Modeling shallow water waves

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DMA

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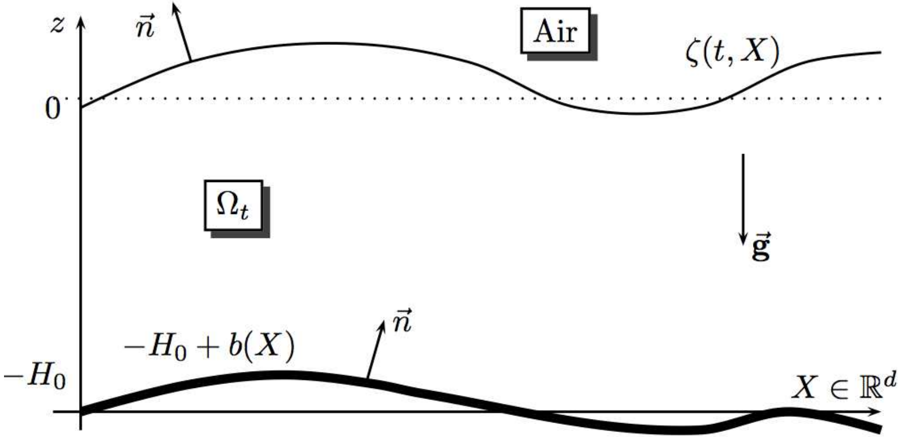
z
$\vec{n}$
Air
$\zeta(t,X)$
0
Ω_t
$\vec{\mathbf{g}}$
$\vec{n}$
$-H_0 + b(X)$
$-H_0$
$X \in \mathbb{R}^d$

- (H1) The fluid is homogeneous and inviscid
- (H2) The fluid is incompressible
- (H3) The flow is irrotational
- (H4) The surface and the bottom can be parametrized as graphs above the still water level
- (H5) The fluid particles do not cross the bottom
- (H6) The fluid particles do not cross the surface
- (H7) There is no surface tension and the external pressure is constant.
- (H8) The fluid is at rest at infinity
- (H9) The water depth is always bounded from below by a nonnegative constant

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(H9)' The water depth does not vanish

(H1)' $\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z})\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z} P - g\mathbf{e_z}$ in Ω_t

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Definition

Equations (H1)'-(H8)' are called free surface Euler equations.

(H1)' $\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla_{X,z})\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}P - g\mathbf{e_z}$ in Ω_t

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$\rightsquigarrow$ THREE unknown functions U on a moving, unknown domain Ω_t

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1. Define $\boxed{\psi(t,X) = \Phi(t,X,\zeta(t,X))}$.
2. ζ and ψ fully determine Φ: indeed, the equation

$$\begin{cases} \Delta_{X,z}\Phi = 0 & \text{in } \Omega_t, \\ \Phi_{|_{z=\zeta}} = \psi, & \partial_{\mathbf{n}}\Phi_{|_{z=-H_0+b}} = 0. \end{cases}$$

has a unique solution Φ.

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Craig-Sulem 93:

Definition (Dirichlet-Neumann operator)

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$\rightsquigarrow$ The equation on ζ can be written

$$\boxed{\partial_t\zeta - G[\zeta,b]\psi = 0}$$

- Equation on ψ. We use (H1)" and (H7)"

$$\partial_t \Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) \qquad \text{AND} \qquad P_{|_{z=\zeta}} = P_{atm}$$

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We now observe that:

$$\begin{aligned}
(\partial_t \Phi)_{|_{z=\zeta}} &= \partial_t \psi - (\partial_z \Phi)_{|_{z=\zeta}} \partial_t \zeta, \\
(\nabla \Phi)_{|_{z=\zeta}} &= \nabla \psi - (\partial_z \Phi)_{|_{z=\zeta}} \nabla \zeta, \\
(\partial_z \Phi)_{|_{z=\zeta}} &= \frac{G[\zeta, b]\psi + \nabla \zeta \cdot \nabla \psi}{1 + |\nabla \zeta|^2}.
\end{aligned}$$

- Equation on ψ. We use (H1)" and (H7)"

$$\partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) \qquad \text{AND} \qquad P_{|_{z=\zeta}} = P_{atm}$$

$$\Downarrow$$

$$\partial_t\Phi_{|_{z=\zeta}} + \frac{1}{2}|\nabla_{X,z}\Phi|^2_{|_{z=\zeta}} + g\zeta = 0$$

