Anholonomic frames in nonholonomic mechanics*

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The Euler-Lagrange vector field

- Let (q^i) be coordinates on a manifold Q, $(q^i, \dot{q}^i)$ on its tangent TQ.
- We will always assume that the Lagrangian $L(q, \dot{q})$ is **regular**, i.e. the matrix $\left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ is everywhere non-singular.

$\rightsquigarrow$ The E-L eq. $\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^j + \dfrac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j - \dfrac{\partial L}{\partial q^i} = 0$ may then be written explicitly

in the form $\quad \ddot{q}^i = f^i(q, \dot{q})$.

We will interpret solutions of the E-L eq. as ***integral curves*** of the associated ***second-order*** differential equations **field** Γ **on** TQ, namely

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}.$$

$\rightsquigarrow$ This vector field is completely determined by the assumption that it is a second-order diff. eq. field and by the equations $\Gamma\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) - \dfrac{\partial L}{\partial q^i} = 0$.

An approach using anholonomic frames

- Two important lifts of a VF $Z = Z^i\partial/\partial q^i$ on Q to a VF on TQ:

  ⋆ **Complete (tangent) lift** $Z^{\mathrm{C}} = Z^i \dfrac{\partial}{\partial q^i} + \dfrac{\partial Z^i}{\partial q^j}\dot{q}^j \dfrac{\partial}{\partial \dot{q}^i} \in \mathfrak{X}(TQ)$.

  (flow of Z^{C} consists of the tangent maps of the flow of Z)

  ⋆ **Vertical lift** $Z^{\mathrm{V}} = Z^i \dfrac{\partial}{\partial \dot{q}^i} \in \mathfrak{X}(TQ)$.

  (tangent to the fibres of $\tau : TQ \to Q$ and on T_qQ coincides with Z_q)

- If $\{Z_i\}$ is a **basis of VF on** Q, $\{Z_i^{\mathrm{C}}, Z_i^{\mathrm{V}}\}$ is a **basis of VF on** TQ.
- An equivalent expression for the E-L eq:

$$\Gamma\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad \Leftrightarrow \qquad \Gamma(Z_i^{\mathrm{V}}(L)) - Z_i^{\mathrm{C}}(L) = 0.$$

- The frame $\{Z_i\}$ defines **quasi-velocities** v^i, s.t. $v_q = v^i Z_i(q) \in T_qQ$.

  ⇝ We can write $\Gamma = v^i Z_i^{\mathrm{C}} + \Gamma^i Z_i^{\mathrm{V}}$, where the functions $\Gamma^i(q, v)$ are to be determined from $\quad \Gamma(Z_i^{\mathrm{V}}(L)) - Z_i^{\mathrm{C}}(L) = 0$.

Hamel's equations

- For an **anholonomic** frame: $[Z_i, Z_j] = R^k_{ij} Z_k$.
- In general, for any VF Z on Q, function f on Q and 1-form θ on Q,

$$Z^{\mathrm{C}}(f) = Z(f), \quad Z^{\mathrm{V}}(f) = 0, \quad Z^{\mathrm{C}}\vec{\theta} = \overrightarrow{\mathcal{L}_Z \theta}, \quad Z^{\mathrm{V}}\vec{\theta} = \tau^* \theta(Z)$$

where $\vec{\theta}$ is the fibre-linear function on TQ defined by the 1-form θ.
(If $\{\vartheta^i\}$ is the dual basis of $\{Z_i\}$, then $\vec{\vartheta}^i = v^i$.)

- In terms of (not-natural) coordinates (q^i, v^i) on TQ we may write, for $Z_i = Z_i^j \partial/\partial q^j \in \mathfrak{X}(Q)$, that

$$Z_i^{\mathrm{C}} = Z_i^j \frac{\partial}{\partial q^j} - R^j_{ik} v^k \frac{\partial}{\partial v^j}, \qquad Z_i^{\mathrm{V}} = \frac{\partial}{\partial v^i},$$

$\rightsquigarrow$ The EL eq $\Gamma(Z_i^{\mathrm{V}}(L)) - Z_i^{\mathrm{C}}(L) = 0$ are then Hamel's equations:

$$\Gamma\left(\frac{\partial L}{\partial v^i}\right) - Z_i^j \frac{\partial L}{\partial q^j} + R^j_{ik} v^k \frac{\partial L}{\partial v^j} = 0.$$

Frames adapted to the action of a symmetry group

Let $\psi^Q : G \times Q \to Q$ be a (free and proper) **action** of a Lie group G on Q.
$\rightsquigarrow$ Then $\pi^Q : Q \to Q/G$ is a principal G-bundle.

