Large Cardinals:

- Who are they?
- What are they doing here?
- Why wont they go away?

Fields Institute
November 7, 2012

Matt Foreman
UC Irvine

Apocryphal quote:

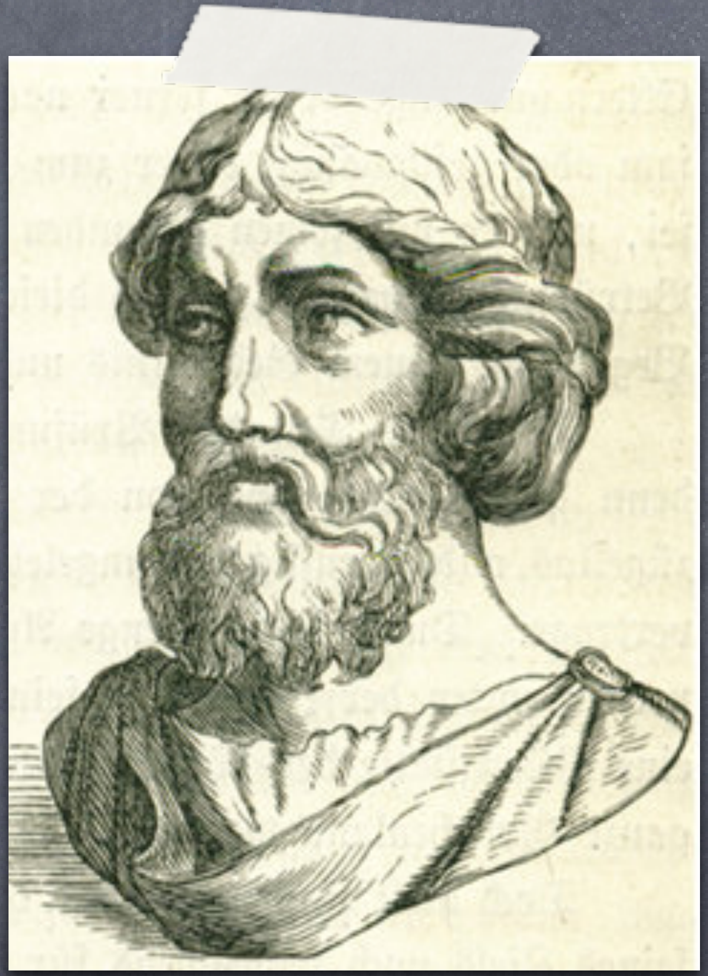
Infinity is a fathomless gulf into which all things vanish.

Marcus Aurelius

121-180 AD

Irrational numbers: Not built from finite objects by algebraic operations

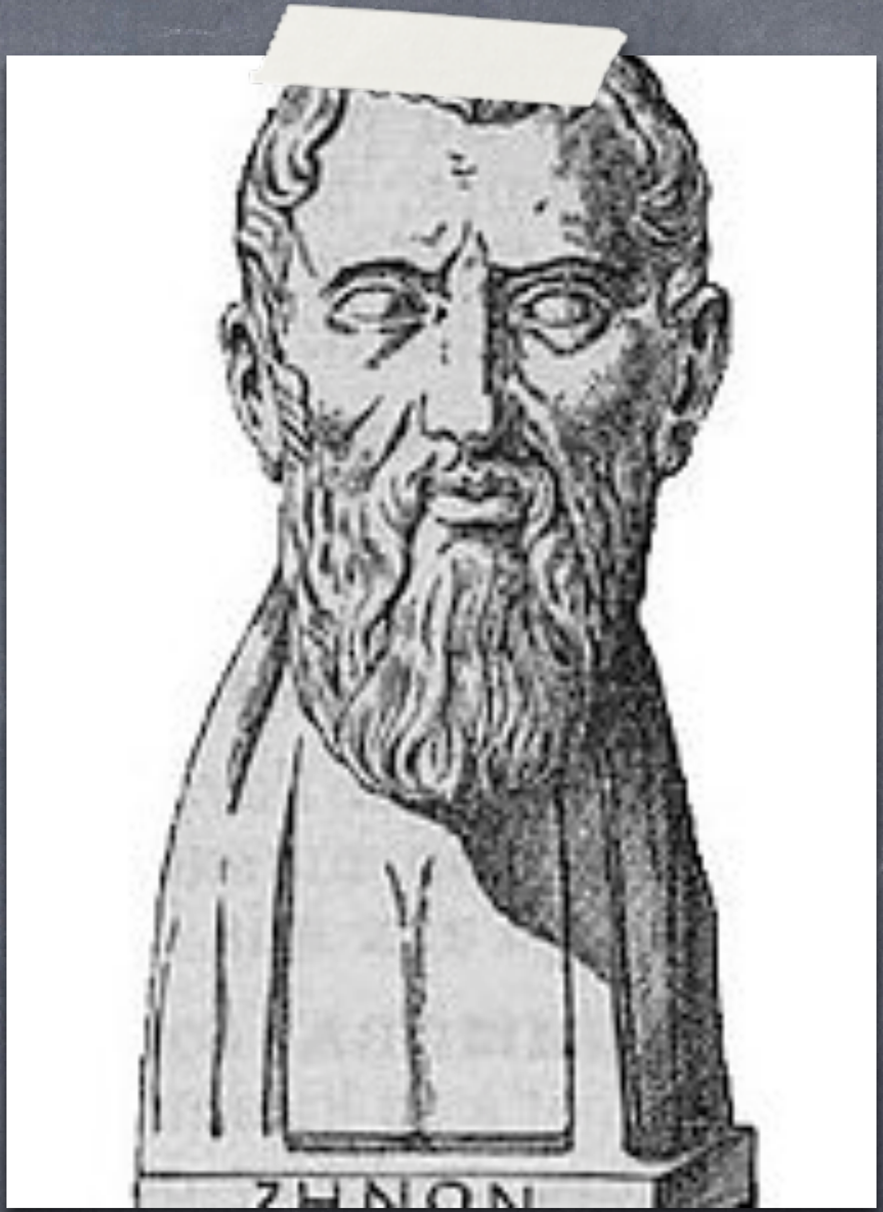
According to tradition Hypassus was thrown into the sea and drowned in reaction to his discovery of irrational numbers.