We now observe that:

$$\begin{aligned}
(\partial_t\Phi)_{|_{z=\zeta}} &= \partial_t\psi - (\partial_z\Phi)_{|_{z=\zeta}}\partial_t\zeta,\\
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$\rightsquigarrow$ The equation on ψ can be written

$$\boxed{\partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0.}$$

The Zakharov-Craig-Sulem equations

$$\begin{cases} \partial_t \zeta - G[\zeta, b]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta, b]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0. \end{cases}$$

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Hamiltonian structure

Zakharov remarked that this system has a Hamiltonian structure in the canonical variables (ζ, ψ):

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\zeta H \\ \partial_\psi H \end{pmatrix},$$

with the Hamiltonian $H = K + P$ and

$$\begin{aligned} K &= \frac{1}{2}\int_\Omega |\mathbf{U}|^2 = \frac{1}{2}\int_\Omega |\nabla_{X,z}\Phi(X,z)|^2 = \frac{1}{2}\int_{\mathbb{R}^d} \psi G[\zeta,b]\psi \\ P &= \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2. \end{aligned}$$

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- The (ZCS) system contains ALL the information
- One can in particular deduce models in $(\zeta, \overline{V})$ with $\overline{V}$ the vertically averaged velocity

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Question

How to get a model in $(\zeta, \overline{V})$ from the (ZCS) system?

Taking the (horizontal) gradient of the equation in ψ we get

$$\begin{cases} \partial_t \zeta - G[\zeta, b]\psi = 0, \\ \partial_t \nabla\psi + g\nabla\zeta + \frac{1}{2}\nabla|\nabla\psi|^2 - \nabla\frac{(G[\zeta, b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{cases}$$

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- Asymptotic models in $(\zeta, \overline{V})$ are found by plugging this approximate expression in the above equations.

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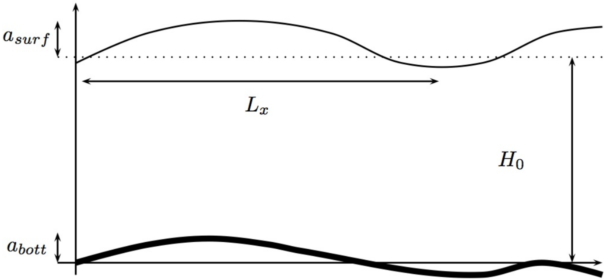

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- In large (or infinite) depth, ε is not a relevant parameter and one rather uses

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$\rightsquigarrow$ Nonlinear effects are expected to be stronger.

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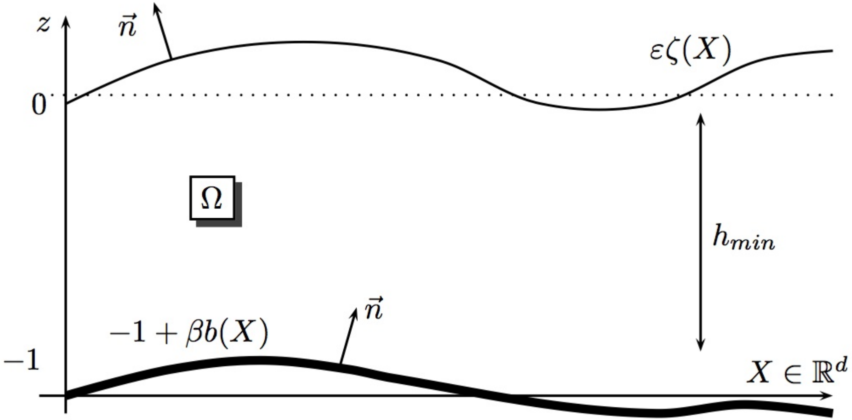
z
$\vec{n}$
$\varepsilon\zeta(X)$
0
Ω
h_{min}
$\vec{n}$
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-1
$X \in \mathbb{R}^d$

In this new set of variables and unknowns the equations become:

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Shallow water asymptotics ($\mu \ll 1$)

- We look for an asymptotic description with respect to μ of $\nabla\psi$ in terms of ζ and $\overline{V}$

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$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t \nabla\psi + \nabla\zeta + \frac{\varepsilon}{2}\nabla|\nabla\psi|^2 - \varepsilon\mu\nabla\frac{(-\nabla\cdot(h\overline{V}) + \nabla(\varepsilon\zeta)\cdot\nabla\psi)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = 0, \end{cases}$$

where in dimensionless form

$$h = 1 + \varepsilon\zeta \quad \text{and} \quad \overline{V} = \frac{1}{h}\int_{-1}^{\varepsilon\zeta} V(x,z)dz.$$

Shallow water asymptotics ($\mu \ll 1$)

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- What is the dimensionless potential equation?