A vector field Z on Q is **invariant** if $[Z, \tilde{\xi}] = 0$, for all infinitesimal generators $\tilde{\xi}$ corresponding to $\xi \in \mathfrak{g}$. Z then defines $\check{Z} \in \mathfrak{X}(Q/G)$.

We introduce a local frame $\{Z_i\} = \{\hat{E}_a, X_\alpha\}$ of G**-invariant** VFs on Q, where

1. $\hat{E}_a$, $a = 1 \ldots$dim(G) are tangent to the fibres of $Q \to Q/G$;
2. X_α, $\alpha = 1 \ldots$dim(Q/G) are transverse to the fibres.

1. Let $\{E_a\}$ be a basis of $\mathfrak{g}$ and $\tilde{E}_a$ (not-inv.) inf. gen.: $[\tilde{E}_a, \tilde{E}_b] = -C^c_{ab}\tilde{E}_c$.

$\rightsquigarrow$ $\hat{E}_a = A^b_a E^Q_b$ is invariant if $[\tilde{E}_a, \hat{E}_b] = \Big(E^Q_a(A^c_b) - C^c_{ad}A^d_b\Big)\tilde{E}_c = 0$
(integrability = Jacobi identity).

$\rightsquigarrow$ There are local solutions, for which $A = (A^b_a)$ is non-singular, and for which A is the identity on some chosen local section of $\pi^Q : Q \to Q/G$.

$\rightsquigarrow$ Let $U \subset Q/G$ be an open set over which Q is locally trivial and let (x^α) be coordinates on Q/G. Then $\pi^Q : U \times G \to U$, and $\psi_g^Q(x,h) = (x, gh)$ and $\hat{E}_a : (x,g) \mapsto (\widetilde{\mathrm{ad}_{g^{-1}}} E_a)(x,g) = T\psi_g^Q\big(\tilde{E}_a(x,e)\big)$.

2. Assume a principal connection on $Q \to Q/G$ given; take X_α to be the horizontal lift of a member of a coordinate basis of vector fields on Q/G.

Reduction of an invariant $Z \in \mathfrak{X}(Q)$. Then $Z = Z^a\hat{E}_a + Z^\alpha X_\alpha$.

$\rightsquigarrow$ Z is invariant if $[Z, \tilde{E}_a] = 0$, for all a. Z then defines $\check{Z} \in \mathfrak{X}(Q/G)$.

$\rightsquigarrow$ Since Z, E_a and X_α are all invariant, so also are Z^a and Z^α.

$\rightsquigarrow$ In particular, Z^α can be regarded as functions on Q/G, and we have

$$\check{Z} = Z^\alpha \frac{\partial}{\partial x^\alpha} \in \mathfrak{X}(Q/G),$$

where the x^α are coordinates on Q/G.

$\rightsquigarrow$ The reduced equations are simply $\quad \dot{x}^\alpha = Z^\alpha(x)$.

The Lagrangian vector field Γ in case of symmetry

ψ^Q induces an **action** $\psi_g^{TQ} = T\psi_g^Q$ **on** TQ. Let $\pi^{TQ} : TQ \to TQ/G$.

We assume L **is invariant**: $L(\psi_g^{TQ} v) = L(v)$.

We have two frames for $\mathfrak{X}(Q)$ at our disposal:

1. $\{Z_i\} = \{X_\alpha, \tilde{E}_a\}$ (moving frame = **non-invariant** frame).
2. $\{Z_i\} = \{X_\alpha, \hat{E}_a\}$ (body-fixed frame = **invariant** frame).

And two equivalent sets of equations from which we can determine Γ:

$$\left\{ \begin{array}{c} \Gamma(X_\alpha^{\rm V}(L)) - X_\alpha^{\rm C}(L) = 0, \\ \Gamma(\hat{E}_b^{\rm V}(L)) - \hat{E}_b^{\rm C}(L) = 0. \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{c} \Gamma(X_\alpha^{\rm V}(L)) - X_\alpha^{\rm C}(L) = 0, \\ \Gamma(\tilde{E}_b^{\rm V}(L)) - \tilde{E}_b^{\rm C}(L) = 0. \end{array} \right.$$

Proposition 1. Γ *is an* **invariant vector field on** TQ*.*

Proof: • **Infinitesimal condition?** Take $\xi \in \mathfrak{g}$.

