A heuristic approach to geometric phase

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Marsden's commitment to teaching mathematics

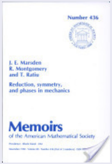
A legacy of writing for students

- JEM was a prolific writer of textbooks and reader-friendly expository monographs.
- He wrote texts on a wide variety of subjects and at many levels. He always wrote in a clear, detailed, and understandable style.

We are strongly influenced by his commitment to educating students about mathematics.

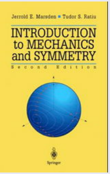
Museum

Marsden's texts and monographs that address geometric phase



Reduction, Symmetry, and Phases in Mechanics [1990] (with R. Montgomery and T. Ratiu)

Lectures on Mechanics [1992]



Introduction to mechanics and symmetry [1994,1999] (with T. Ratiu)

Motivation

Void in the lower level literature

- We could find no mathematics books on more elementary levels that address geometric phase
- The only undergraduate physics book we found that mentioned phase [D. Griffiths, *Introduction to Quantum Mechanics*, 2/e (2004)] describes Berry's phase in QM but not classical phase.

Goal

To use geometric phase examples to introduce lower level undergraduate mathematics students to some basic geometric mechanics and elementary differential geometry

Outline

Objectives

- Describe a conceptualization of **phase** for periodic functions
- Provide a prototype to illustrate **geometric phase**
- Determine when phase is geometric
- Develop a heuristic for computing geometric phase
- Peek at what's behind the curtain: a descriptive approach to reduction and reconstruction
- Apply the heuristic in another application with geometric phase
- Indicate the important role that differential geometry plays

What follows is a preliminary draft of a treatment for lower-level students.

A simple illustration of phase

The Wet-Dog Shake

The physics of the frequency of shaking dry vs. body size
[A. Dickerson, et al., *The Wet-Dog Shake*, `arXiv.org` 2010]

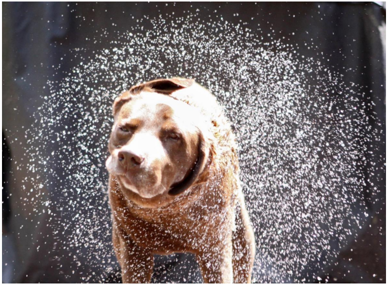


Photo credit: A. Dickerson and H. Hu, `engageengineering.org`

Shake frequencies

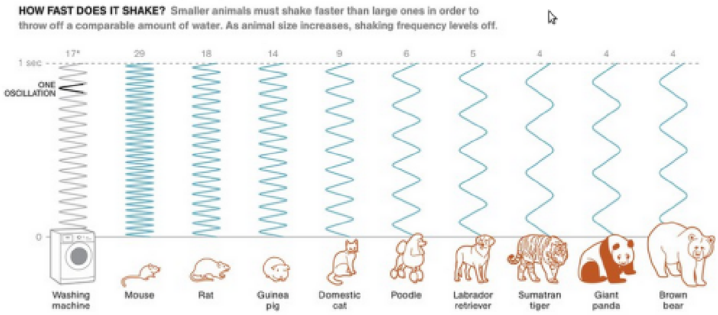


Photo credit: National Geographic Magazine, Sept. 2011

Dogs and cats

Let's compare a domestic cat (9 Hz) against a poodle (6 Hz).

What is phase?

Cat-dog phase

For periodic functions of *different* periods, the **phase** is defined as the value (in radians) that one function goes through if the other function completes exactly one cycle.

The domestic cat shakes at 9 Hz (or $T_1 = 1/9$ sec).
The poodle shakes at 6 Hz (or $T_2 = 1/6$ sec).

The phase (or *phase difference*) is

$$\gamma := 2\pi\,(T_1/T_2) = 2\pi(6/9) = 4\pi/3\,.$$

Thus, when the cat finishes one shake, the dog has turned $4\pi/3$ radians en route to his first shake.

Description of geometric phase

Identifying geometric phase

Geometric phase can often be visualized as the interplay between the two characteristic periods which go in and out of “synch”.

Phase is geometric if it is *independent of the speed at which the the cycles are traversed*.

It’s moot to ask whether cat-dog phase is geometric.

Gastronomical illustration of geometric phase

Matt's heartburn

Matt eats pizza for dinner every third day.
This results in heartburn every seventh day.