(dimensional) $\rightsquigarrow$ (dimensionless)

$$\begin{cases} \Delta\Phi + \partial_z^2\Phi = 0, \\ \Phi_{|_{z=\zeta}} = \psi, \\ \partial_z\Phi_{|_{z=-H_0}} = 0 \end{cases} \rightsquigarrow \begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0, \\ \Phi_{|_{z=\varepsilon\zeta}} = \psi, \\ \partial_z\Phi_{|_{z=-1}} = 0 \end{cases}$$

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- We look for an approximate solution to the dimensionless potential equation under the form

$$\Phi_{app} = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \dots$$

Method

We plug

$$\Phi_{app} = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \dots$$

into

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0, \\ \Phi_{|_{z=\varepsilon\zeta}} = \psi, \qquad \partial_z\Phi_{|_{z=-1}} = 0, \end{cases}$$

and choose the Φ_j to cancel the leading order terms in μ.

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and choose the Φ_j to cancel the leading order terms in μ.

- Order $O(1)$.

$$\begin{cases} \partial_z^2\Phi_0 = 0, \\ \Phi_0\,_{|_{z=\varepsilon\zeta}} = \psi, \qquad \partial_z\Phi_0\,_{|_{z=-1}} = 0, \end{cases}$$

$$\Rightarrow \boxed{\Phi_0(X,z) = \psi(X)}.$$

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$$\begin{cases} \partial_z^2\Phi_1 = -\Delta\Phi_0, \\ \Phi_1{}_{|_{z=\varepsilon\zeta}} = 0, \qquad \partial_z\Phi_1{}_{|_{z=-1}} = 0, \end{cases}$$

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$$\Rightarrow \boxed{\Phi_1(X,z) = \frac{1}{2}[h^2 - (z+1)^2]\Delta\psi} \qquad (h = 1 + \varepsilon\zeta).$$

$$\Phi(X,z) = \psi + \mu\frac{1}{2}[h^2 - (z+1)^2]\Delta\psi + O(\mu^2).$$

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$$\begin{aligned} V(X,z) &= \nabla\Phi(X,z) \\ &= \nabla\psi + \mu\frac{1}{2}\nabla\{[h^2 - (z+1)^2]\Delta\psi\} + O(\mu^2) \end{aligned}$$

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- We deduce an expansion for the averaged velocity

$$\begin{aligned} \overline{V}(x) &= \frac{1}{h}\int_{-1}^{\varepsilon\zeta} V(X,z)\partial_z \\ &= \overline{V}_0(X) + \mu\overline{V}_1(X) + O(\mu^2) \end{aligned}$$

with

$$\boxed{V_0 = \nabla\psi, \qquad V_1 = -\mathcal{T}[h]\nabla\psi}$$

and

$$\mathcal{T}[h]V = -\frac{1}{3h}\nabla(h^3\nabla\cdot V).$$

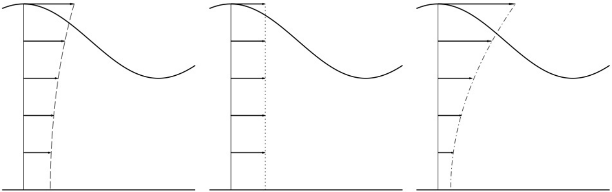

FIGURE 3.1. Shallow water approximation of the horizontal velocity field $V(x_0, \cdot)$ in the fluid domain when $\mu = 1$. Exact velocity field (dash), zero order approximation (dots) and first order approximation (dash-dots)

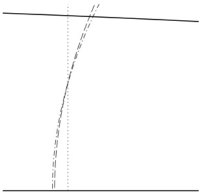

FIGURE 3.2. Shallow water approximation of the horizontal velocity field $V(x_0, \cdot)$ in the fluid domain when $\mu = 0.1$ (zoom). Exact velocity field (dash), zero order approximation (dots) and first order approximation (dash-dots)

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  - For (large) waves on a beach, this is not recommanded.