$\rightsquigarrow$ The fundamental VF $\tilde{\xi}$ of the action ψ^Q on Q is the infinitesimal generator of the 1-par. group of transformations $\psi^Q_{\exp(t\xi)}$.

$\rightsquigarrow$ The fundamental VFs of the induced action $T\psi^Q_g$ on TQ is the infin. generator of $T\psi^Q_{\exp(t\xi)}$, and is thus $\tilde{\xi}^{\mathrm{C}}$, the **complete lift** of $\tilde{\xi}$!

$\rightsquigarrow$ To prove: If $\tilde{E}^{\mathrm{C}}_a(L) = 0$, then $[\tilde{E}^{\mathrm{C}}_a, \Gamma] = 0$, $\{E_a\}$ basis of $\mathfrak{g}$.

• The Euler-Lagrange equations become:

$$\begin{cases} \Gamma(X^{\mathrm{V}}_\alpha(L)) - X^{\mathrm{C}}_\alpha(L) = 0, \\ \Gamma(\hat{E}^{\mathrm{V}}_b(L)) - \hat{E}^{\mathrm{C}}_b(L) = 0. \end{cases}$$

It follows that

$$\begin{aligned} 0 &= \tilde{E}^{\mathrm{C}}_b(\Gamma(X^{\mathrm{V}}_\alpha(L))) - \tilde{E}^{\mathrm{C}}_b(X^{\mathrm{C}}_\alpha(L)) \\ &= [\tilde{E}^{\mathrm{C}}_b, \Gamma](X^{\mathrm{V}}_\alpha(L)) + \Gamma(\tilde{E}^{\mathrm{C}}_b(X^{\mathrm{V}}_\alpha(L))) - [\tilde{E}^{\mathrm{C}}_b, X^{\mathrm{C}}_\alpha](L) - X^{\mathrm{C}}_\alpha(\tilde{E}^{\mathrm{C}}_b(L)) \\ &= [\tilde{E}^{\mathrm{C}}_b, \Gamma](X^{\mathrm{V}}_\alpha(L)) + \Gamma([\tilde{E}^{\mathrm{C}}_b, X^{\mathrm{V}}_\alpha](L)) + \Gamma(X^{\mathrm{V}}_\alpha(\tilde{E}^{\mathrm{C}}_b(L))) \\ &= [\tilde{E}^{\mathrm{C}}_b, \Gamma](X^{\mathrm{V}}_\alpha(L)). \end{aligned}$$

Thus, $[\tilde{E}_b^{\mathrm{C}}, \Gamma](X_\alpha^{\mathrm{V}}(L)) = 0$. Likewise, $[\tilde{E}_b^{\mathrm{C}}, \Gamma](\hat{E}_c^{\mathrm{V}}(L)) = 0$.

• Since Γ is a second-order differential equation field:

$[\tilde{E}_b^{\mathrm{C}}, \Gamma] = B_b^\alpha X_\alpha^{\mathrm{V}} + B_b^a \hat{E}_a^{\mathrm{V}},$ for some functions B_b^α, B_b^a on TQ

Therefore:

$$B_b^\alpha X_\alpha^{\mathrm{V}}(X_\beta^{\mathrm{V}}(L)) + B_b^a \hat{E}_a^{\mathrm{V}}(X_\beta^{\mathrm{V}}(L)) = 0$$
$$B_b^\alpha X_\alpha^{\mathrm{V}}(\hat{E}_c^{\mathrm{V}}(L)) + B_b^a \hat{E}_a^{\mathrm{V}}(\hat{E}_c^{\mathrm{V}}(L)) = 0,$$

and thus, due to the regularity of L, $B_b^\alpha = 0$ and $B_b^a = 0$.

Explicit expression of the reduced VF $\check{\Gamma}$ on TQ/G

- **Reduction?** L reduces to a function $\check{L}$ on $TQ/G \rightsquigarrow L = \check{L} \circ \pi^{TQ}$;
  Γ reduces to a VF $\check{\Gamma}$ on $TQ/G \rightsquigarrow T\pi^{TQ} \circ \Gamma = \check{\Gamma} \circ \pi^{TQ}$.

