{R}^d), \qquad \widehat{\mathcal{H}_\mu \mathbf{v}}(\xi) = -\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|}(i\xi)\cdot\widehat{\mathbf{v}}(\xi). \tag{29}$$

Without loss of generality, we may assume that

$$\sqrt{\mu}\nu = 1, \quad \epsilon = \varepsilon\sqrt{\mu} \leq \epsilon_{\text{max}} \leq 1, \quad \mu \geq \mu_{\text{min}} > 0. \tag{30}$$

This model has been proven by D. Lannes (see his forthcoming AMS book 2013) to be consistent with the full water wave system. To our knowledge, no (local or on large time) well-posedness result for the Cauchy problem seems to be available. Our goal is to derive an equivalent system, which is also consistent with the water wave system, and for which we prove the large time well-posedness of the Cauchy problem. The new system, which has the same accuracy as the original one, is obtained after a (nonlinear and nonlocal) change of the two independent variables and turns out to be symmetrizable yielding the large time existence on the *hyperbolic* time scale $1/\epsilon$. This method is inspired by Bona-Colin-Lannes where it was used in the (simpler) case of Boussinesq systems having a skew-adjoint linear dispersive part.

One can now write the FD system as

$$\begin{cases} \partial_t \zeta - \mathcal{H}_\mu \mathbf{v} + \epsilon\big(\mathcal{H}_\mu(\zeta \nabla \mathcal{H}_\mu \mathbf{v}) + \nabla\cdot(\zeta \mathbf{v})\big) = 0, \\ \partial_t \mathbf{v} + \nabla\zeta + \epsilon\big(\frac{1}{2}\nabla(|\mathbf{v}|^2) - \nabla\zeta \mathcal{H}_\mu \nabla\zeta\big) = \mathbf{0}, \end{cases} \tag{31}$$

where $\mathcal{H}_\mu = \tanh(\sqrt{\mu}|D|)\mathcal{H}$ for $d=1$ while $\mathcal{H}_\mu = (\mathcal{H}_{\mu,1}, \mathcal{H}_{\mu,2})^T$ with $\mathcal{H}_{\mu,j} = \tanh(\sqrt{\mu}|D|)\mathcal{R}_j$ for $d=2$. Here $\mathcal{H} = -\frac{\partial_x}{|\partial_x|}$ is the Hilbert transform and $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)^T = -\frac{\nabla}{|D|}$ is the Riesz transform. In what follows, we shall use the notations

$$\mathcal{R}\mathbf{u} = \sum_{j=1}^{2} \mathcal{R}_j u_j, \quad \mathbf{u}\cdot\mathcal{R}f = \sum_{j=1}^{2} u_j \mathcal{R}_j f. \tag{32}$$

The same notations are valid for $\mathcal{H}_\mu$.

- Note that the dispersive part (order zero part) is the linearized water waves system at zero velocity and flat surface.
- As previously noticed, no local well-posedness result seems to be known for this system.
- We will perform a change of variables leading to a new (consistent) Full dispersion system for which we will prove a long time existence result.

- The one-dimensional case

In this case, we consider the nonlinear changes of variables

$$\widetilde{v} = v + \frac{\epsilon}{2} v \mathcal{H}_\mu \partial_x \zeta, \qquad \widetilde{\zeta} = \zeta - \frac{\epsilon}{4}|v|^2. \tag{33}$$

Deleting the $O(\epsilon^2)$ terms, we obtain the following system (omitting the tildes)

$$\begin{cases} \partial_t \zeta - \mathcal{H}_\mu v + \frac{\epsilon}{2}\mathcal{H}_\mu(v\mathcal{H}_\mu \partial_x \zeta) + \frac{\epsilon}{2} v \partial_x \zeta + \epsilon\Big(\mathcal{H}_\mu(\zeta \mathcal{H}_\mu \partial_x v) + \zeta \partial_x v\Big) = 0, \\ \partial_t v + \partial_x \zeta - \frac{\epsilon}{2}\partial_x \zeta \mathcal{H}_\mu \partial_x \zeta + \frac{3\epsilon}{2} v \partial_x v - \frac{\epsilon}{2} v \mathcal{H}_\mu^2 \partial_x v = 0. \end{cases} \tag{34}$$

- The two-dimensional case

We consider now the nonlinear changes of variables

$$\widetilde{\mathbf{v}} = \mathbf{v} + \frac{\epsilon}{2}\mathbf{v}\mathcal{H}_\mu\nabla\zeta, \qquad \widetilde{\zeta} = \zeta - \frac{\epsilon}{4}|\mathbf{v}|^2, \tag{35}$$

Discarding the $O(\epsilon^2)$ terms, we finally get (omitting the tildes)

$$\begin{cases} \partial_t\zeta - \mathcal{H}_\mu\mathbf{v} + \frac{\epsilon}{2}\mathcal{H}_\mu(\mathbf{v}\mathcal{H}_\mu\nabla\zeta) + \frac{\epsilon}{2}\mathbf{v}\cdot\nabla\zeta + \epsilon\Big(\mathcal{H}_\mu(\zeta\nabla\mathcal{H}_\mu\mathbf{v}) + \zeta\nabla\cdot\mathbf{v}\Big) = 0, \\ \partial_t\mathbf{v} + \nabla\zeta - \frac{\epsilon}{2}\nabla\zeta\mathcal{H}_\mu\nabla\zeta - \frac{\epsilon}{2}\mathbf{v}(\mathcal{H}_\mu\nabla\mathcal{H}_\mu + \mathrm{div}\,)\mathbf{v} \\ \qquad + \epsilon\begin{pmatrix} 2v_1\partial_1 v_1 + v_2\partial_2 v_1 \\ 2v_2\partial_2 v_2 + v_1\partial_1 v_2 \end{pmatrix} + \frac{\epsilon}{2}\begin{pmatrix} v_2\partial_1 v_2 + v_1\partial_2 v_2 \\ v_1\partial_2 v_1 + v_2\partial_1 v_1 \end{pmatrix} = \mathbf{0}. \end{cases} \tag{36}$$

- In both cases the new systems are consistent with the original ones (assuming that curl $\mathbf{v} = O(\varepsilon)$ in $2D$, a condition which is satisfied when deriving the FD system).
- It turns out that the new systems are symmetrizable leading to existence (with uniforms bounds) on time scales of order $1/\varepsilon$.

Final comments and open questions

- WP for the original Full Dispersion system ?
- Similar results for other systems for surface or internal water waves, in various regimes.
- So far we have not found a relevant water waves **system** for which the existence time is larger than the *hyperbolic* one $1/\epsilon$.
- Again observe the difference with the *one directional models* such as KdV, BBM, BO, KP,...
- In many physical systems, dispersion is weak and cannot be used alone...
- Hyperbolic+ weak dispersion : F. Linares, D. Pilod, JCS, (arXiv 2013 +*in progress*)...