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  - We start from the full equations

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  - We drop all $O(\mu)$ terms

Saint-Venant
$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t\overline{V} + \nabla\zeta + \frac{\varepsilon}{2}\nabla|\overline{V}|^2 = 0. \end{cases}$$

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$$\overline{V} = (1-\mu\mathcal{T}[h])\nabla\psi + O(\mu^2)$$

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$$\rightsquigarrow \quad \nabla\psi = (1+\mu\mathcal{T}[h])\overline{V} + O(\mu^2).$$

  - We replace $\nabla\psi$ in the full equations by this approximation

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t(1+\mu\mathcal{T}[h])\overline{V} + \nabla\zeta + \frac{\varepsilon}{2}\nabla|(1+\mu\mathcal{T}[h])\overline{V}|^2 \\ \qquad -\varepsilon\mu\nabla\frac{(-\nabla\cdot(h\overline{V}) + \nabla(\varepsilon\zeta)\cdot\overline{V})^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = O(\mu^2), \end{cases}$$

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- We drop all $O(\mu^2)$ terms (and do some computations!) Serre/Green-Naghdi equations:

$$\boxed{\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ (1+\mu\mathcal{T}[h])\big(\partial_t \overline{V} + \nabla\zeta + \varepsilon\overline{V}\cdot\nabla\overline{V}\big) + \varepsilon\mu\mathcal{Q}[h](V) = 0 \end{cases}}$$

with

$$\begin{aligned} \mathcal{T}[h]V &= -\tfrac{1}{3h}\nabla\cdot(h^3\nabla\cdot V), \\ \mathcal{Q}[h](V) &= \tfrac{2}{3h}\nabla\big[h^3(\partial_1 V\cdot\partial_2 V^\perp + (\nabla\cdot V)^2)^2\big]. \end{aligned}$$

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  - (almost) columnar motion,

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ (1 + \mu \mathcal{T}[h])\big(\partial_t \overline{V} + \nabla \zeta + \varepsilon \overline{V} \cdot \nabla \overline{V}\big) + \varepsilon \mu \mathcal{Q}[h](V) = 0 \end{cases}$$

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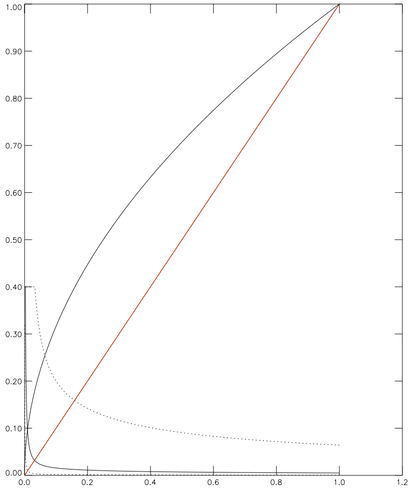
  are a consequence of the shallow water assumption $\mu \ll 1$.
- No assumption has been made on the size of ε: $\varepsilon = O(\mu)$
- Dropping the $O(\mu)$ terms from the Green-Naghdi system, one gets

$$\begin{cases} \partial_t \zeta + \nabla\cdot(h\overline{V}) = 0 \\ \partial_t \overline{V} + \nabla\zeta + \varepsilon\overline{V}\cdot\nabla\overline{V} = 0 \end{cases}$$

which is of course the Saint-Venant system.

Simpler models can be achieved by making smallness assumptions on the nonlinearity parameter ε.



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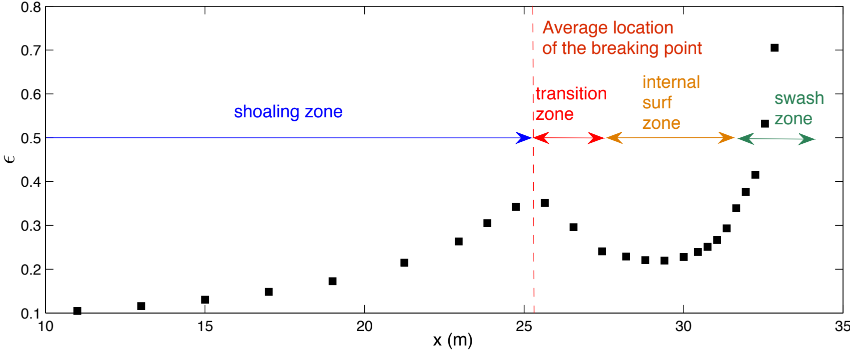

(by M. Tissier, based on experiments by Van Dongeren et al)

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$$\boxed{\varepsilon = O(\mu)}.$$

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(Linear wave equation) $$\boxed{\left\{ \begin{array}{l} \partial_t \zeta + \nabla \cdot \overline{V} = 0 \\ \partial_t \overline{V} + \nabla \zeta = 0 \end{array} \right.}$$

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- The speed in 1 in dimensionless variables, and $\sqrt{gH_0}$ in variables with dimensions.