  $\rightsquigarrow$ The defining relation for the reduced vector field $\check{\Gamma}$ are simply :

$$\begin{cases} \check{\Gamma}(\check{X}_\alpha^{\mathrm{V}}(\check{L})) - \check{X}_\alpha^{\mathrm{C}}(\check{L}) = 0, \\ \Gamma(\check{E}_b^{\mathrm{V}}(\check{L})) - \check{E}_b^{\mathrm{C}}(\check{L}) = 0. \end{cases}$$

- Denote the quasi-coordinates w.r.t. $\{X_\alpha, \hat{E}_a\}$ by (v^α, w^a) and let (x^α) be coordinates on Q/G.

  $\rightsquigarrow$ Since x^α, v^α and w^a are invariant functions on TQ, they induce **coordinates on** TQ/G.

  $\rightsquigarrow$ Any VF $\check{W}$ on TQ/G is determined by its action on x^α, v^α and w^a.

- If we set $[\hat{E}_a, \hat{E}_b] = C^c_{ab}\hat{E}_c, \quad [X_\alpha, X_\beta] = K^a_{\alpha\beta}\hat{E}_a, \quad [X_\alpha, \hat{E}_a] = \Upsilon^b_{\alpha a}\hat{E}_b$, then,

$$\begin{aligned}
\check{E}^{\mathrm{C}}_a &= \left(\Upsilon^b_{\alpha a}v^i + C^b_{ac}w^c\right)\frac{\partial}{\partial w^b}, & \check{E}^{\mathrm{V}}_a = \frac{\partial}{\partial w^a}, \\
\check{X}^{\mathrm{C}}_\alpha &= \frac{\partial}{\partial x^\alpha} - \left(K^a_{\alpha\beta}v^\beta + \Upsilon^a_{\alpha b}w^b\right)\frac{\partial}{\partial w^b}, & \check{X}^{\mathrm{V}}_\alpha = \frac{\partial}{\partial v^\alpha}.
\end{aligned}$$

- We have $\Gamma = w^a\hat{E}^{\mathrm{C}}_a + v^\alpha X^{\mathrm{C}}_\alpha + \Gamma^a\hat{E}^{\mathrm{V}}_a + \Gamma^\alpha X^{\mathrm{V}}_\alpha$.
  - ⇝ Each term is invariant, so Γ^a and Γ^α define functions on TQ/G.
  - ⇝ We have

$$\begin{aligned}
\check{\Gamma} &= w^a(\Upsilon^b_{\alpha a}v^\alpha + C^b_{ac}w^c)\frac{\partial}{\partial w^b} + v^\alpha\frac{\partial}{\partial x^\alpha} \\
&\quad - v^\alpha\left(K^a_{\alpha\beta}v^\beta + \Upsilon^a_{\alpha b}w^b\right)\frac{\partial}{\partial w^b} + \Gamma^a\frac{\partial}{\partial w^a} + \Gamma^\alpha\frac{\partial}{\partial v^\alpha} \\
&= v^\alpha\frac{\partial}{\partial x^\alpha} + \Gamma^\alpha\frac{\partial}{\partial v^\alpha} + \Gamma^a\frac{\partial}{\partial w^a}.
\end{aligned}$$

The reduced equations become

$$\check{\Gamma}\left(\frac{\partial l}{\partial v^\alpha}\right) - \frac{\partial l}{\partial x^\alpha} = (K^a_{\alpha\gamma} v^\gamma + \Upsilon^a_{\alpha b} w^b)\frac{\partial l}{\partial w^a}$$
$$\check{\Gamma}\left(\frac{\partial l}{\partial w^a}\right) = (\Upsilon^b_{\alpha a} v^\alpha + C^b_{ac} w^c)\frac{\partial l}{\partial w^b}.$$

⇝ This is **Lagrange-Poincaré reduction**! (see e.g. [Cendra et al, 2001])

⇝ The two equations correspond to splitting the equations according to the so-called Atiyah sequence,

$$0 \to (Q \times \mathfrak{g})/G \to TQ/G \to T(Q/G) \to 0.$$

Routh reduction

• Take now $\{\tilde{E}_a, X_\alpha\}$ as the basis for VF on Q.

$\rightsquigarrow$ The E-L eq. are now $\left\{ \begin{array}{l} \Gamma(X_\alpha^{\mathrm{V}}(L)) - X_\alpha^{\mathrm{C}}(L) = 0, \\ \Gamma(\tilde{E}_b^{\mathrm{V}}(L)) - \tilde{E}_b^{\mathrm{C}}(L) = 0 \end{array} \right.$.

$\rightsquigarrow$ Put, in short, $p_a = \tilde{E}_b^{\mathrm{V}}(L)$ for the **momentum**.