Hypassus of Metapontum
5th century BCE

Zeno's Paradoxes

- Achilles and the Tortoise
- The Arrow paradox



Zeno of Elia
490-430 BCE

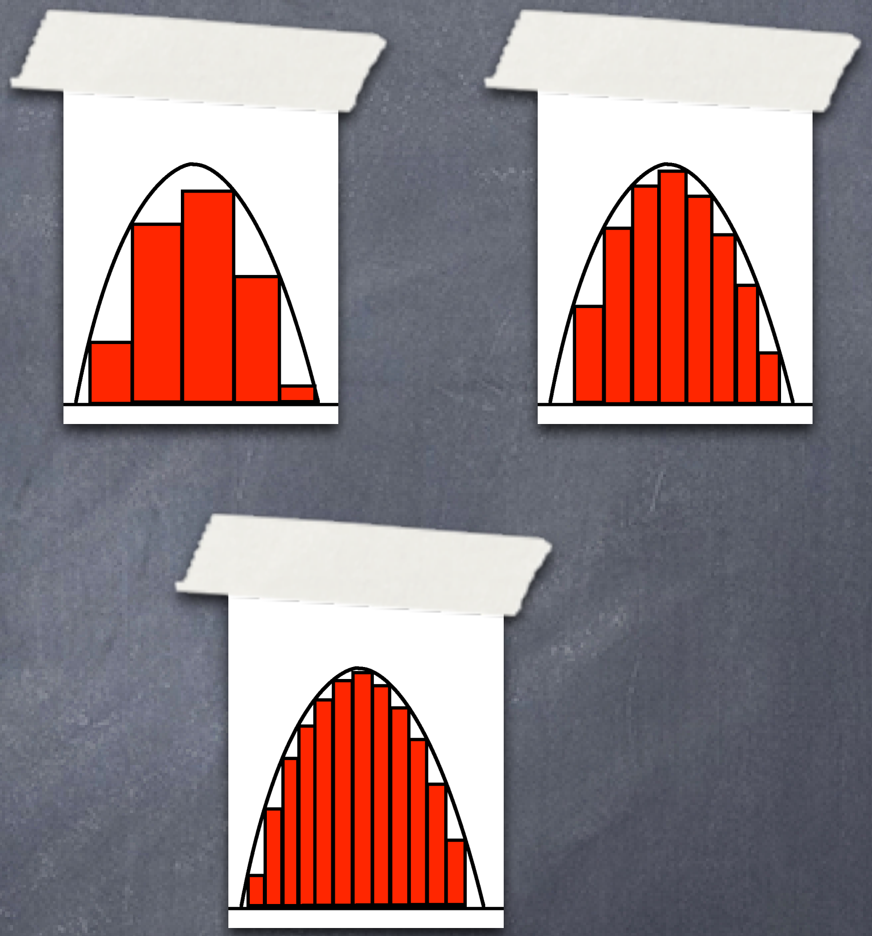
Eudoxus of Cnidus

- Became worried about the nature of mathematical objects
- Rebelled against arithmetization and preferred to use purely geometrical notions taking ideas such as "magnitude" as primitive

Eudoxus of Cnidus
410-355 BCE (?)

Eudoxus of Cnidus

- In doing so he hoped to be rid of "incommensurables" (irrational numbers)
- However, was a proponent of the method of exhaustion



method of Exhaustion

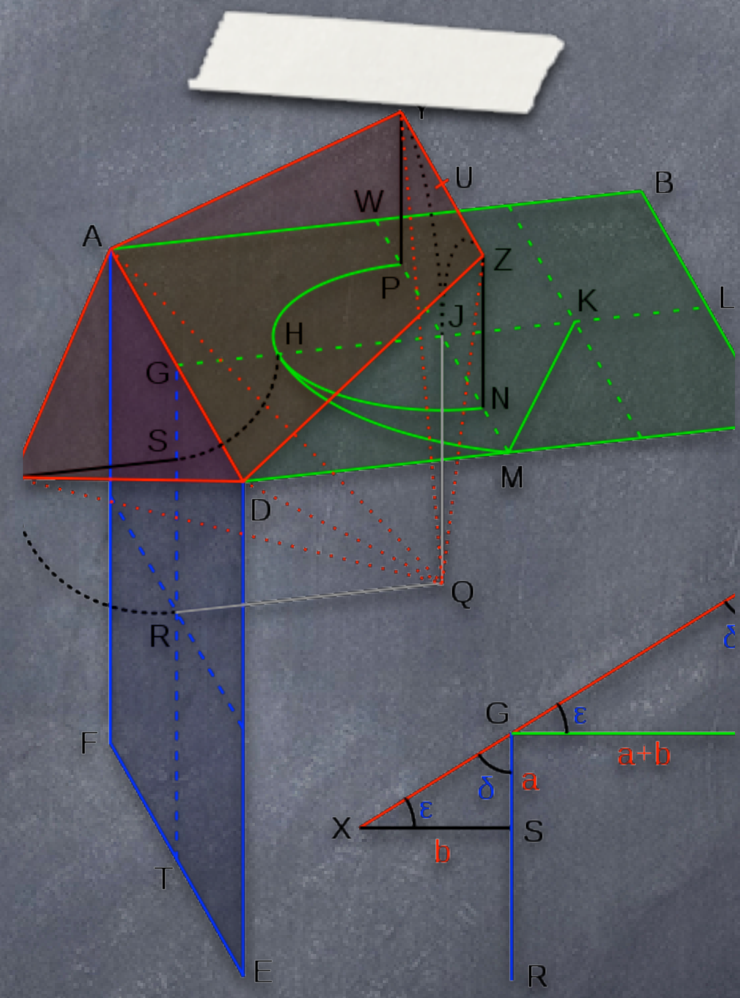
Aristotle

- What is a legitimate argument?
- Is this an objective question?
- Syllogistic method

Aristotle
384-322 BCE

Euclid of Alexandria

- Attempted to axiomatize mathematics (meaning geometry)
- Axioms were supposed to be self evident
- Axioms were supposed to be complete

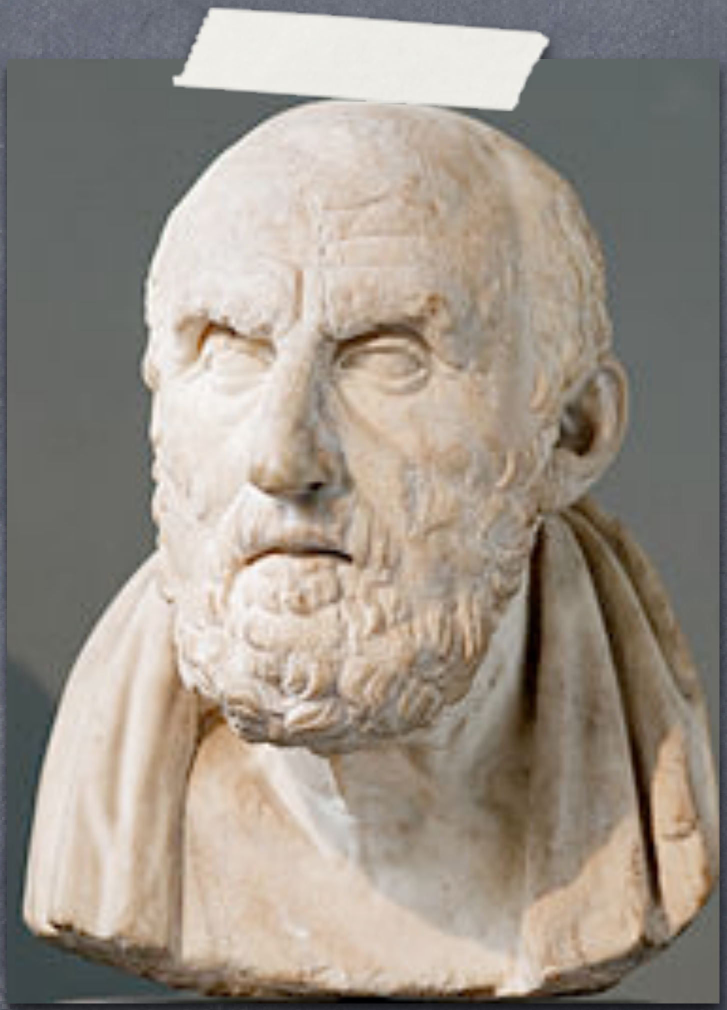


Dodecahedron

Aristotle's logic was not adequate, even for Euclid's Geometry

Chrysippus of Soli

First "modern" logical system is due to the Stoics who developed the Propositional Calculus