Small amplitude regime

$$\boxed{\varepsilon = O(\mu)}.$$

- First order model (precision $O(\mu)$). The Saint-Venant system

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0 \\ \partial_t \overline{V} + \nabla\zeta + \varepsilon \overline{V} \cdot \nabla \overline{V} = 0 \end{cases} \qquad (h = 1 + \varepsilon\zeta)$$

becomes after dropping the terms that are $O(\mu)$ in this regime

$$\text{(Linear wave equation)} \qquad \boxed{\begin{cases} \partial_t \zeta + \nabla \cdot \overline{V} = 0 \\ \partial_t \overline{V} + \nabla\zeta = 0 \end{cases}}$$

- The speed in 1 in dimensionless variables, and $\sqrt{gH_0}$ in variables with dimensions.
- Very rough model OK only if $\varepsilon \ll 1$, $\mu \ll 1$ (e.g. tsunami in the ocean)

Small amplitude regime

$$\boxed{\varepsilon = O(\mu)}.$$

Small amplitude regime

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• Second order model (precision $O(\mu^2)$). The Serre/Green-Naghdi equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ (1 + \mu \mathcal{T}[h])\big(\partial_t \overline{V} + \nabla\zeta + \varepsilon \overline{V} \cdot \nabla \overline{V}\big) + \varepsilon\mu \mathcal{Q}[h](V) = 0 \end{cases}$$

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$$\text{(Boussinesq)} \qquad \boxed{\begin{cases} \partial_t \zeta + \nabla \cdot \big((1 + \varepsilon\zeta)\overline{V}\big) = 0 \\ (1 - \frac{\mu}{3}\Delta)\partial_t \overline{V} + \nabla \zeta + \varepsilon \overline{V} \cdot \nabla \overline{V} = 0 \end{cases}}$$

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$$\begin{aligned} \rightsquigarrow -\frac{\mu}{3}\Delta\partial_t V &= \frac{\mu}{3}\Delta\nabla\zeta + O(\mu^2) \\ &= (1-\alpha)\frac{\mu}{3}\Delta\nabla\zeta - \alpha\frac{\mu}{3}\Delta\partial_t V + O(\mu^2) \end{aligned}$$

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- We plug this into the equations and drop $O(\mu^2)$ terms

$$\boxed{\begin{cases} \partial_t \zeta + \nabla \cdot ((1+\varepsilon\zeta)\overline{V}) = 0 \\ (1 - \alpha\frac{\mu}{3}\Delta)\partial_t \overline{V} + \nabla\zeta + \varepsilon \overline{V}\cdot\nabla\overline{V} + \frac{\mu}{3}(1-\alpha)\Delta\nabla\zeta = 0 \end{cases}}$$

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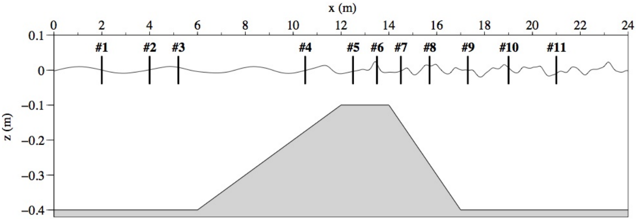
x (m)
z (m)
#1
#2
#3
#4
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#6
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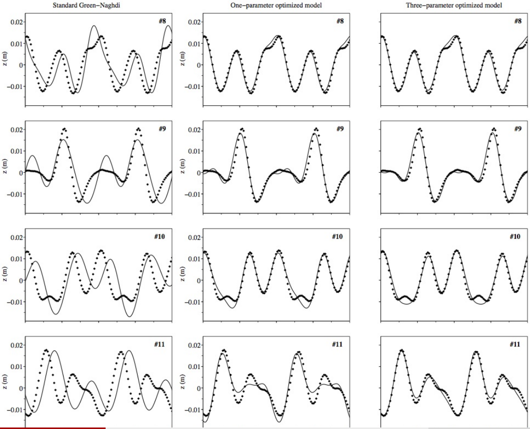
#11


Standard Green–Naghdi
One–parameter optimized model
Three–parameter optimized model
#8
#9
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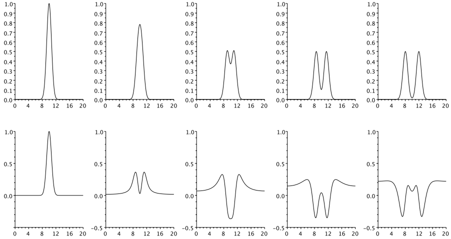

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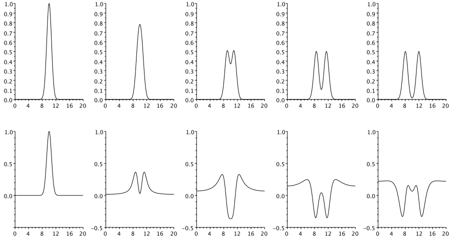

Question

Is it possible to find a scalar model describing the right-going wave?