Thus Matt has pizza *and* heartburn once every 21 days.
The geometric phase is

$$\gamma = 2\pi \left(T_1 / T_2 \right) = 2\pi \left(3/7 \right) = 6\pi/7 \, .$$

The phase is geometric because the ratio 3:7 stays fixed:
If he has pizza more often, he gets heartburn more often.

Gastronomical illustration of geometric phase

Matt's football

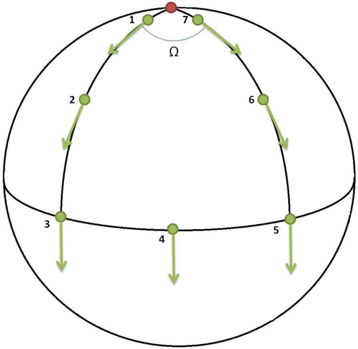
Matt eats pizza for dinner every third day but also watches Monday Night Football every seventh day.

Unfortunately, eating pizza more frequently does not create more Monday night games. This phase is *not* geometric.

Geometric phase can come from curvature

Parallel transport on a sphere

The phase γ of the parallel-transported tangent vector along a cycle is equal to the solid angle Ω subtended by the cycle. It is independent of the speed at which the path is traversed. Thus, it is geometric.



Heuristic approach to geometric phase

The heuristic approach

How can we calculate the geometric phase in *specific* mechanical systems?

Every example so far is a system with more than one characteristic period, in which the periods "synchronize" at regular intervals.

This suggests that geometric phase can always be defined for a (classical) system with at least two characteristic periods *as long as the periods are dependent in some way*.

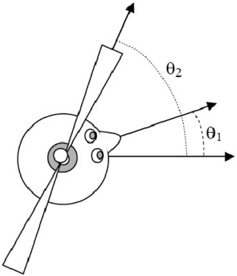
Elroy's beanie

Elroy [*The Jetsons*, 1962] spends a lot of time in weightlessness. In order to turn he spins his propeller, causing his body to rotate in the opposite direction. When he achieves the desired angle, he reaches up and stops the propeller, which stops his spin.

Image: Hanna-Barbera

Origin of this model: Marsden, Montgometry, Ratiu [1990]

Coordinates and parameters for the free Elroy's beanie



θ_1 = angle of Elroy's frontal-dorsal axis measured counterclockwise from the x-axis of an inertial frame

θ_2 = the angle of the central axis of the propeller from the x-axis

I_1 = Elroy's moment of inertia (second moment)

I_2 = the moment of inertia of the propeller

Dynamics of the free system

Lagrangian and Legendre transformation

The energy functional (**Lagrangian**) for the free system is the kinetic energy

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}\left(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2\right), \text{ where } \dot{\theta}_i := d\theta_i/dt\,.$$

The Legendre transformation obtains the momenta using the change of variable

$$p_i = \frac{\partial L}{\partial \dot{\theta}_i} = I_i\dot{\theta}_i,\ i = 1, 2\,.$$

Note that both θ_1 and θ_2 do not appear explicitly in L (i.e., they are are **cyclic** variables, so their respective momenta are constant.

New and improved coordinates for free system

Coordinates that isolate the dynamics

The only thing that matters physically is the angle between these axes of the two rigid bodies. This suggests a new set of coordinates,

$$\theta = \theta_1 \text{ and } \psi = \theta_2 - \theta_1 \,.$$

Now θ_1 is physically arbitrary: It's just the orientation with respect to the x-axis.

The difference coordinate ψ contains all of the dynamics.

New coordinates

Energy and momenta in new coordinates

In these new coordinates, the Lagrangian becomes

$$L(\theta, \psi, \dot{\theta}, \dot{\psi}) = \frac{1}{2}\left(I_1\dot{\theta}^2 + I_2(\dot{\theta} + \dot{\psi})^2\right) .$$

The new momenta are

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = I_1\dot{\theta} + I_2(\dot{\theta} + \dot{\psi}), \text{ and} \\ p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = I_2(\dot{\theta} + \dot{\psi}) , \end{aligned}$$

The new coordinates $\{\theta, \psi\}$ are also cyclic variables so the motion is given by the (**Euler-Lagrange**) equations:

$$\dot{p_\theta} = 0 \quad \text{and} \quad \dot{p_\psi} = 0 .$$

Brute force method to compute geometric phase

Geometric phase from the equations of motion

The equations of motion (EOM) are the 2nd order ODE system.