Chrysippus of Soli
279-206 BCE

Truth functional connectives and the "Five indemonstrables"

The Basic Questions

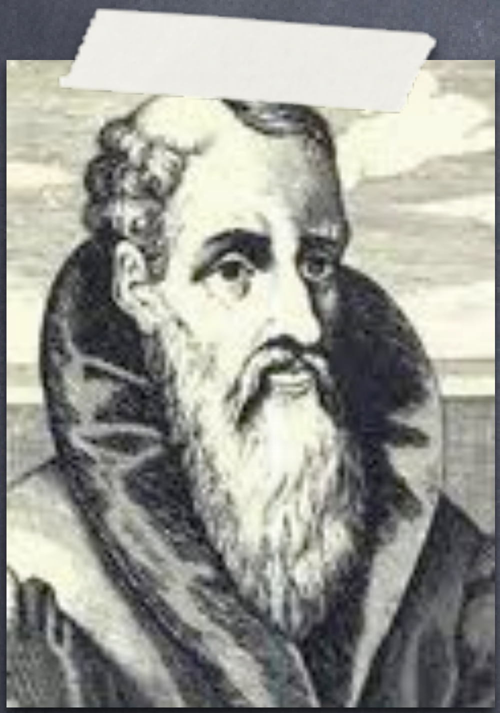
- Are there fundamental truths which form a basis for mathematical knowledge?
- If there are, how does mathematics flow from these truths? (What is a proof?)
- How does geometry relate to arithmetic? Is it legitimate to argue using inherently infinite objects?

The Basic Questions

- What is a proof?
- What are the assumptions one starts with? (What are the Axioms?)
- How does one unify mathematics in one set of assumptions?

Skip ahead a couple of millennia

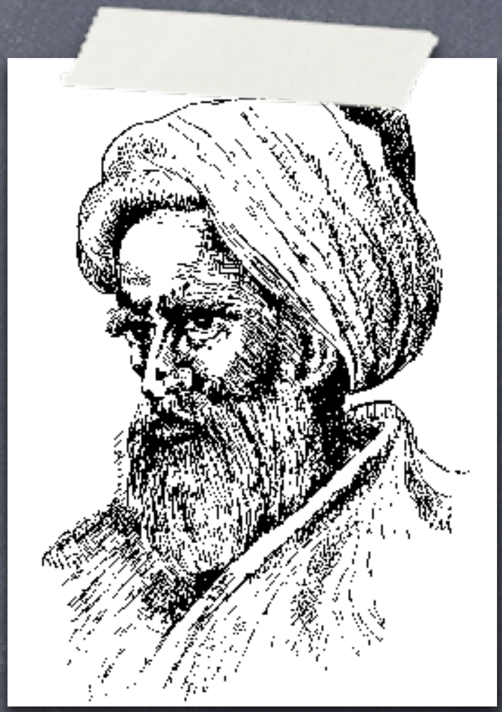
(ignoring some truly romantic figures)



Boethius

Al-Khwarizmi

Peter Abelaard



Alhazen

Rene Descartes

- Developed "Analytical Geometry"
- Heavy emphasis on "demonstration" as a means of discovery

Rene Descarte
1596-1650

Not Everyone was convinced

"Numbers imitate space, which is of such a different nature"

Blaise Pascal
1623-1662

Meanwhile ... Mathematics goes on.

But the issues become more and more difficult to ignore.

19th Century mathematics

- Completeness properties of the real numbers …
- What is a function?
- Alternatives to Euclidean Geometry
- Do we need to PROVE that 2+2=4? From what??

19th Century mathematics

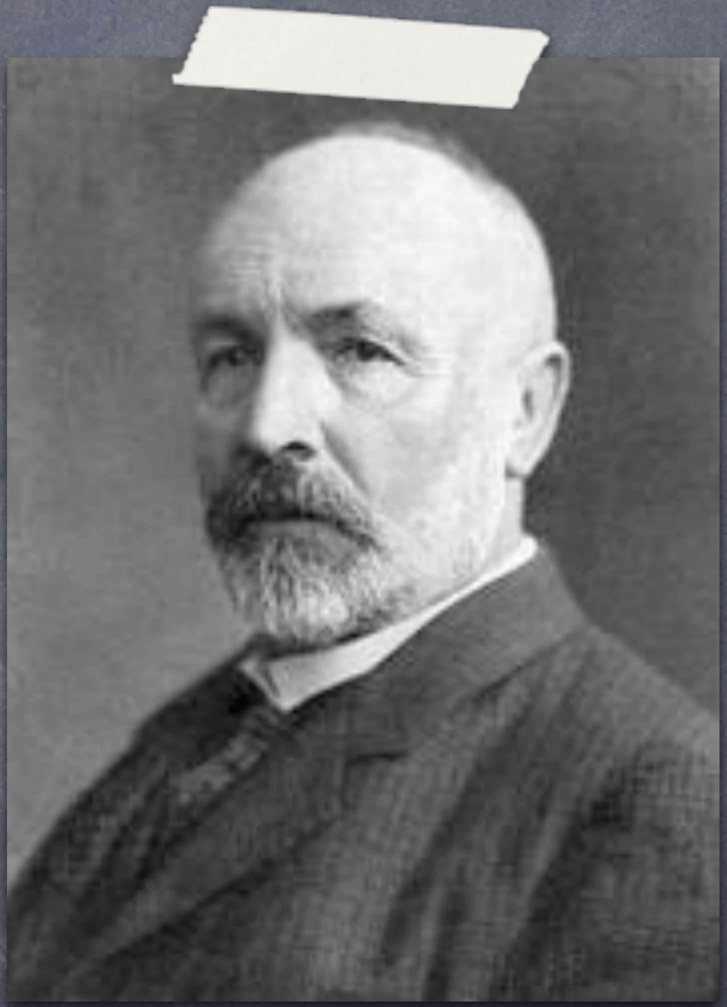
- The widespread acceptance of "imaginary" numbers
- Abstract mathematical structures with no obvious "physical" interpretations (e.g. Groups)
- Power series solutions to equations

- Formal objects (Such as formal power series)
- constructions of functions as limiting objects
- The definitions of "limit" even for a sequence of real numbers.
- etc.
- etc.

And just when it seemed like things couldnt get any worse ....

Studying properties of trigonometric series, Cantor made a dramatic discover:

There are different sizes of infinity!!

