Universality in classes of Banach spaces and compact spaces

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Outline

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- $B_{l_2(\kappa)}$ is injectively universal for $\mathbb{UE}_\kappa$. In particular $C(B_{l_2(\kappa)})$ is weakly universal for $\overline{\mathcal{H}}_\kappa$.
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- **(M. Bell; 2002) There is a compact $K \in \mathbb{UE}_{\omega_1}$ which is not a continuous image of any space $A(\kappa)^{\mathbb{N}}$. In particular the space $A(\kappa)^{\mathbb{N}}$ is not universal in $\mathbb{UE}_\kappa$ for an uncountable κ.**

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- (M. Bell; 2000) It is consistent that there is no universal uniform Eberlein compact space of weight ω_1
- **(M. Bell; 2000) Assume $\kappa = 2^{<\kappa}$. Then, there is a universal uniform Eberlein compact space UE_κ in $\mathbb{UE}_\kappa$ and so $C(UE_\kappa)$ is isometrically universal Banach space for $\overline{\mathcal{H}}_\kappa$. In particular, the above results hold for $\kappa = 2^\omega = \omega_1$ if we assume the continuum hypothesis.**

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- **(C. Brech, P.K.; To appear in PAMS) It is consistent that there is no Banach space of density 2^ω or ω_1 which contains isomorphic copies of all Banach spaces from $\mathbb{UE}_{2^\omega}$ or from $\mathbb{UE}_{\omega_1}$ respectively. In particular, it is consistent that there is no universal Banach space neither in $\overline{\mathcal{H}}_{\omega_1}$ nor in $\overline{\mathcal{H}}_{2^\omega}$**

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Questions

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- **Is it consistent that there is a universal graph of size ω_1 (2^ω) but there is no universal Banach space for $\mathcal{B}_{\omega_1}$ ($\mathcal{B}_{2^\omega}$)? Or any other class we consider here?**

Scattered compact space and Asplund Banach spaces

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- **(Szlenk 1968, Wojtaszczyk 1970, Hajek, Lancien, Montesinos 2007) For any cardinal κ there is no universal reflexive or Asplund Banach space of density $\leq \kappa$**

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- **(A. Avilés; 2005) strongly associated if $\kappa < \mathfrak{b}$ and not strongly associated if $\kappa \geq \mathfrak{b}$;**

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Eberlein Compact and weakly compactly generated (WCG) Banach spaces I

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- **(Amir, Lindenstrauss; 1968; Rosenthal; 1974) $\mathbb{E}_\kappa$ and $\mathcal{WCG}_\kappa$, are K-associated and are not strongly associated.**

Eberlein Compact and weakly compactly generated (WCG) Banach spaces II

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- **(Argyros, Benyamini; 1987) If $\kappa^\omega = \kappa$ or $\kappa = \omega_1$ then there is no weakly universal Eberlein compact of weight κ nor a universal WCG Banach space of density κ. If κ is a strong limit cardinal of countable cofinality, then there is a universal Eberlein compact of weight κ, and so, there is a universal WCG Banach space of density κ.**

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- **(Argyros, Mercourakis; 1993)** $K \in \mathbb{CM}_\kappa$ **if and only if** $C(K) \in \mathcal{WLD}_\kappa$

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- **(Argyros, Mercourakis, Negrepontis; 1988) Assuming MA$+\neg$CH the classes $\mathbb{C}_\kappa = \mathbb{CM}_\kappa$ and $\mathcal{WLD}_\kappa$ are strongly associated for any uncountable cardinal κ.**

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- (Argyros, Mercourakis; 1993) $K \in \mathbb{CM}_\kappa$ if and only if $C(K) \in \mathcal{WLD}_\kappa$
- (Argyros, Mercourakis; 1993) $X \in \mathcal{WLD}_\kappa$ if and only if $B_{X^*} \in \mathbb{C}_\kappa$,
- (Argyros, Mercourakis, Negrepontis; 1988) Assuming MA$+\neg$CH the classes $\mathbb{C}_\kappa = \mathbb{CM}_\kappa$ and $\mathcal{WLD}_\kappa$ are strongly associated for any uncountable cardinal κ.
- Assuming *CH*
  1. (Argyros, Mercourakis, Negrepontis; 1988) $\mathbb{C}_{2^\omega}$ is not associated with any class of Banach spaces
  2. (Kalenda, Plebanek; 2002) $\mathcal{WLD}_{2^\omega}$ is not associated with any class of compact spaces
  3. (Kalenda, Plebanek; 2002) $\mathcal{L}_{2^\omega}$ is not associated with any class of compact spaces
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Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces III

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- **There is no universal weakly Lindelöf Banach space of density 2^ω.**

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