$$\begin{aligned} (I_1 + I_2)\ddot{\theta} + I_2\ddot{\psi} &= 0, \text{ and} \\ I_2\ddot{\theta} + I_2\ddot{\psi} &= 0\,, \end{aligned}$$

which can be solved to give

$$\ddot{\theta} = 0 = \ddot{\psi}\,.$$

Thus the angular velocities are the constant frequencies

$$\omega_\theta = \dot{\theta}(0) \quad \text{and } \omega_\psi = \dot{\psi}(0)\,.$$

Brute force method to compute geometric phase

Geometric phase from EOM

In the zero total angular momentum case, when $p_\theta = 0$, we have

$$0 = p_\theta = I_1\omega_\theta + I_2(\omega_\theta + \omega_\psi)$$

or

$$\frac{\omega_\theta}{\omega_\psi} = \frac{-I_2}{I_1 + I_2}.$$

So the geometric phase is

$$\gamma = 2\pi\left(\frac{T_\psi}{T_\theta}\right) = 2\pi\left(\frac{\omega_\theta}{\omega_\psi}\right) = -2\pi\left(\frac{I_2}{I_1 + I_2}\right).$$

Brute force method: analysis

Reality check on geometric phase.

If the moments of inertia are equal then we get

$$\gamma = -2\pi \frac{I_1}{I_1 + I_1} = -\pi \,.$$

This means that after a full cycle of the difference angle ψ Elroy's orientation has rotated by π radians in the opposite direction.

What's so special about p_θ?

The following are all integrals of the motion: L (energy), p_1, p_2, in the old coordinates; $p_\theta = p_1 + p_2$ (total angular momentum) and p_ψ in the new coordinates. So why not energy conservation? Or another conserved quantity?

Symmetry of the system

Circle group symmetry

To describe the dynamics, it suffices to find only an expression for $\psi(t)$. The angle θ merely tracks the orientation of the system of the coupled rigid bodies with the x-axis of the inertial frame.

Thus, the system has the symmetry of a circle: Merely rotating the entire configuration by a fixed angle α has no effect upon the dynamics.

We can describe this as a **group action**. Indeed, the (abelian) action of the planar rotation group $SO(2)$ on the configurations is

$$\alpha \cdot (\theta, \psi) = (\theta + \alpha, \psi) .$$

Note: In our original coordinates the action is

$$\alpha \cdot (\theta_1, \theta_2) = (\theta_1 + \alpha, \theta_2 + \alpha) .$$

Group symmetry

So why p_θ?

θ is the coordinate affected by the group action.

$\therefore$ *The momentum generated by the circle group action* is p_θ.

Reduction by symmetry and reconstruction

We have effectively used a one-dimensional symmetry group to **reduce** the dynamical system from two degrees of freedom to one.

Once we know the dynamics of ψ we can **reconstruct** the entire system merely by declaring the initial orientation of the system from the x-axis.

Geometric phase using the momentum of the symmetry

Solution using conservation law matching the symmetry

The total (conserved) angular momentum is given by $\mu = p_\theta$. We can use this to construct a one-form.

$$
\begin{aligned}
\mu &= (I_1 + I_2)\dot{\theta} + I_2\dot{\psi}\,, \\
\mu\, dt &= (I_1 + I_2)d\theta + I_2 d\psi\,, \\
d\theta &= \frac{\mu}{I_1 + I_2}\,dt - \frac{I_2}{I_1 + I_2}d\psi\,.
\end{aligned}
$$

Now, we restrict ourselves to $\mu = 0$ and integrate around one closed circuit in the angle ψ (where $\int_C d\psi = 2\pi$) to give us geometric phase as the precession in θ,

$$
\gamma = \Delta\theta = -\frac{I_2}{I_1 + I_2}2\pi\,.
$$

A glimpse behind the curtain: Reduction by symmetry

The **configuration space** M is the torus

$$T^2 = S^1 \times S^1 \text{ with coordinates } \{\theta, \psi\}$$

The **symmetry group** is the Lie group $SO(2)$ - the circle group, and the *action* $SO(2) \times T^2 \to T^2$ is given by

$$(\alpha, (\theta, \psi)) \to \alpha \cdot (\theta, \psi) = (\theta + \alpha, \psi)$$