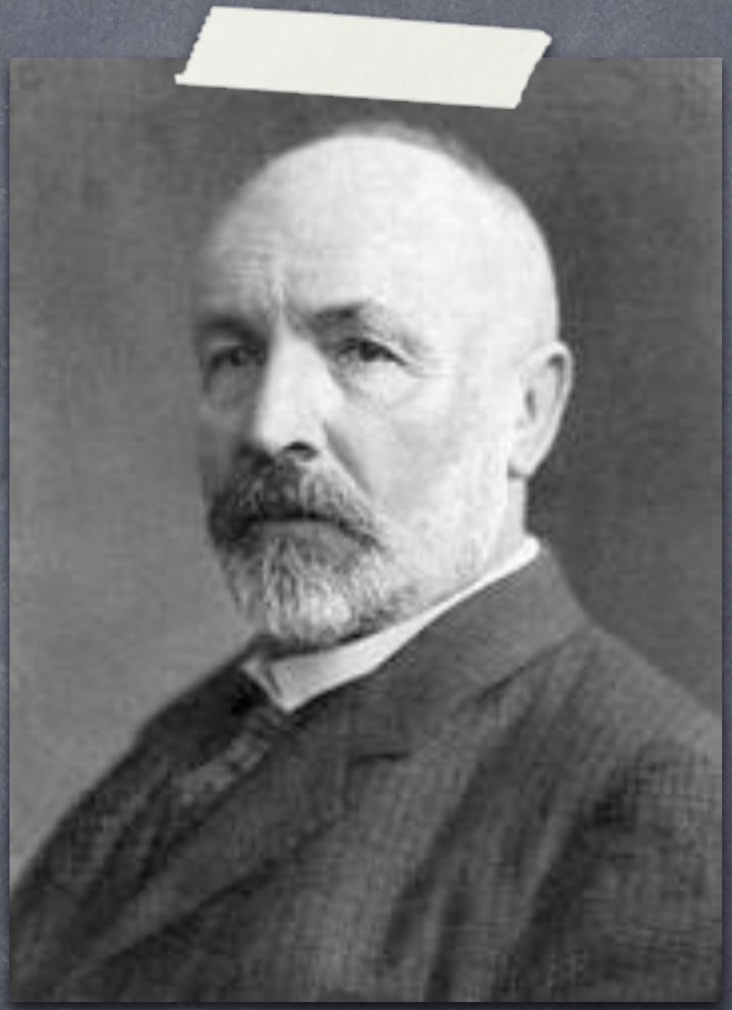


George Cantor
1845-1918

And more:

There are (at least) two different kinds of infinite number:

- Cardinals
- Ordinals



George Cantor
1845-1918

The Well-ordering principle

Every set can be well-ordered

An attempt at a Solution to the three puzzles

Frege had developed a broad conception of logic, in which Arithmetic was part of logic and didn't need axioms.

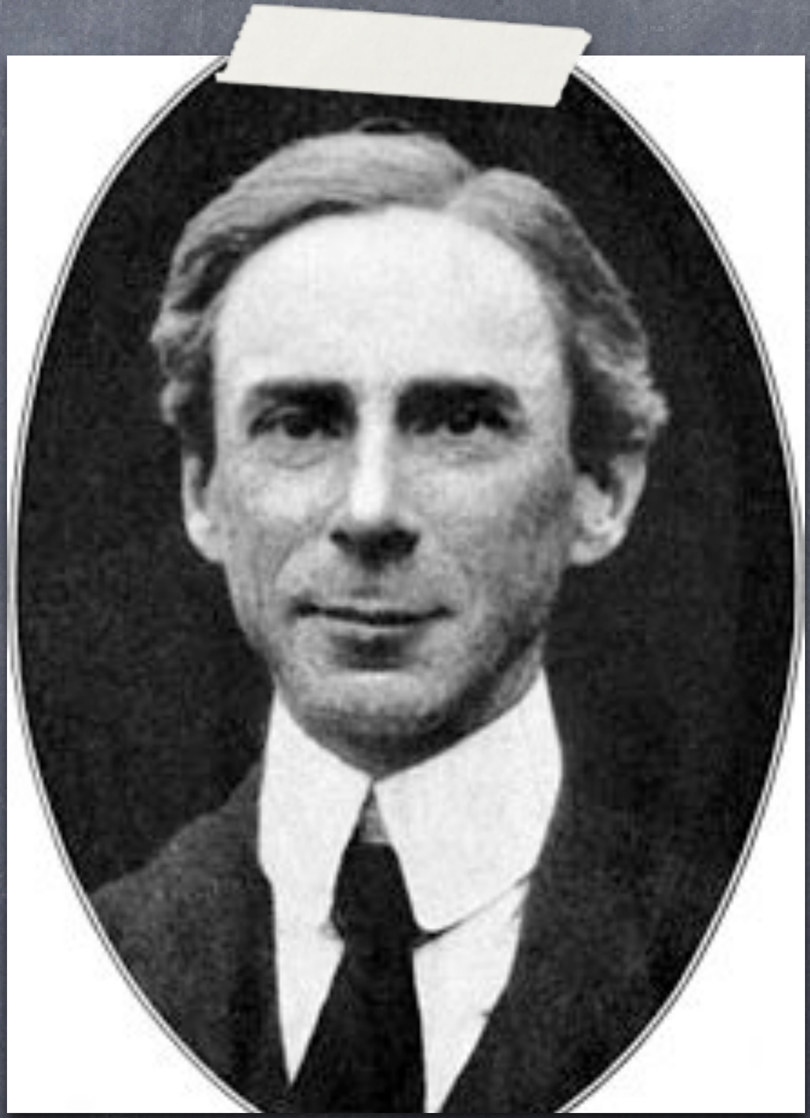
Frege
1848-1925

But it doesn't work

Russell adapted arguments of Cantor to show that Frege's system is

INCONSISTENT.

"Russell's Paradox"