- We have seen that the first order model, in shallow water, and for small amplitude waves is, in dimension 1

$$\text{(Linear wave equation)} \qquad \begin{cases} \partial_t \zeta + \partial_x \overline{v} = 0 \\ \partial_t \overline{v} + \partial_x \zeta = 0 \end{cases}$$

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- Adding and substracting this two lines, we get

$$\begin{cases} (\partial_t + \partial_x)\dfrac{\zeta + \overline{v}}{2} = 0 \\ (\partial_t - \partial_x)\dfrac{\zeta - \overline{v}}{2} = 0, \end{cases} \Rightarrow \begin{cases} \dfrac{\zeta + \overline{v}}{2}(t,x) = \dfrac{\zeta^0 + \overline{v}^0}{2}(x-t) \\ \dfrac{\zeta - \overline{v}}{2}(t,x) = \dfrac{\zeta^0 - \overline{v}^0}{2}(x+t) \end{cases}$$

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- If initially $\zeta^0 = \overline{v}^0$ then $\zeta = \overline{v}$ for all times
- The solution to the linear wave equation can be written

$$(\partial_t + \partial_x)\zeta = 0, \qquad \overline{v} = \zeta.$$

Right-going solutions to the linear wave equation:

$$\begin{cases} \partial_t \zeta + \partial_x \overline{v} = 0 \\ \partial_t \overline{v} + \partial_x \zeta = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} (\partial_t + \partial_x)\zeta = 0, \\ \overline{v} = \zeta. \end{cases}$$

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Question

What about the next order?
For instance, in the small amplitude regime, what are the "right going waves"?

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Question

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For instance, in the small amplitude regime, what are the "right going waves"?

$\rightsquigarrow$ We look for right-going solutions to the Boussinesq system:

$$\begin{cases} \partial_t \zeta + \partial_x \big((1+\varepsilon\zeta)\overline{v}\big) = 0 \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t \overline{v} + \partial_x \zeta + \varepsilon \overline{v}\partial_x \overline{v} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} (\partial_t + \partial_x)\zeta = ???, \\ \overline{v} = \zeta + ???. \end{cases}$$

$$\text{(Boussinesq)} \quad \begin{cases} \partial_t \zeta + \partial_x\big((1+\varepsilon\zeta)\overline{v}\big) = 0 \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t \overline{v} + \partial_x \zeta + \varepsilon \overline{v}\partial_x \overline{v} = 0 \end{cases}$$

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The idea is to

- Modify the transport equation satisfied by ζ for a RGW

$$\text{(KdV)} \quad \partial_t\zeta + \partial_x\zeta + \varepsilon\beta\zeta\partial_x\zeta + \mu\gamma\partial_x^3\zeta = 0,$$

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- Add a correction to the reconstruction formula for $\overline{v}$,

$$(\text{corr}) \quad \overline{v} = \zeta + \varepsilon w,$$

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- The coefficients β and γ, and the corrector w are chosen such that

$$\begin{array}{ll} & \zeta \text{ solution of (KdV)} \\ + & \overline{v} \text{ given by (corr)} \end{array} \Rightarrow (\zeta, \overline{v}) \text{ solves (Boussinesq) at order } O(\mu^2)$$

$$(\text{Boussinesq}) \quad \begin{cases} \partial_t\zeta + \partial_x\overline{v} + \varepsilon\partial_x(\zeta\overline{v}) = 0 \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t\overline{v} + \partial_x\zeta + \varepsilon\overline{v}\partial_x\overline{v} = 0 \end{cases}$$

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and therefore

$$\boxed{\varepsilon w = \varepsilon \frac{1}{2}(\beta - 2)\zeta^2 + \mu \gamma \partial_x^2 \zeta}$$

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$$\rightsquigarrow \boxed{\beta = \frac{3}{2}, \qquad \gamma = \frac{1}{6}}$$

The right-going solutions of the Boussinesq equation

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- We of course recover the transport equation by neglecting $O(\mu)$ terms.