This forms an equivalence relation on the configuration space

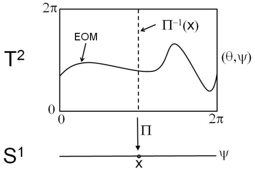
$$(\theta', \psi') \sim (\theta, \psi) \text{ if there exists an } \alpha \text{ where } (\theta', \psi') = \alpha \cdot (\theta, \psi)$$

The **shape space** $T^2/\sim =: T^2/SO(2)$ is the *quotient* of the map

$$\pi : T^2 \to S^1 : \quad (\theta, \psi) \mapsto \psi$$

Momentum mapping and phase

The solution curves on T^2 project to S^1.



The **momentum mapping** p_θ is the momentum in the direction of the action, that is, in the *fiber*

$$\pi^{-1}(\psi_0) = \{(\theta, \psi_0) \,|\, \theta \in S^1\} \simeq SO(2)$$

The geometric phase comes from computing on the zero-level set of the momentum mapping ($p_\theta = 0$). More rigorously, ...

Mechanical connection

The equation $p_\theta = 0$ may be written as

$$A := d\theta + \frac{I_2}{I_1 + I_2} d\psi = 0\,.$$

The one-form A on T^2 is called a **mechanical connection**.

The one-form, A is a linear functional on tangent vectors to T^2. From coordinates $\{\theta, \psi\}$ we construct a basis of the tangent space to T^2 at the point (θ_0, ψ_0),

$$\{\partial_\theta := \frac{\partial}{\partial\theta}\Big|_{(\theta_0,\psi_0)}\,,\ \partial_\psi := \frac{\partial}{\partial\psi}\Big|_{(\theta_0,\psi_0)}\}\,.$$

Note that $d\pi(\partial_\theta) = 0$ and $d\pi(\partial_\psi) = \frac{\partial}{\partial\psi}\Big|_{\psi_0}$.

Reconstruction

To reverse the projection map is not well defined, even if we insist that we stay on the fiber $\pi^{-1}(\psi_0)$. To stay on the fiber dictates that the lifted vector field ξ be of the form

$$\xi = \partial_\psi - f(\theta, \psi)\partial_\theta \,.$$

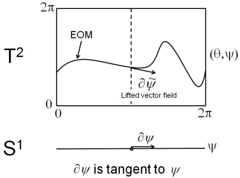
Horizontal lift

The **horizontal lift** is the unique tangent vector in the kernel of the connection, that is, the unique vector $\xi_{\theta,\psi}$ in the tangent space of T^2 that satisfies $A(\xi) = 0$.

A quick computation shows that

$$\xi = \partial_\psi - f(\theta_0, \psi_0)\partial_\theta = \partial_\psi - \frac{I_2}{I_1 + I_2}\partial_\theta \,.$$

Reconstructed EOM

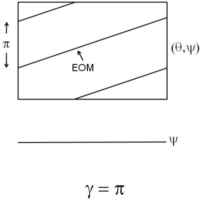


The reconstructed EOM must follow the horizontal lift.

We confirm the geometric phase to be

$$\Delta\Theta = -2\pi f(\theta_0) = -\frac{2\pi I_2}{I_1 + I_2}\,.$$

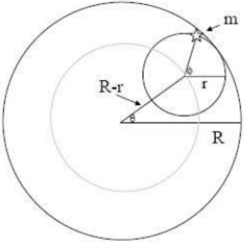
The picture for $I_1 = I_2$ is



Annular teacup ride

We call this system the “annular teacup ride” because the trajectory of a rider is restricted to an annulus.

Mad Tea Party at Disneyland



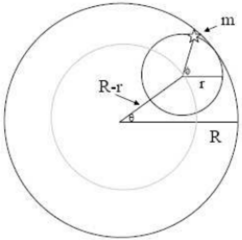
Idealization of ride

The turntable is a disc of radius R. The teacup disc is of radius $r < R/2$. The rider of mass m sits in a teacup at point $\star$.

Annular teacup ride

Disregard the rotation of the teacup itself (for now).

View this as a linkage: A massless (for now) rod of length $r_1 = R - r$ rotates freely about a fixed point. At the end of the rod is a pivot joint, to which is attached another massless rod of length $r_2 = r$. At the end of *this* rod is a point mass m.



A natural coordinate choice is $\{\theta, \phi\}$, both angles measured with respect to an inertial frame.