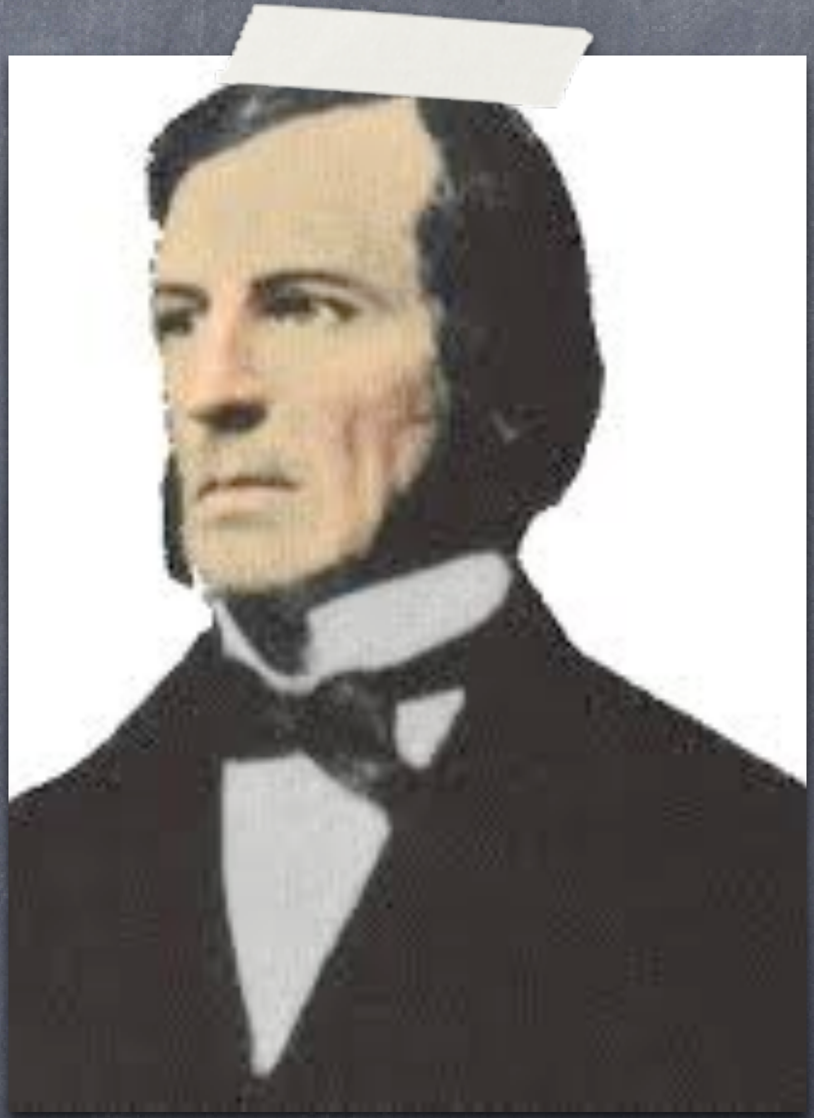


Russell
1872-1970

Mathematics $\not\subseteq$ Logic

Rescuing logic from theology

- Boole realized that "laws of thought" can be studied mathematically.
- Boole explained how the Propositional Calculus (and more) can be understood in algebraic structures: Boolean Algebras



George Boole
1815-1864

Earlier in the 19th century ...

Modern First Order Logic

What emerged from the work of Boole, Frege, Skolem and others was an understanding of what "formal logic" means.

A special case became the "gold standard":

Modern First Order Logic

First order logic has a rigorous well-defined mathematical notion of proof.

"A proof of B from assumption A is a finite string of symbols such that ...."

where "...." is concrete and uncontroversial.

Semantics of First order Logic

Q: If a proposition is a formal mathematical object, what does it mean for a proposition to be "True" in a structure?

Tarski clarified this by giving a mathematical definition of "truth".

Alfred Tarski
1901-1983

Godel's Completeness Theorem

Let A and B be propositions. Godel showed that:

If every structure satisfying A also satisfies B, then

there is a first order PROOF that B follows from A.

Kurt Godel
1906-1978

First Order Logic

- Proofs and propositions are easily and uncontroversially recognizable.
- There is a clear understanding of the relationship between a mathematical structure and the formal propositions that hold in that structure.
- It gives a satisfactory model of what mathematicians actually "do". They give rigorous proofs that have formal proofs as normative ideals
- If B always holds when A does, then there is a proof of B assuming A.

Three Puzzles

- What is a proof?
- Proof FROM WHAT ASSUMPTIONS?
- Assumptions be comprehensive enough to include all standard mathematical objects

We've solved one:

- What is a proof?

A formal proof means a proof in

First Order Logic

Mathematical knowledge

=

First order logic +

Assumptions

What are the assumptions?

What SHOULD happen?

Assumptions SHOULD

involve a simple primitive notion that is easy to understand and can be used to "build" or develop all standard mathematical objects,

- be evident,
- be complete in that they settle all mathematical questions,
- be easily recognized as part of a recursive schema.

Zermelo-Frankel Set Theory with AC

- there is an infinite set
- if X exists then ∪X exists
- if X, Y exist then so does {X,Y}
- if X exists the P(X) exists
- if X exists and f is a definable functional then with domain X, then range of f exists
- X=Y iff X and Y have the same elements
- AC
- For all X there is a Y∈X with X∩Y empty

Why Sets?

Why these axioms?

- mostly self evident
- really a compromise

We have logic, we have axioms, but
do we have mathematics??

We need to make a common playground for all mathematical objects: it is a place where the arithmetic and the geometric can interact.

An imperfect, but helpful analogy

Operating system		Set Theory
————————	≈	————————
High level programming Language		Mathematics

Is this the end of the story?

For example: Are all mathematical truths provable in ZFC?

Is ZFC the final arbiter of mathematical truth?

An collection of assumptions is

COMPLETE

if it either proves or refutes every mathematical statement.

The opposite of completeness is independence: a proposition P is

independent

of a collection A of assumptions if A does NOT resolve P

Godel's Incompleteness Theorems

- If A is a recursive, complete collection of assumptions then A is inconsistent.
- If A is recursive, consistent and strong enough to derive basic number theory then

A cannot prove the statement:
A is consistent.

Et Alors??

- We've conservatively constructed an axiom system that evidently consistent--not worried about that.
- Maybe the only unresolvable statements are "philosophical".

Hilbert's First Problem

- Cantor's diagonal argument shows that the real numbers have larger cardinality than the natural numbers.

- Slightly different arguments show that there must be an uncountable ORDINAL.

David Hilbert
1862-1943

The Continuum Question

Is there a bijection between the real numbers and the first uncountable cardinal?

The Continuum Question

Equivalently:

Is there an subset X of the real numbers of cardinality **between** the natural numbers and the real numbers?

Godel's L

In the 1930's Godel showed that IF there is an example of ZF then there is a canonical minimal example of ZFC.

"L" is plays a role in set theory analogous to the Rationals for characteristic 0 fields.

Godel's L

Godel showed that L satisfies both the Continuum Hypothesis and the Axiom of Choice.

Forcing

Paul Cohen invented a general method for transforming one example of ZF (or ZFC) to another. The method is called Forcing.

In many ways it is analogous to adding a root of a polynomial to a field.

Paul Cohen
1934-2007

The first use of forcing

Cohen used forcing to show the following result:

Any model of ZFC can be transformed into a model of ZFC where the continuum hypothesis fails.

A REAL independence result

The continuum hypothesis cannot be settled by the axioms of set theory

(ZFC).

How widespread is this problem?

Virtually every area of mathematics that inherently involves infinite combinatorics is now known to suffer from independence phenomena.

More about this in the second lecture

How do we deal with this?

