HALF COHEN

Jindrich Zapletal
University of Florida
Academy of Sciences, Czech Republic

Motivating problem.

Iterating proper posets with countable support, if iterands add no Cohen reals, can the iteration add Cohen reals?

- successor stage
- stage ω

Infinitely equal real

Definition. A function $x \in \omega^\omega$ in a generic extension is an *infinitely equal real* if it has nonempty intersection with every function in the ground model.

Fact. (Bartoszynski) Adding an infinitely equal real twice produces a Cohen real.

Question. Is there a proper poset adding an infinitely equal real but not a Cohen real?

YES.

σ-ideals σ-generated by closed sets

Let X be a Polish space, let I be a σ-idealon X σ-generated by closed sets, let P_I be the poset of I-positive Borel sets with inclusion.

Fact. (Solecki) G_δ sets are dense in P_I.

Fact. The quotient poset P_I is

- proper;
- preserves Baire category;
- every intermediate forcing extension is given by a single Cohen real.

Examples

- I is countable subsets of 2^ω–Sacks forcing;
- I is σ-generated by compact subsets of ω^ω–Miller forcing;
- I is σ-generated by sets of finite packing measure–forcing is bounding, adds no independent reals;
- I is σ-generated by closed Lebesgue null sets–forcing is not bounding, adds independent reals.

Main theorem

Let K be any compact metric space, infinite-dimensional, with every closed subset either zero-dimensional or infinite dimensional. (Henderson, Zarelua, Walsh, Dranishnikov. . . 1960's and onward)

Theorem. Let I be the σ-ideal on K σ-generated by zero-dimensional compact sets. Then the quotient poset P_I is proper, adds an infinitely equal real, and no Cohen real. In fact, the P_I extension is a minimal forcing extension.

Open questions. Is there a reasonable combinatorial characterization of P_I? Does P_I depend on the initial choice of K? How?

Adding infinitely equal real

(Banakh and coauthors) There is a Borel bijection $\pi : \omega^\omega \to [0,1)$ such that for every $x \in \omega^\omega$, $\pi''\{y \in \omega^\omega : x \cap y = 0\}$ is nowhere dense–and so its closure is zero-dimensional.

Consider $\rho = \pi^\omega : \omega^{\omega\times\omega} \to [0,1)^\omega$. For every $x \in \omega^{\omega\times\omega}$, $\rho''\{y \in \omega^{\omega\times\omega} : x \cap y = 0\}$ is a subset of a product of compact zero-dimensional spaces, which is compact zero-dimensional.

Embed K into $[0,1/2]^\omega$, and consider the name for ρ^{-1} of the generic point. It is a name for an infinitely equal real.

Not adding Cohen real

Calibration. The ideal I is *calibrated*: If $C \subset K$ is closed I-positive and $D_n : n \in \omega$ are closed in I, then there is a closed I-positive $D' \subset D \setminus \bigcup_n D_n$. *Proof.* $\bigcup_n D_n$ is zero-dimensional, and so covered by a G_δ zero-dimensional set. The rest of C is non-zero-dimensional and F_σ.

Minimal real. (Pol–Zakrzewski) Calibrated σ-ideals of closed sets on compact spaces have one-to-one or constant property: every Borel function on I-positive Borel set is one-to-one or constant on an I-positive Borel subset. (KSZ) In fact, total canonization of analytic equivalence relations. *Proof.* A demanding fusion argument.

Finite dimension would not work

- Suppose that K is finite dimensional, so $K \subset [0,1]^n$ for some $n \in \omega$.

- Let $\langle x_i : i \in n \rangle$ be the generic point. I claim that one of x_i must be a Cohen generic point in $[0,1]$.

- otherwise, there would be closed nowehere dense sets C_i in the ground model such that $x_i \in C_i$. But then, $\langle x_i : i \in n \rangle \in \prod_i C_i$ which is compact and zero-dimensional in the ground model. Contradiction.

Related generalities

Theorem. For every calibrated σ ideal I of closed sets, there is an I-positive G_δ-set B such that relatively-in-B closed sets are dense in P_I below B.

Theorem. If I is a σ-ideal σ-generated by closed sets on Polish X such that no infinitely equal real is added by P_I, then every alternative Polish topology with same Borel structure coincides with the original one on a positive G_δ set.