Annular teacup ride

Again, the $SO(2)$ action affects both coordinates equally. Like Elroy's beanie, use coordinates (θ, ψ) defined by

$$\theta = \theta \text{ and } \psi = \phi - \theta\,,$$

As before, the dynamics on T^2 reduce to $T^2/SO(2) \simeq S^1$. The free Lagrangian is

$$\begin{aligned} L(x, y) &= \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \\ L(\theta, \psi, \dot{\theta}, \dot{\psi}) &= \tfrac{m}{2}\left({r_1}^2\dot{\theta}^2 + {r_2}^2(\dot{\theta}+\dot{\psi})^2 + 2r_1r_2\dot{\theta}(\dot{\theta}+\dot{\psi})\cos\psi\right)\,. \end{aligned}$$

Annular teacup ride

We note that θ is cyclic while ψ is not; the conservation law we require is that p_θ is conserved. Since

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m{r_1}^2\dot\theta + m{r_2}^2(\dot\theta + \dot\psi) + m r_1 r_2(\dot\theta + \dot\psi)\cos\psi + m r_1 r_2 \dot\theta \cos\psi,$$

we get the mechanical connection

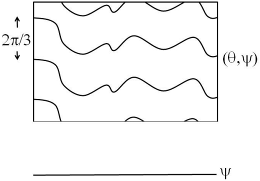
$$A = \left(m{r_1}^2 + m{r_2}^2 + 2 m r_1 r_2 \cos\psi\right) d\theta + \left(m r_1 r_2 \cos\psi\right) d\psi\,.$$

As before, restrict to the zero angular momentum level set, obtaining the phase difference in θ for one full cycle of ψ:

$$\Delta\theta = -\int_C \left(\frac{r_1 r_2 \cos\psi}{{r_1}^2 + {r_2}^2 + 2 r_1 r_2 \cos\psi}\right) d\psi.$$

Consider the case $r_1 = 2r_2$ (i.e., $R = 4r$). The geometric phase is then

$$\Delta\theta = -\int_0^{2\pi} \left(\frac{2\cos\psi}{5 + 4\cos\psi} \right) d\psi = -\frac{2\pi}{3}.$$



$\gamma = 2\pi/3$

Why is the mechanical connection the correct connection?

Consider the position-velocity space to be a metric space with metric given by the energy functional $L(\theta, \psi, \dot{\theta}, \dot{\psi})$.

The EOM are simply the geodesics of the metric, that is, the integrals of the horizontally lifted vector fields. The connection is the device that calculates how to use the metric to perform the horizontal lift.

We deem this metric connection "mechanical" because the metric comes from the energy.

Holonomy and curvature

Holonomy

The geometric phase is the **holonomy** of the connection A over a closed path.

The general form of a connection one-form is

$$A = d\theta + f(\theta, \psi)d\psi = 0\,.$$

Compare the connection for Elroy's beanie, where

$$f(\theta, \psi) = \frac{l_2}{l_1 + l_2}$$

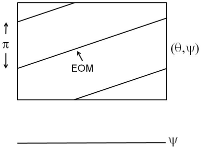
to that of the teacup ride, where

$$f(\theta, \psi) = \frac{r_1 r_2 \cos\psi}{({r_1}^2 + {r_2}^2 + 2 r_1 r_2 \cos\psi)}\,.$$

For Elroy, $f(\theta, \psi)$ is constant throughout T^2, indicating a **flat connection**. The flows of the geodesics are lines due to the flatness of the torus.

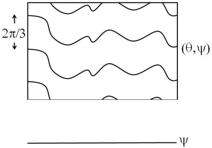
For the teacup, $f(\theta, \psi)$ is nonconstant, so it is **curved**, not flat. It is integrable because $\frac{\partial f}{\partial \theta} = 0$.

Elroy



$\gamma = \pi$

Teacup



$\gamma = 2\pi/3$

Conclusions

Summary

Visual toy models such as Elroy's beanie and the teacup ride may capture students' attention. These are simple examples to illustrate geometric concepts, such as metric connection, holonomy, and curvature.

Student activities to reinforce concepts

- Spirographs
- Add a clock spring between beanie and propeller
- Phase experiment: Gather data at an amusement park
- Ice skaters: The arm motions produce geometric phase
- Averaged phase in quasiperiodic systems