Replace previous goals for our axiom system. Find assumptions that:

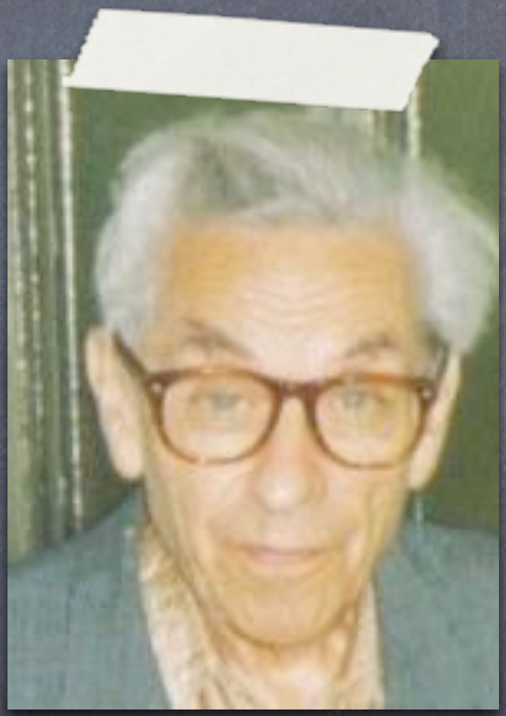
- are in accord with the intuitions of mathematicians well versed in the appropriate subject matter
and
- describe mathematics to as large an extent as is possible.

Extend ZFC in appropriate ways

Find assumptions that are robust and parsimonious and that have consequences that accord with the general picture of the mathematical world.

Starting in the early 20th century, set theory developed two distinct streams, exemplified by:

Nikolai Luzin
1883-1950



Paul Erdos
1913-1996

Corresponding to these two traditions were two extensions of ZFC

- Descriptive Set Theory: "Determinacy Axioms"
- Combinatorial Set Theory: "Large Cardinal Axioms"

Determinacy Axioms

Let A be a subset of the unit interval. Two players take turns playing either 0 or 1.

The result is an infinite sequence x of 0's and 1's. Player 1 wins if the number whose binary sequence is coded x belongs to A.

Determinacy Axioms

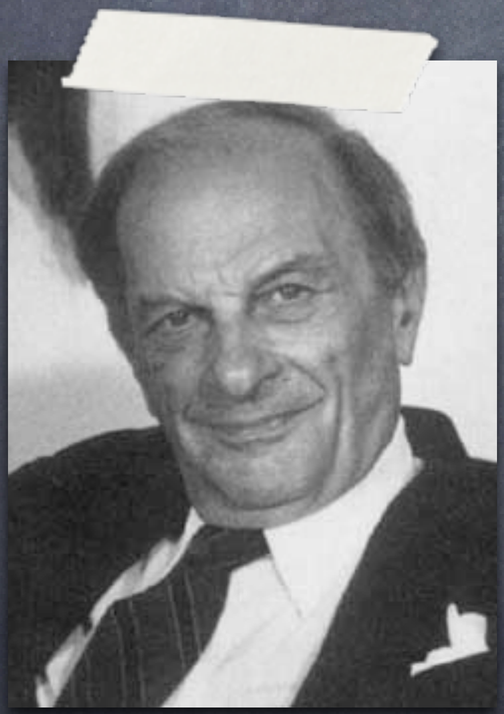
The Axiom of determinacy for a collection S of subsets of the unit interval says:

For each set $A \in S$, either player I or player II has a winning strategy.

Large Cardinal Axioms

Large Cardinal Axioms posit sets that have many of the properties of the whole mathematical universe.

Some representative figures



Stanislaw Ulam
1909–1984

Jan Mycielski

Robert Solovay

Virtues and Drawbacks

Determinacy:

- Virtues: easy to state, settles most problems in Descriptive Set Theory
- Drawbacks: strong versions are inconsistent with AC. Moreover, it is hard to argue for a priori.

Virtues and Drawbacks

Large Cardinals:

- Virtues: A priori arguments in their favor; continue the tradition of the expansion of mathematical objects
- Drawbacks: They involve very large sets (Duh...)

Worst possible situation

Competing axiom systems, no apparent connection, each with its own mathematical constituency.

Happy Ending

Donald Martin

John Steel

W. Hugh Woodin

Unification!

Large Cardinals imply the

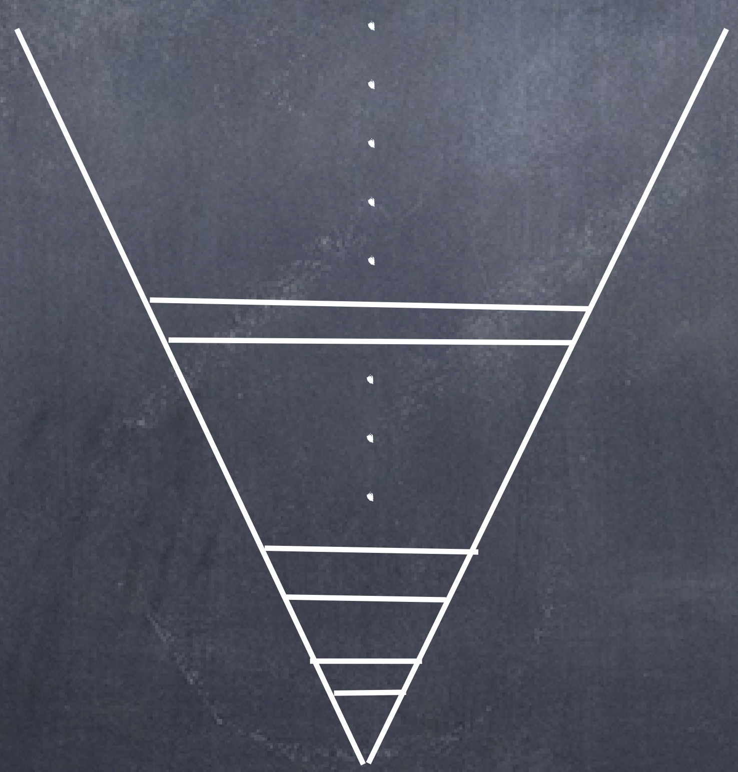
Axioms of Determinacy!!

A little more color

Give a very loose description of Large Cardinal Axioms

Start with a basic description of the mathematical universe

The mathematical universe is built by starting with the emptyset and iterating the Power set operation transfinitely.