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- First order models. The Saint-Venant system

$$\text{(Saint-Venant)} \qquad \begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t \overline{V} + \nabla \zeta + \frac{\varepsilon}{2} \nabla |\overline{V}|^2 = 0. \end{cases}$$

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- The scalar (CH) equation for right going waves of medium amplitude

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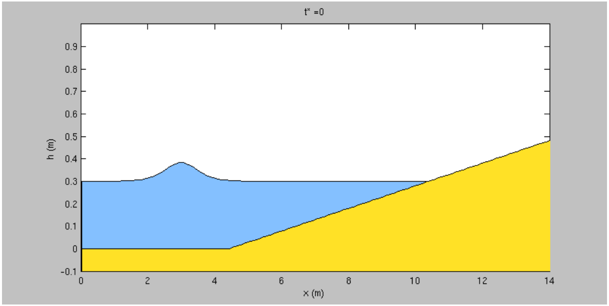
is nonlinear enough to contain "wave breaking" singularities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x (m)
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0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
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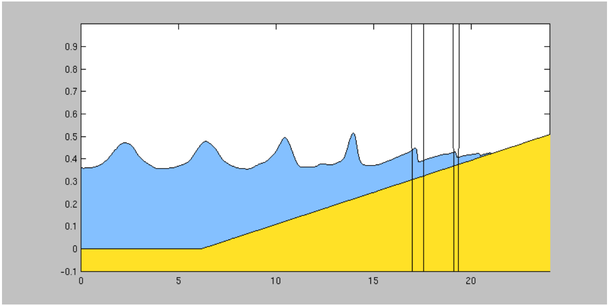
2
4
6
8
10
12
14


0.9
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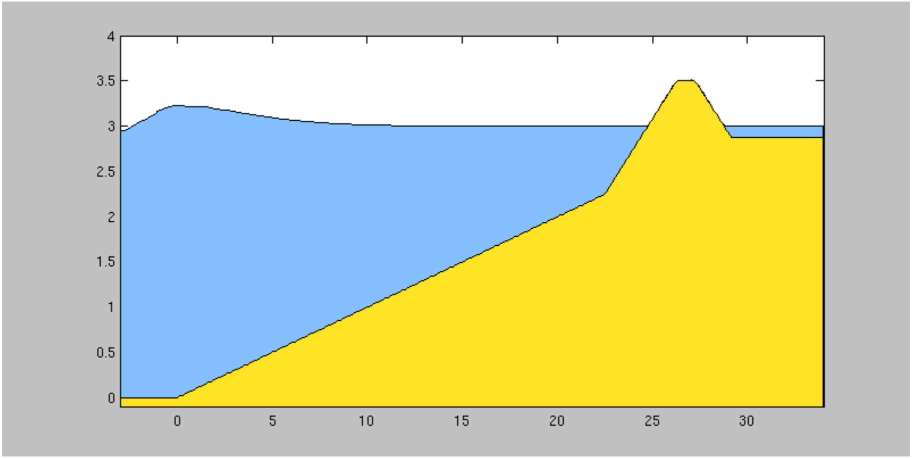
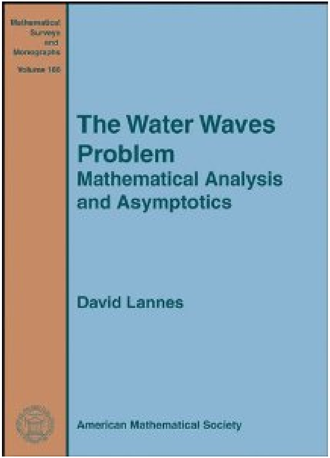


Tissier et al. 2012


Mathematical
Surveys
and
Monographs
The Water Waves Problem
Mathematical Analysis and Asymptotics
David Lannes
American Mathematical Society

(H1) The fluid is homogeneous and inviscid

(H2) The fluid is incompressible

(H3) ~~The flow is irrotational~~

(H4) The surface and the bottom can be parametrized as graphs above the still water level

(H5) The fluid particles do not cross the bottom

(H6) The fluid particles do not cross the surface

(H7) There is no surface tension and the external pressure is constant.