• Since L is **invariant** ($\tilde{E}_b^{\mathrm{C}}(L) = 0$) solutions lie on a **fixed level set** T_μ of the momentum: $p_a = \mu_a$.

• From the second eq., Γ is **tangent** to all level sets T_μ.

$\rightsquigarrow$ The G-action on TQ restricts to a G_μ-action on T_μ.

$\rightsquigarrow$ When restricted to T_μ, Γ is G_μ-invariant ($[\tilde{\xi}^{\mathrm{C}}, \Gamma] = 0, \forall \xi \in \mathfrak{g}_\mu$).

$\rightsquigarrow$ It reduces to a VF $\check{\Gamma}$ on T_μ/G_μ.

$\rightsquigarrow$ The equations for the integral curves of $\check{\Gamma}$ are differential equations in all variables on Q, except for those associated with G_μ!

• Let $R = L - v^a p_a$ be the **Routhian**, and R^μ its restriction to T_μ. One may rewrite the reduced equations in terms of $\check{R}^\mu$ (the Routh equations).

Systems with linear nonholonomic constraints

The constraints define a distribution $\mathcal{D}$ on Q (and associated subman of TQ).

Choose a frame $\{Z_i\} = \{X_\alpha, X_a\}$ whose first m members $\{X_\alpha\}$ span $\mathcal{D}$.
Denote the corresponding quasi-velocities by (v^α, v^a).

$\rightsquigarrow$ $v_q \in \mathcal{D}$ iff $v^a = 0$.

$\rightsquigarrow$ A VF Γ on $\mathcal{D}$ is tangent to $\mathcal{D}$ if and only if $\Gamma(v^a) = 0$.

$\rightsquigarrow$ A VF Γ on $\mathcal{D}$ is of second-order type (i.e. satisfy $\tau_{*(q,u)}\Gamma = u$, $\forall (q,u) \in \mathcal{D}$) and is tangent to $\mathcal{D}$ iff it is of the form $\Gamma = v^\alpha X_\alpha^{\mathrm{C}} + \Gamma^\alpha X_\alpha^{\mathrm{V}}$.

Proposition 2. *If L is regular w.r.t. $\mathcal{D}$ (if $\left(X_\alpha^{\mathrm{V}}(X_\beta^{\mathrm{V}}(L))\right)$ is nonsingular on $\mathcal{D}$), there is a unique Γ on $\mathcal{D}$ which is of second-order type, is tangent to $\mathcal{D}$, and is such that on $\mathcal{D}$*

$$\Gamma(Z^{\mathrm{V}}(L)) - Z^{\mathrm{C}}(L) = 0, \qquad \forall Z \in \mathcal{D}.$$

It may be determined from the equations $\Gamma(X_\alpha^{\mathrm{V}}(L)) - X_\alpha^{\mathrm{C}}(L) = 0$ *(on $\mathcal{D}$).*

$\rightsquigarrow$ The non-zero functions $\lambda_a := \Gamma(X_a^{\mathrm{V}}(L)) - X_a^{\mathrm{C}}(L)$ are Lagrangian multipliers.

We will always assume that L is regular with respect to both $\mathcal{D}$ and TQ.

Invariance of nonholonomic systems

Assume that L and $\mathcal{D}$ are invariant under the induced action of G on TQ.

Proposition 3. *The VF Γ is invariant under the induced action of G on $\mathcal{D}$.*

$\rightsquigarrow$ Since Γ is invariant, it reduces to a vector field $\check{\Gamma}$ on $\mathcal{D}/G$.

How to define a frame adapted to this situation? What is the analogue of the Atiyah sequence?

- Since $\mathcal{D}$ is invariant it defines a distribution $\bar{\mathcal{D}}$ on Q/G by $\bar{\mathcal{D}}_{\pi(q)} = \pi_*(\mathcal{D}_q)$. Let us assume that $\bar{\mathcal{D}}$ has constant dimension.