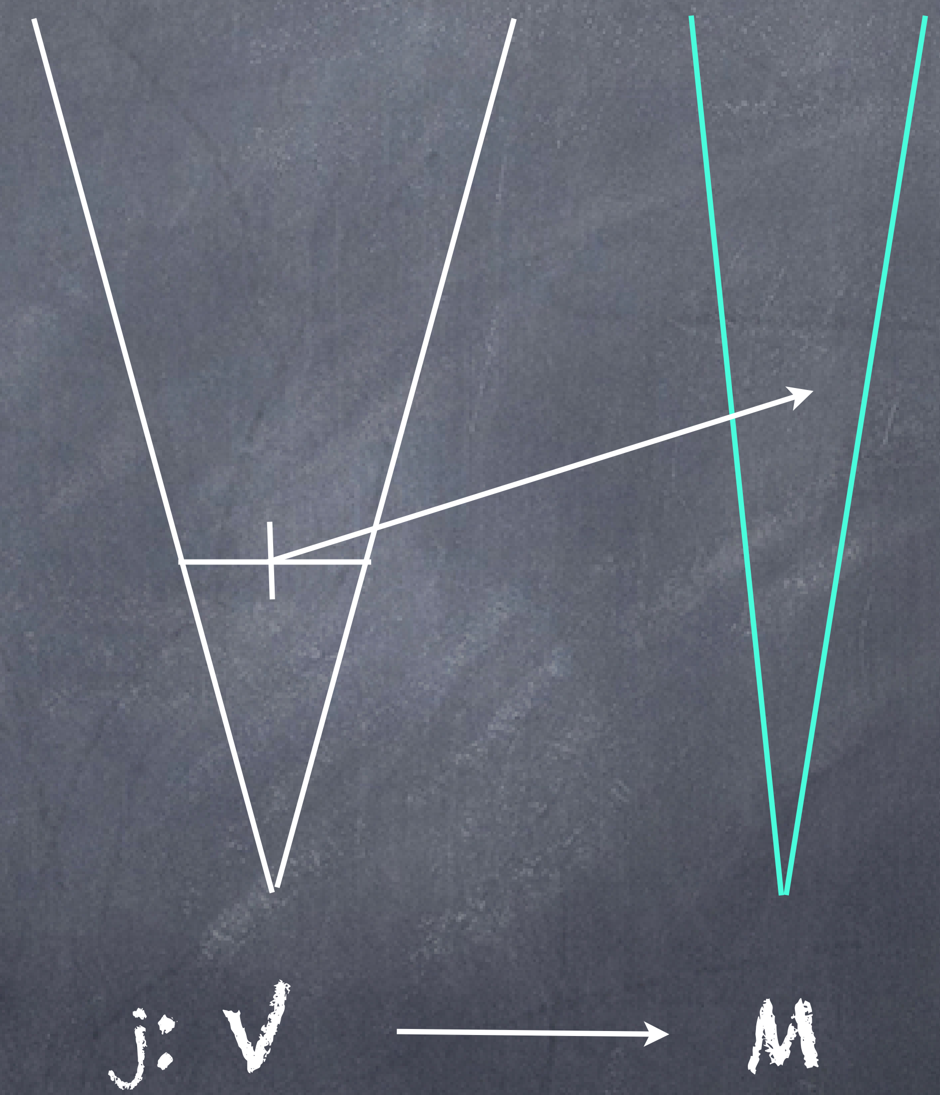


"V"

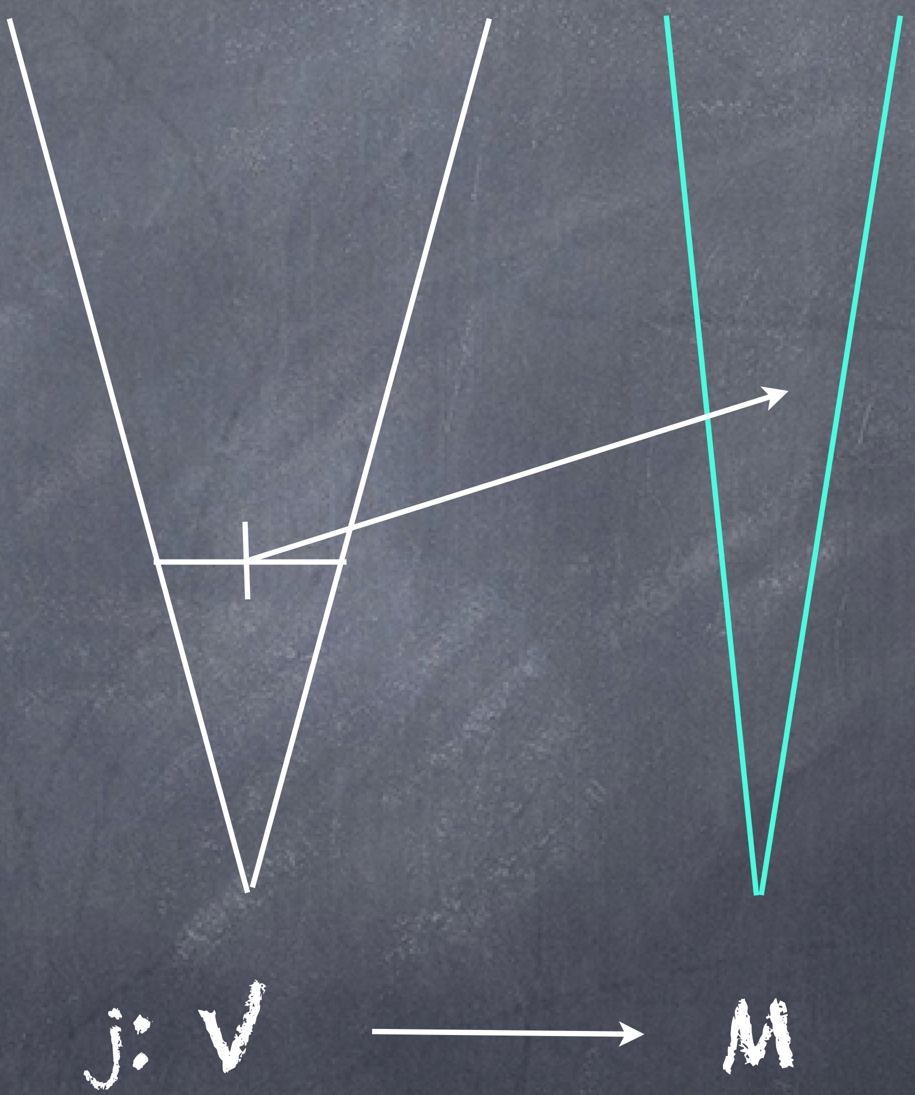
Standard Form of a Large Cardinal assumption

The basic building blocks of a cofinal set of large cardinals are elementary embeddings from V the universe of sets to a transitive model M.

Think of these as non-trivial injections of V into a proper subclass.