(H8) The fluid is at rest at infinity

(H9) The water depth is always bounded from below by a nonnegative constant

$$\begin{aligned} \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} &= -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z, \\ \nabla_{X,z} \cdot \mathbf{U} &= 0, \\ P_{|_{z=\zeta}} &= P_{atm} \end{aligned}$$

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Irrotational case

- We get from curl $\mathbf{U} = 0$ that $\mathbf{U} = \nabla_{X,z}\Phi$

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- We get from curl $\mathbf{U} = 0$ that $\mathbf{U} = \nabla_{X,z} \Phi$
- We replace Euler's equation on $\mathbf{U}$ by Bernoulli's equation on Φ

$$\partial_t \Phi + \frac{1}{2} |\nabla_{X,z} \Phi|^2 + gz = -\frac{1}{\rho} (P - P_{atm})$$

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- We eliminate the pressure by taking the trace on the interface

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- We reduce the problem to an equation on ζ and $\psi(t,X) = \Phi(t,X,\zeta(t,x))$.

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One has curl $\mathbf{U} = \omega \neq 0$ and

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- One can remark that

$$
\begin{aligned}
(\nabla_{X,z}P)_{|_{z=\zeta}} &= \begin{pmatrix} \nabla(P_{|_{z=\zeta}}) \\ 0 \end{pmatrix} + N\partial_z P_{|_{z=\zeta}}\\
&= 0 + N\partial_z P_{|_{z=\zeta}}
\end{aligned}
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$\rightsquigarrow$ One can eliminate the pressure by

1. Taking the trace of Euler's equation at the surface
2. Take the vectorial product of the resulting equation with N.

$\rightsquigarrow$ This leads to an equation on the tangential part of the velocity at the surface

One has

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g\mathbf{e}_z$$

and

$$(\nabla_{X,z} P)_{|_{z=\zeta}} = N \partial_z P_{|_{z=\zeta}}, \qquad \text{with} \quad N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}.$$

⇝ One can eliminate the pressure by

1. Taking the trace of Euler's equation at the surface
2. Take the vectorial product of the resulting equation with N.

⇝ This leads to an equation on the tangential part of the velocity at the surface

Notation

With $\underline{U} = (\underline{V}, \underline{w}) = \mathbf{U}_{|_{z=\zeta}}$, we write

$$U_{\|} = \underline{V} + \underline{w} \nabla\zeta \quad \text{so that} \quad \underline{U} \times N = \begin{pmatrix} -U_{\|}^{\perp} \\ -U_{\|}^{\perp} \cdot \nabla\zeta \end{pmatrix}$$

$$\left\{\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_{X,z}\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}P - g\mathbf{e}_z\right\}_{|z=\zeta} \times N$$

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$\Downarrow$ (with some computations)

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How do we generalize to the rotational case?

We decompose $U_{\parallel}$ into

$$\boxed{U_{\parallel} = \nabla\psi + \nabla^{\perp}\widetilde{\psi}}$$

We have found

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$$\boxed{\partial_t(\underline{\omega}\cdot N - \nabla^\perp\cdot U_\parallel) = 0},$$

which is trivially true and does not bring any information

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$$(\text{ZCS}) \qquad \begin{cases} \partial_t \zeta - \underline{U} \cdot N = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\underline{U}\cdot N + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0 \\ \omega = 0. \end{cases}$$

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- We are therefore led to solve

$$\begin{cases} \mathrm{curl}\ \mathbf{U} = & \omega & \text{in} \quad \Omega \\ \mathrm{div}\ \mathbf{U} = & 0 & \text{in} \quad \Omega \\ U_{\parallel} = & \nabla\psi + \nabla^{\perp}\Delta^{-1}(\underline{\omega} \cdot N) & \text{at the surface} \\ \mathbf{U}_{|_{z=-H_0}} \cdot N_b = & 0 & \text{at the bottom.} \end{cases}$$

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It is possible to solve the div-curl problem and therefore we wave

Theorem (A. Castro, D. L. '13)

1. *The $(ZCS)_{gen}$ equations are in closed form.*
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$\rightsquigarrow$ The rotational part of the velocity is then approximated by

$$\mathbf{U}^\mu_{rot} \sim \nabla^\mu \times \mathbf{A}_{app}$$

- We have therefore an asymptotic expansion of the velocity field in the fluid domain,

$$V(X,z) = \underline{V}(X) + \sqrt{\mu}\int_z^{\varepsilon\zeta} \omega_h^{\perp}(X,z)dz + O(\mu),$$

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- We only need to know ω_h at order $O(\sqrt{\mu})$.

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- No physical assumption has been made. The only assumption is $\mu \ll 1$.

Conclusion

In general, the standard St-Venant model is not correct at order $O(\mu)$ in presence of vorticity.