- Let $\mathcal{V}_q = \ker \pi_{*q}$ and $\mathcal{S}_q := \ker \pi_{*q}|_{\mathcal{D}_q} = \mathcal{D}_q \cap \mathcal{V}_q$.
  - $\star$ We may identify $\mathcal{S}_q$ with $\mathfrak{g}^q = \{A \in \mathfrak{g} \,|\, \tilde{A}_q \in \mathcal{S}_q\} \subset \mathfrak{g}$.
  - $\star$ Consider $\mathfrak{g}^{\mathcal{D}} = \{(q, A) \,|\, A \in \mathfrak{g}^q\}$.
  - $\star$ There is an action of G on $\mathfrak{g}^{\mathcal{D}}$ given by $(q, A) \mapsto (\psi_g(q), \mathrm{ad}(g^{-1})A)$.
  - $\star$ Its quotient $\mathfrak{g}^{\mathcal{D}}/G$ is a vector subbundle of $(Q \times \mathfrak{g})/G \to Q/G$.

Proposition 4. *We have the following short exact sequence of vector bundles over* Q/G*:*

$$0 \to \bar{\mathfrak{g}}^{\mathcal{D}} \to \mathcal{D}/G \to \bar{\mathcal{D}} \to 0.$$

(This is a version for nonholonomic systems of the so-called Atiyah sequence,

$$0 \to (Q \times \mathfrak{g})/G \to TQ/G \to T(Q/G) \to 0.)$$

We can use this to divide the reduced nonhol equations into two sets.

We can choose an **invariant** frame $\{X_i\} = \{X_\alpha, X_a\}$ with the next properties:

- $\{X_\alpha\}$ a basis of $\mathcal{D}$ and is of the form $\{X_\rho, X_\kappa\}$ where $\{X_\rho\}$ is a basis for $\mathcal{S}$.
- $\{X_a\}$ takes the form $\{X_c, X_k\}$ where the X_c are vertical.
- $\{X_\rho, X_c\}$ is a basis $\{X_r\}$ of the vertical vector fields (as $\{\hat{E}_a\}$ was in the unconstrained case).
- $\{X_\kappa, X_k\} = \{X_I\}$ is transverse to the fibres of $Q \to Q/G$ and is invariant, e.g. the horizontal lifts of VFs Y_I on Q/G w.r.t. some principal connection.
- The vector fields Y_κ form a basis for $\bar{\mathcal{D}}$.

The Lagrange-d'Alembert equations $\Gamma(X^{\mathrm{V}}_\alpha(L)) - X^{\mathrm{C}}_\alpha(L) = 0$ reduce to the following equations on $\mathcal{D}/G$:

$$\check{\Gamma}\left(\frac{\partial l}{\partial v^\rho}\right) = \left(\Upsilon^r_{\kappa\rho} v^\kappa - \bar{C}^r_{\rho\sigma} v^\sigma\right) \frac{\partial l}{\partial v^r}$$

$$\check{\Gamma}\left(\frac{\partial l}{\partial v^\kappa}\right) - Y_\kappa(l) + R^I_{\kappa\lambda} v^\lambda \frac{\partial l}{\partial v^I} = \left(K^r_{\kappa\lambda} v^\lambda - \Upsilon^r_{\kappa\rho} v^\rho\right) \frac{\partial l}{\partial v^r}.$$

The constrained Lagrangian L_c is invariant, and defines a function l_c on $\mathcal{D}/G$.

Proposition 5. *The Lagrange-d'Alembert-Poincaré equations are given by*

$$\check{\Gamma}\left(\frac{\partial l_c}{\partial v^\rho}\right) = \left(\Upsilon^r_{\kappa\rho} v^\kappa - \bar{C}^r_{\rho\sigma} v^\sigma\right) \left.\frac{\partial l}{\partial v^r}\right|_{\mathcal{D}/G}$$

$$\check{\Gamma}\left(\frac{\partial l_c}{\partial v^\kappa}\right) - Y_\kappa(l_c) + R^\lambda_{\kappa\mu} v^\mu \frac{\partial l_c}{\partial v^\lambda} = -R^k_{\kappa\lambda} v^\lambda \left.\frac{\partial l}{\partial v^k}\right|_{\mathcal{D}/G} + \left(K^r_{\kappa\lambda} v^\lambda - \Upsilon^r_{\kappa\rho} v^\rho\right) \left.\frac{\partial l}{\partial v^r}\right|_{\mathcal{D}/G}.$$

The first eq. is (a version of) the reduced momentum equation (see e.g. [Bloch et al, 1996]).

Routh type reduction for nonholonomic systems

Assume there exists $H \subset G$, such that $\tilde{A} \in \mathcal{D}$ for all $A \in \mathfrak{h}$ and $\mathcal{S}_q (= \mathcal{D}_q \cap \mathcal{V}_q)$ is given by $\{\tilde{A}(q) \,|\, A \in \mathfrak{h}, q \in Q\}$

(i.e. assume there is a horizontal symmetry group, [Bloch et al, 1996], [Cortes, 2002]).