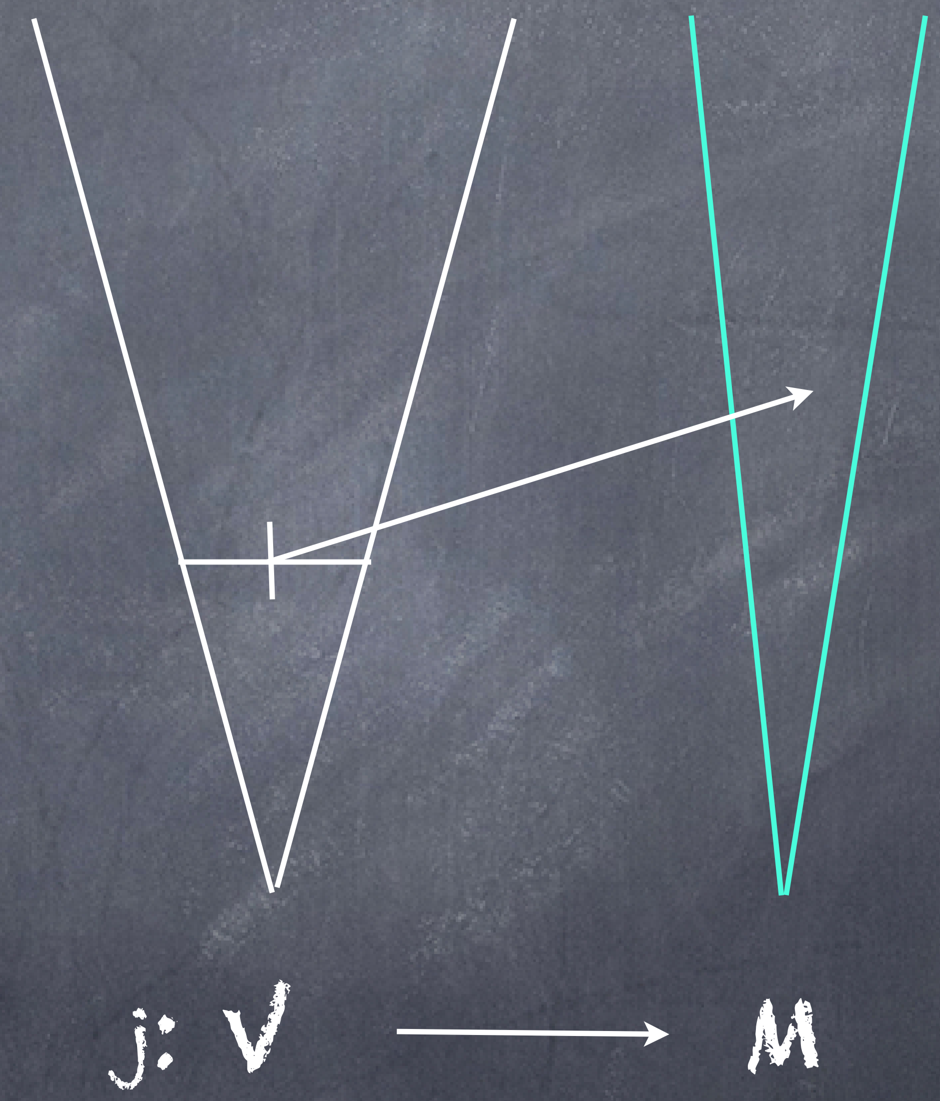
Any elementary embedding of V into a transitive class M **must** move an ordinal. The least ordinal moved is the **large cardinal.**



Two parameters determine the strength of the large cardinal:

- Where ordinals are moved

- The extent to which M resembles V



Remember Godel's theorem?

- Godel's theorem said that no consistent theory can prove it's own consistency
- This gives a hierarchy of consistency strength of assumptions:

A < B

if and only if the consistency of B implies the consistency of A

Remarkable Facts of Nature

- Large Cardinals form an essentially linear hierarchy of assumptions in this ordering.
- As far as is known, all natural assumptions extending ZFC fit on this hierarchy.

The singular cardinal hypothesis

If λ is a singular strong limit cardinal cardinal then

$$2^{\lambda} = \lambda^{+}.$$

Magidor and Jensen

- Magidor: If there is a supercompact cardinal then it is consistent that
$$2^{\aleph_\omega} > \aleph_{\omega+1}$$
- Jensen: If this happens then there are fairly strong large cardinals.

Menachem Magidor

Ronald Jensen

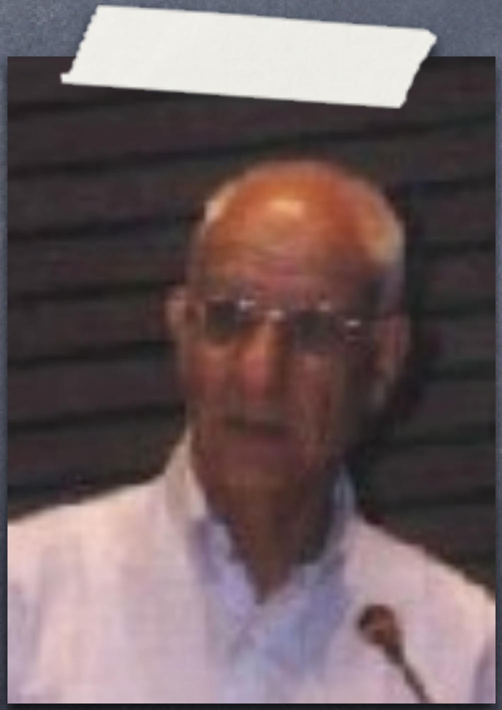
First clear example of the inevitability of Large Cardinals

Is this the end of the story?

Well … no.

The Levy-Solovay theorem

Large cardinals are preserved under "small forcing".



Azriel Levy

Robert Solovay

In particular

Large Cardinals cannot settle questions involving small sets:

e.g.

The continuum Hypothesis

Where the action is

Find axioms that settle the CH.

Then settle the rest ....

Avenues of Research: Forcing Axioms

- Martin's Maximum
- Proper Forcing axiom

Saharon Shelah

Menachem Magidor

Stevo Todorcevic

Forcing Axioms

- Prove that the real numbers are the second uncountable cardinal
- Give an essentially complete theory of sets of size ω_1
- In particular they settle most (all?) combinatorial questions

Generic large Cardinals

- These are axioms that combine large cardinal embeddings with forcing.
- The elementary embedding of V is revealed in a forcing extension of V
- Include ordinary large cardinals as special cases
- Settle essentially all questions.

Other Approches

- Specify entirely the mathematical universe by describing it as the result of a specific construction.

"Ultimate L"

- Give meta-mathematical arguments involving stronger logics.

"Omega Logic"

W. Hugh Woodin

Why won't they go away?

- As strengthenings of ZFC they are canonical
  (at least in the consistency hierarchy)
- But... if you've got an idea, let's hear it!

Recall:

Mathematical knowledge

=

First order logic +

Assumptions

Two lines of attack that
don't involve strengthening
the axioms

First Attack: the logic (either strengthen or weaken)

- Intuitionism/constructivism
- Second order logic
- A different strengthening of First order logic

We dont really need infinite sets (we dont really need uncountable sets)

- Everything "real" is finite
- Everything "real" is countable

The key word is NEED

- Logical need

  Given a result (say the Hahn-Banach theorem) that uses the Axiom of Choice in an essential role. Is there a related result that plays the same role in some application that can be proved using only finite sets? Countable sets?

Often the answer is yes.

The mathematics needed to design an aircraft probably can be derived in a purely finitist way.

But could airplanes be built if calculus didnt exist?

Mathematical Finance

The Fundamental Theorem of Asset Pricing

is proved using the Hahn-Banach theorem.

It CAN be proved using an "effective" version of HB. But would it have been? Would the researcher been able to find the right version and verify the hypothesis?

Asset Pricing

The basic theory of asset pricing (in a continuous context) is based on

Brownian Motion.

Essential to BM are continuous nowhere differential functions and abstract measure theory.

None of this is possible in very weak theories.

In each case

A fortiori-- one can go back and find an effective version of the theorem and an effective version of the proof.

However the set theoretic infrastructure was conceptually necessary for the mathematical development.

What would it mean?

If the conceptual framework of set theory is necessary for mathematics to proceed shouldn't we take it at face value?

Logically necessary

or

Conceptually necessary?

Thank You!