$\rightsquigarrow$ Then $\mathfrak{h} = \mathrm{ad}(g^{-1})\mathfrak{h}$, i.e. $\mathfrak{h}$ is an ideal (or H is a normal subgroup).

As before: all one needs to do is to choose an appropriate frame:

(assume (for simplicity) that $\mathcal{V}_q + \mathcal{D}_q = T_qQ$.)

- Let $\{X_\kappa\}$ be the invariant vector fields we had before.
- let $\{E_r\} = \{E_\rho, E_c\}$ be a basis of $\mathfrak{g}$ whose first members $\{E_\rho\}$ span $\mathfrak{h}$

$\rightsquigarrow$ We can use $\{X_\alpha\} = \{X_\kappa, \tilde{E}_\rho\}$ as a (now **not-invariant**) frame for $\mathcal{D}$.

$\rightsquigarrow$ We can use $\{X_a\} = \{\tilde{E}_\rho, \tilde{E}_c\}$ as a basis of $\mathcal{V}$.

$\rightsquigarrow$ We can use $\{X_\alpha, X_a\} = \{X_\kappa, \tilde{E}_\rho, \tilde{E}_c\}$ as a complete basis for vector fields on Q (with corresponding quasi-velocities $(v^\kappa, \tilde{v}^\rho, \tilde{v}^c)$).

In this frame, the Lagrange-d'Alembert equations $\Gamma(X^{\rm V}_\alpha(L)) - X^{\rm C}_\alpha(L) = 0$ are

$$\left\{ \begin{array}{l} \Gamma(X^{\rm V}_\kappa(L)) - X^{\rm C}_\kappa(L) = 0, \\ \Gamma(\tilde{E}^{\rm V}_\rho(L)) - \tilde{E}^{\rm C}_\rho(L) = 0. \end{array} \right.$$

$\rightsquigarrow$ Given that $\tilde{E}^{\rm C}_\rho(L) = 0$, we get $\tilde{E}^{\rm V}_\rho(L) = \mu_\rho$, on $\mathcal{D}$. Denote a level set by N_μ.

$\rightsquigarrow$ The remaining equations can be rewritten in terms of a Routhian.

Reduction.

- Restrict Γ to a level set N_μ.
- The action of G on $\mathcal{D}$ restricts to an action of H_μ on N_μ in $\mathcal{D}$. Indeed:

$$0 = A^\sigma \tilde{E}^{\rm C}_\sigma(\tilde{E}^{\rm V}_\rho(L)) = A^\sigma C^\tau_{\rho\sigma} \tilde{E}^{\rm V}_\tau(L) = A^\sigma C^\tau_{\rho\sigma}\mu_\tau \quad \Leftrightarrow \quad A = A^\sigma E_\sigma \in \mathfrak{h}_\mu.$$

$\rightsquigarrow$ We can reduce Γ to a vector field $\check{\Gamma}_1$ on N_μ/H_μ.

- But: Since H is normal, the G-action on $\mathcal{D}$ restricts to a G_μ-action on N_μ:

$$0 = A^r \tilde{E}^{\rm C}_r(\tilde{E}^{\rm V}_\rho(L)) = A^r C^s_{\rho r} \tilde{E}^{\rm V}_s(L) = A^r C^\sigma_{\rho r} \tilde{E}^{\rm V}_\sigma(L) = A^r C^\sigma_{\rho r}\mu_\sigma \Leftrightarrow A = A^r E_r \in \mathfrak{g}_\mu.$$

$\rightsquigarrow$ Γ restricts to a G_μ-invariant vector field on N_μ, which we can reduce to a vector field $\check{\Gamma}_2$ on N_μ/G_μ.

The link with $\check{\Gamma}_1$? There are two choices:

(1) Do a direct reduction by G_μ,

(2) Do a reduction in two stages. One may define an induced action of G_μ/H_μ on N_μ/H_μ. The vector field $\check{\Gamma}_1$ will be invariant under that action and we can do a 2nd reduction.

(We will not give expressions for these reduced vector fields and their corresponding differential equations.)

The Cartan form approach to symmetries

- Let $\theta_L = \dfrac{\partial L}{\partial v^i} dx^i$, $\omega_L = d\theta_L$ be the **Cartan** 1- and 2-form of a Lagrangian L.
- Let $\tilde{\mathcal{D}}$ be the distribution on $\mathcal{D}$ which is projectable to Q, and $\tau_{|\mathcal{D}*}\tilde{\mathcal{D}} = \mathcal{D}$.
- For f a function on $\mathcal{D}$, let Z_f be the unique (Hamiltonian-type) vf on $\mathcal{D}$ such that $Z_f \in \tilde{\mathcal{D}}$ and $Z_f \lrcorner \iota^* \omega_L - df \in \tilde{\mathcal{D}}^\circ$.

Proposition 6. *The function f is a first integral of Γ if and only if $Z_f(\iota^* E_L) = 0$.*

Proposition 7. *Let $Z \in \tilde{\mathcal{D}}$ be such that $\tilde{\mathcal{D}} \lrcorner \mathcal{L}_Z(\iota^* \omega_L) \subset \tilde{\mathcal{D}}^\circ$, $\mathcal{L}_Z(\tilde{\mathcal{D}}) \subset \tilde{\mathcal{D}}$, $\mathcal{L}_Z(\iota^* \omega_L) \in d(\tilde{\mathcal{D}}^\circ)$, and $Z(\iota^* E_L) = 0$.*

$\rightsquigarrow$ *Then Z is a symmetry of Γ, and there is, at least locally, a function f on $\mathcal{D}$ such that $Z = Z_f$ and $\Gamma(f) = 0$.*

$\rightsquigarrow$ *The set of vector fields Z satisfying these conditions forms a Lie algebra $\mathcal{S}$.*

$\rightsquigarrow$ *For $Z_1, Z_2 \in \mathcal{S}$, with first integrals f_1, f_2, we have $Z_1(f_2) = -Z_2(f_1)$, and the first integral of $[Z_1, Z_2]$ is (up to an additive constant) $Z_1(f_2)$.*

(related results: [Bates and Śniatycki, 1993],[Cushman et al, 1995],[Giachetta, 2000], [Zenkov, 2002], ...)

The nonholonomic Noether theorem

Let $\varepsilon(X) = \Gamma(X^{\mathrm{V}}(L)) - X^{\mathrm{C}}(L)$ for X a vector field on Q.

The next statement is the *nonholonomic Noether theorem* of e.g. [Fasso et al, 2007].

Proposition 8. *For a vector field Z on Q any two of the following three conditions imply the third: (1) $Z^{\mathrm{C}}(L) = 0$ on $\mathcal{D}$; (2) $\varepsilon(Z) = 0$; (3) $Z^{\mathrm{V}}(L)|_{\mathcal{D}}$ is a first integral of Γ.*

An analogue of Proposition 8, that makes use of Proposition 6:

Proposition 9. *For any vector field Z tangent to $\mathcal{D}$ and for any function f on $\mathcal{D}$ such that $Z \lrcorner \iota^*\omega_L - df \in \tilde{\mathcal{D}}^\circ$, we have*

$$\Gamma(f) = Z(\iota^* E_L) - \iota^*\epsilon(Z);$$

and if any two of the terms vanish so does the third.

Application

Let (M, g) be a Riemannian manifold, with Levi-Civita connection ∇.

Set $K_Z(u, v) = g(\nabla_u Z, v)$.

Proposition 10. *Any two of the following conditions implies the third:*

1. *Z is orthogonal to the second fundamental form of N;*

2. *the restriction of K_Z to TN is skew;*

3. *$g(Z, \dot{c})$ is constant along every geodesic of N.*

This result is a special case of the nonholonomic Noether theorem where the constraints are actually holonomic!

We have also extended this proposition to Lagrangians of mechanical type and we have given conditions for quadratic first integrals.

Some of our papers:

- M. Crampin and T. Mestdag, Routh's procedure for non-Abelian symmetry groups, *J. Math. Phys.* **49** (2008) 032901.
- T. Mestdag and M. Crampin, Invariant Lagrangians, mechanical connections and the Lagrange-Poincaré equations, *J. Phys. A: Math. Theor.* **41** (2008) 344015.
- M. Crampin and T. Mestdag, Reduction of invariant constrained systems using anholonomic frames, Journal of Geometric Mechanics 3 (2011) 23-40.
- M. Crampin and T. Mestdag, The Cartan form for constrained Lagrangian systems and the nonholonomic Noether theorem. Int. J. Geom. Methods. Mod. Phys. 8 (2011) 897-923.

See: http://users.ugent.be/∼tmestdag

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